J. Appl. Math. & Informatics Vol. **35**(2017), No. 1 - 2, pp. 11 -24
https://doi.org/10.14317/jami.2017.011

EXISTENCE AND MULTIPLICITY OF WEAK SOLUTIONS FOR SOME $p(x)$-LAPLACIAN-LIKE PROBLEMS VIA VARIATIONAL METHODS

G.A. AFROUZI, S. SHOKOOH, N.T. CHUNG*

ABSTRACT. Using variational methods, we study the existence and multiplicity of weak solutions for some $p(x)$-Laplacian-like problems. First, without assuming any asymptotic condition neither at zero nor at infinity, we prove the existence of a non-zero solution for our problem. Next, we obtain the existence of two solutions, assuming only the classical Ambrosetti-Rabinowitz condition. Finally, we present a three solutions existence result under appropriate condition on the potential F.

AMS Mathematics Subject Classification: 35J55, 35J65.
Key words and phrases: Variable exponent Sobolev spaces; Weak solutions; $p(x)$-Laplacian-like problems; Variational methods.

1. Introduction

The aim of this paper is to study the following $p(x)$-Laplacian-like problem

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & x\in\Omega, \\ u=0, \qquad x\in\partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, $p\in C(\overline{\Omega})$ with

$$N<p^-:=\inf_{x\in\overline{\Omega}}p(x)\leq p^+:=\sup_{x\in\overline{\Omega}}p(x)<+\infty$$

and f is an L^1-Carathéodory function.

Received January 17, 2016. Revised April 27, 2016. Accepted May 18, 2016. *Corresponding author.

The equation

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(t,u) \quad \text{in}\, \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

plays, as is well known, a role in differential geometry and in the theory of relativity. Existence, non-existence and multiplicity of positive solutions of problem (2) have been discussed by several authors in the last decades.

Obersnel and Omari in [20] studied the existence of positive solutions of the parametric problem

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(t,u) \quad \text{in}\, \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open subset with sufficiently smooth boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function whose potential satisfies a suitable oscillating behaviour at zero.

W. Ni and J. Serrin in [18, 19] initiated the study of ground states for equations of the form

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u) \quad \text{in}\ \mathbb{R}^N, \tag{4}$$

with very general right hand side f. Radial solutions of the problem (4) have been studied in the context of the analysis of capillary surfaces for a function f of the form $f(u) = ku$, for $k > 0$ (for more details see [10, 13]).

Recently, the study of various mathematical problems with variable exponent growth condition has received considerable attention in recent years; see e.g. [2, 7, 14, 15, 23].

M. Rodrigues in [23] established the existence of non-trivial solutions for problem (1) by using Mountain Pass lemma (see [9]) and Fountain theorem (see Theorem 3.6 in [24]). The results were extended by G. Bin in [3], where the Ambrosetti-Rabinowitz type condition did not hold. Some further results on this type of problems can be found in the papers [17, 22].

In this paper, at first, we prove the existence of a non-zero solution for problem (1) without assuming any asymptotic condition neither at zero nor at infinity (see Theorem 3.1). Next, assuming only the classical Ambrosetti-Rabinowitz condition, we obtain the existence of two solutions (see Theorems 3.2 and 3.4). Finally, we present a three solutions existence result under appropriate condition on the potential F (see Theorem 3.5).

Our main tools are Bonanno critical point theorems contained in [5] and [8], which are powerful analytic tools for multiplicity results in nonlinear problems with a variational structure. In the next section, we recall this theorems for the reader's convenience.

The paper is organized as follows. In Section 2, we recall some properties of variable exponent spaces and our main tools. In Section 3, we discuss the existence of one, two and three weak solutions for the problem (1).

2. Preliminaries

In this section, we recall definitions and theorems which will be used in this paper.

Definition 2.1. Let $(X, \|\cdot\|)$ be a real Banach space and $J, I : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; put $g = J - I$ and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$. We say that the functional g satisfies the Palais-Smale condition cut off lower at r_1 and upper at r_2 ($^{[r_1]}(\mathrm{PS})^{[r_2]}-$condition) if any sequence $\{u_n\} \subset X$ such that

- $\{g(u_n)\}$ is bounded,
- $\lim_{n\to+\infty} \|g'(u_n)\|_{X^*} = 0$,
- $r_1 < J(u_n) < r_2, \quad \forall n \in \mathbb{N}$,

has a convergent subsequence. If $r_1 = -\infty$ and $r_2 = +\infty$, it coincides with the classical (PS)-condition, while if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$-condition.

First, we recall a result of local minimum obtained in [5], which is based on [4, Theorem 5.1].

Theorem 2.2 ([5]). *Let X be a real Banach space, let $J, I : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X J = J(0) = I(0) = 0$. Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < J(\bar{u}) < r$, such that*

$$\frac{\sup_{u\in J^{-1}(]-\infty,r[} I(u)}{r} < \frac{I(\bar{u})}{J(\bar{u})} \tag{5}$$

and, for each $\lambda \in \Lambda := \left] \frac{J(\bar{u})}{I(\bar{u})}, \frac{r}{\sup_{u\in J^{-1}(]-\infty,r[)} I(u)} \right[$ the functional $T_\lambda := J - \lambda I$ satisfies the $(\mathrm{PS})^{[r]}$-condition. Then, for each $\lambda \in \Lambda$, there is $u_\lambda \in J^{-1}(]0, r[)$ (hence, $u_\lambda \neq 0$) such that $T_\lambda(u_\lambda) \le T_\lambda(u)$ for all $u \in J^{-1}(]0, r[)$ and $T'_\lambda(u_\lambda) = 0$.

Now we point out another result, which ensures the existence of at least three critical points, that has been obtained in [8] and it is a more precise version of [6, Theorem 3.2].

Theorem 2.3 ([8]). *Let X be a reflexive real Banach space, let $J : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, moreover*

$$J(0) = I(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < r < J(\bar{u})$, such that

(i) $\frac{\sup_{u \in J^{-1}(]-\infty, r[)} I(u)}{r} < \frac{I(\bar{u})}{J(\bar{u})}$,

(ii) *for each* $\lambda \in \Lambda := \left] \frac{J(\bar{u})}{I(\bar{u})}, \frac{r}{\sup_{u \in J^{-1}(]-\infty, r[)} I(u)} \right[$ *the functional* $T_\lambda = J - \lambda I$ *is coercive.*

Then, for each $\lambda \in \Lambda$, *the functional* T_λ *has at least three distinct critical points in* X.

Let us introduce some notation which will be used later. Here and in the sequel, we suppose that $p \in C(\overline{\Omega})$ satisfies the following condition:

$$N < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < +\infty. \tag{6}$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega \Big| u(x) \Big|^{p(x)} dx < \infty \right\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Define the Sobolev space with variable exponent

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

It is well known [12] that, in view of (6), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces. We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. For any $u \in X$, define $\|u\| = |\nabla u|_{p(x)}$. It is easy to see that $W_0^{1,p(x)}(\Omega)$ endowed with the above norm is a separable, reflexive Banach space. In $W_0^{1,p(x)}(\Omega)$ the Poncarè inequality holds, so $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.4 ([12]). *Set* $\rho(u) = \int_\Omega |u|^{p(x)}\, dx$. *For* $u, u_n \in L^{p(\cdot)}(\Omega)$, *we have*

(1) $|u|_{p(\cdot)} < (=;>) 1 \Leftrightarrow \rho(u) < (=;>) 1$,
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^+}$,
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho(u) \leq |u|_{p(\cdot)}^{p^-}$,
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$,
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.4, for $u \in L^{p(\cdot)}(\Omega)$ the following inequalities hold:

$$\|u\|^{p^-} \leq \int_\Omega |\nabla u(x)|^{p(x)}\, dx \leq \|u\|^{p^+} \quad \text{ if } \|u\| > 1; \tag{7}$$

$$\|u\|^{p^+} \leq \int_\Omega |\nabla u(x)|^{p(x)}\,dx \leq \|u\|^{p^-} \quad \text{if } \|u\| < 1. \tag{8}$$

Proposition 2.5 ([16]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding we deduce that $W_0^{1,p(x)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N < p^-$.*

From Proposition 2.5, there exists a positive constant m depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \leq m\|u\|, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{9}$$

Put

$$J(u) := \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{\sqrt{1+|\nabla u|^{2p(x)}}}{p(x)}\right) dx, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

It is known that $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and

$$J'(u)(v) = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right) \nabla v\,dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega).$$

Moreover, we have J is convex, sequentially weakly lower semi-continuous and $J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is an homeomorphism, see [23].

Set

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. For each $u \in W_0^{1,p(x)}(\Omega)$, we let the functional $I : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ be defined by

$$I(u) := \int_\Omega F(x,u(x))\,dx.$$

Proposition 2.5 guarantees that the functional I is well defined, continuously Gâteaux differentiable with compact derivative and whose Gâteaux derivative is given by

$$I'(u)(v) = \int_\Omega f(x,u(x))v(x)\,dx$$

for any u, $v \in W_0^{1,p(x)}(\Omega)$.

Now, let us introduce the energy functional T_λ related to problem (1)

$$T_\lambda(\cdot) := J(\cdot) - \lambda I(\cdot)$$

and we observe that, for each $\lambda > 0$, the critical points u of T_λ are the weak solutions of (1), i.e.,

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\nabla v\,dx - \lambda \int_\Omega f(x,u(x))v(x)\,dx = 0$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

3. Main results

Before introducing our result we observe that, setting

$$\varrho(x) = \sup\{\varsigma > 0 : B(x, \varsigma) \subseteq \Omega\}$$

for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0, \gamma) \subseteq \Omega$, where

$$\gamma = \sup_{x\in\Omega} \varrho(x). \tag{10}$$

Fixed $r > 0$, we also denote by

$$\omega_r := r^N \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})},$$

the measure of the N-dimensional ball of radius r where Γ is the Gamma function. Put

$$\beta_+ := 2^{p^+-N+1}\omega_N\gamma^{N-p^+}(2^N - 1) + |\Omega|, \tag{11}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω, and

$$\beta^\circ := 2^{p^--N}\omega_N\gamma^{N-p^-}(2^N - 1). \tag{12}$$

Now, we present our main results. First, we establish the existence of one non-trivial solution for problem (1).

Theorem 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that there exist $c \geq m$ and $d \geq \max\{1, \gamma\}$ with $\beta_+ d^{p^+} < \frac{p^-}{p^+}\left(\frac{c}{m}\right)^{p^-}$ such that*

$$\frac{\int_\Omega \max_{|t|\leq c} F(x,t)\,dx}{\left(\frac{c}{m}\right)^{p^-}} < \frac{p^- \int_\Omega F(x,d)\,dx}{p^+\beta_+ d^{p^+}}. \tag{13}$$

Then, for each

$$\lambda \in \Lambda := \left] \frac{\beta_+ d^{p^+}}{p^- \int_\Omega F(x,d)\,dx}, \frac{\left(\frac{c}{m}\right)^{p^-}}{p^+ \int_\Omega \max_{|t|\leq c} F(x,t)\,dx} \right[, \tag{14}$$

problem (1) admits at least one non-trivial weak solution $\bar{u}_1 \in W_0^{1,p(x)}(\Omega)$ such that

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Proof. Our aim is to apply Theorem 2.2 to problem (1) to the space $X := W_0^{1,p(x)}(\Omega)$ with the usual norm and to the functionals $J, I : X \to \mathbb{R}$ defined as

$$J(u) := \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{\sqrt{1 + |\nabla u|^{2p(x)}}}{p(x)} \right) dx,$$

and

$$I(u) := \int_\Omega F(x, u(x))\, dx$$

for all $u \in X$.

The functional $J, I \in C^1(X, \mathbb{R})$, as it was said in the previous section. This ensures that the functional $T_\lambda = J - \lambda I$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r > 0$ (see [4, Proposition 2.1]). Define

$$w(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \gamma), \\ d & \text{if } x \in B(x_0, \frac{\gamma}{2}), \\ \frac{2d}{\gamma}(\gamma - |x - x_0|) & \text{if } x \in B(x_0, \gamma) \setminus B(x_0, \frac{\gamma}{2}), \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. Put

$$r := \frac{1}{p^+}\big(\frac{c}{m}\big)^{p^-}.$$

Clearly, $w \in X$, and from the condition $\beta_+ d^{p^+} < \frac{p^-}{p^+}\big(\frac{c}{m}\big)^{p^-}$ one has

$$\begin{aligned} J(w) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla w|^{p(x)} + \sqrt{1 + |\nabla w|^{2p(x)}}\right) dx \\ &\leq \frac{2}{p^-}\int_{B(x_0,\gamma)\setminus B(x_0,\frac{\gamma}{2})} |\nabla w|^{p(x)}\, dx + \frac{1}{p^-}|\Omega| \\ &\leq \frac{1}{p^-} 2^{p^+ - N + 1}(2^N - 1)\gamma^{N - p^+}\omega_N d^{p^+} + \frac{1}{p^-}|\Omega| d^{p^+} \\ &= \frac{1}{p^-}\beta_+ d^{p^+} < r. \end{aligned}$$

For all $u \in X$ with $J(u) < r$, owing to (7) and (8), definitively one has

$$\min\{\|u\|^{p^+}, \|u\|^{p^-}\} < rp^+.$$

Then

$$\|u\| < \max\{(rp^+)^{\frac{1}{p^+}}, (rp^+)^{\frac{1}{p^-}}\} = \frac{c}{m},$$

and so by (9),

$$\max_{x\in\Omega} |u(x)| \leq m\|u\| < c.$$

Therefore,

$$\frac{\sup_{u\in J^{-1}(]-\infty,r[)} I(u)}{r} \leq \frac{\displaystyle\int_\Omega \max_{|t|\leq c} F(x,t)\, dx}{\frac{1}{p^+}\big(\frac{c}{m}\big)^{p^-}}.$$

So, by the assumption (13), the condition (5) of Theorem 2.2 is verified. Hence, all the assumptions of Theorem 2.2 are satisfied, and it follows that for each

$$\lambda \in \Lambda \subseteq \left] \frac{J(w)}{I(w)}, \frac{r}{\sup_{u\in J^{-1}(]-\infty,r[)} I(u)} \right[,$$

the functional T_λ has at least one non-zero critical point $\bar{u}_1 \in X$ such that $\max_{x\in\Omega} |\bar{u}_1(x)| < c$ that is a non-trivial weak solution of problem (1). $\square$

The following result, in which the global Ambrosetti-Rabinowitz condition is also used, ensures the existence of at least two non-trivial weak solutions for problem (1).

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold. Furthermore, suppose that $f(\cdot, 0) \neq 0$ in Ω and*

(AR) *there exist two positive constants $\mu > 2p^+$ and $R > 0$ such that for all $x \in \Omega$ and $|s| \geq R$,*

$$0 < \mu F(x,s) \leq sf(x,s).$$

Then, for each $\lambda \in \Lambda$, where Λ is given by (14), problem (1) has at least two non-trivial weak solutions $\bar{u}_1$, $\bar{u}_2 \in X$ such that

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Proof. Fix λ as in the conclusion. Theorem 3.1 ensures that problem (1) admits at least one non-trivial weak solution $\bar{u}_1$ which is a local minimum of the functional T_λ. Now, we prove the existence of the second local minimum distinct from the first one. To this end, we must prove that the functional T_λ satisfies the hypotheses of the mountain pass theorem. Clearly, the functional T_λ is of class C^1 and $T_\lambda(0) = 0$. We can assume that $\bar{u}_1$ is a strict local minimum for T_λ in X. Therefore, there is $\rho > 0$ such that $\inf_{\|u-\bar{u}_1\|=\rho} T_\lambda(u) > T_\lambda(\bar{u}_1)$, so condition [21, (I_1), Theorem 2.2] is verified. From (AR), by standard computations, there is a positive constant C such that

$$F(x,s) \geq C|s|^\mu \tag{15}$$

for all $x \in \Omega$ and $|s| > R$. In fact, putting $\theta(x) = \min_{|\xi|=R} F(x,\xi)$ and

$$\phi_s(t) = F(x,ts), \quad \forall t > 0, \tag{16}$$

by (AR), for every $x \in \Omega$ and $|s| > R$, one has

$$0 < \mu\phi_s(t) = \mu F(x,ts) \leq tsf(x,ts) = t\phi_s'(t), \quad \forall t > 0.$$

Therefore,

$$\int_{R/|s|}^1 \frac{\phi_s'(t)}{\phi_s(t)} dt \geq \int_{R/|s|}^1 \frac{\mu}{t} dt.$$

Then

$$\phi_s(1) \geq \phi_s\left(\frac{R}{|s|}\right)|s|^\mu.$$

By (16), we obtain

$$F(x,s) \geq F\left(x, \frac{R}{|s|}s\right)|s|^\mu \geq \theta(x)|s|^\mu \geq C|s|^\mu, \tag{17}$$

and (15) is proved. Now, by choosing any $u \in X \setminus \{0\}$ and $t > 1$, one has

$$
\begin{aligned}
T_\lambda(tu) &= (J - \lambda I)(tu) \\
&= \int_\Omega \frac{1}{p(x)} \left(t^{p(x)} |\nabla u|^{p(x)} + \sqrt{1 + t^{2p(x)} |\nabla u|^{2p(x)}} \right) dx - \int_\Omega f(x, tu(x))\, dx \\
&\leq t^{p^+} \int_\Omega \frac{1}{p(x)} \left(2t^{p(x)} |\nabla u|^{p(x)} + 1 \right) dx - ct^\mu \lambda \int_\Omega |u(x)|^\mu\, dx \to -\infty
\end{aligned}
$$

as $t \to \infty$, since $\mu > 2p^+$. So, condition [21, (I_2), Theorem 2.2] is verified. Therefore, T_λ satisfies the geometry of mountain pass.

Now, let $\{u_n\}$ be a Palais-Smale sequence for the functional T_λ in X; i.e.,

$$
T_\lambda(u_n) \to \bar{c} < \infty \ \text{ and } \ T'_\lambda(u_n) \to 0 \ \text{ in } \ X^* \ \text{ as } \ n \to \infty, \tag{18}
$$

where X^* is the dual space of X. Let us show that $\{u_n\}$ is bounded in X. We have

$$
\begin{aligned}
&\mu T_\lambda(u_n) - \|T'_\lambda(u_n)\|_{X^*} \|u_n\| \\
&\geq \mu T_\lambda(u_n) - T'_\lambda(u_n)(u_n) \\
&= \mu J(u_n) - \lambda \mu I(u_n) - J'(u_n)(u_n) + \lambda I'(u_n)(u_n) \\
&\geq \mu \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} - \lambda \int_\Omega \Big(\mu F(x, u_n) - f(x, u_n) u_n(x) \Big)\, dx \\
&- 2 \int_\Omega |\nabla u_n|^{p(x)} dx \\
&\geq \left(\frac{\mu}{p^+} - 2 \right) \|u_n\|^{p^-} - \lambda \int_\Omega \Big(\mu F(x, u_n) - f(x, u_n) u_n(x) \Big)\, dx \\
&\geq \left(\frac{\mu}{p^+} - 2 \right) \|u_n\|^{p^-}.
\end{aligned}
$$

Since $\mu > 2p^+$, from the above inequality we know that $\{u_n\}$ is bounded in X. Hence, the classical theorem of Ambrosettti and Rabinowitz ensures a critical point $\bar{u}_2$ of T_λ such $T_\lambda(\bar{u}_2) > T_\lambda(\bar{u}_1)$. So, since $f(\cdot, 0) \neq 0$ in Ω, $\bar{u}_1$ and $\bar{u}_2$ are two distinct non-trivial weak solutions of the problem (1) and the proof is complete. □

Here we give the following result as a direct consequence of Theorem 3.2 in the autonomous case.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)\, d\xi$ for all $t \in \mathbb{R}$. Under the following conditions*

(i) *there exist $d \geq \max\{1, \gamma\}$ and $c \geq m$ with $\beta_+ d^{p^+} < \frac{p^-}{p^+} \left(\frac{c}{m}\right)^{p^-}$, such that*

$$
\frac{\max_{|t| \leq c} F(t)}{\left(\frac{c}{m}\right)^{p^-}} < \frac{p^- F(d)}{p^+ \beta_+ d^{p^+}};
$$

(ii) *there exist two positive constants* $\mu > 2p^+$ *and* $R > 0$ *such that for all* $|s| \geq R$,

$$0 < \mu F(s) \leq sf(s),$$

and for each

$$\lambda \in \Lambda := \left] \frac{\beta_+ d^{p^+}}{p^-|\Omega| F(d)}, \frac{\left(\frac{c}{m}\right)^{p^-}}{p^+|\Omega| \max_{|t|\leq c} F(t)} \right[,$$

the problem

$$\begin{cases} -\mathrm{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(u), \quad x \in \Omega, \\ u = 0, \qquad x \in \partial\Omega \end{cases}$$

admits at least two non-trivial weak solutions $\bar{u}_1,\ \bar{u}_2 \in X$ *such that*

$$\max_{x\in\Omega} |\bar{u}_1(x)| < c.$$

Now, we point out the following result of three weak solutions.

Theorem 3.4. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be an* L^1*-Carathéodory function satisfying*

(f_1) *there exist* $a_1, a_2 \in [0, +\infty[$ *and* $q \in C(\overline{\Omega})$ *with* $1 < q(x) < p^-$ *for each* $x \in \overline{\Omega}$, *such that*

$$|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1}$$

for each $(x,t) \in \Omega \times \mathbb{R}$,

and $f(\cdot, 0) \neq 0$ *in* Ω. *Assume that there exist* $d \geq \gamma$ *and* $c \geq m$ *with* $\beta^\circ d^{p^-} > \left(\frac{c}{m}\right)^{p^-}$, *such that the assumption (13) in Theorem 3.1 holds. Then for each* $\lambda \in \Lambda$, *where* Λ *is given by (14), problem (1) has at least three non-trivial weak solutions.*

Proof. Our goal is to apply Theorem 2.3. The functionals w, I and J defined in the proof of Theorem 3.1 satisfy all regularity assumptions requested in the Theorem 2.3. So, our aim is to verify (i) and (ii). Arguing as in the proof of Theorem 3.1, put $r := \frac{1}{p^+}\left(\frac{c}{m}\right)^{p^-}$ one has

$$\begin{aligned} J(w) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla w|^{p(x)} + \sqrt{1+|\nabla w|^{2p(x)}}\right) dx \\ &\geq \frac{1}{p^+}\int_\Omega |\nabla w|^{p(x)}\, dx \\ &\geq \frac{1}{p^+} 2^{p^- - N}(2^N - 1)\gamma^{N-p^-}\omega_N d^{p^-} \\ &= \frac{1}{p^+}\beta^\circ d^{p^-} > r > 0. \end{aligned}$$

Therefore, the assumption (i) of Theorem 2.3 is satisfied.

Now, we prove that the functional $T_\lambda = J - \lambda I$ is coercive for all $\lambda > 0$. By using Hölder inequality and condition (f_1), for all $u \in X$ such that $\|u\| \geq 1$, we have

$$\begin{aligned} I(u) &= \int_\Omega F(x, u(x))\, dx \\ &= \int_\Omega \Big(\int_0^{u(x)} f(x,t)dt \Big)\, dx \\ &\leq \int_\Omega \Big(a_1 |u(x)| + \frac{a_2}{q(x)} |u(x)|^{q(x)} \Big)\, dx \\ &\leq a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{q^-} \int_\Omega |u(x)|^{q(x)}\, dx. \end{aligned}$$

On the other hand, there is a constant $C' = \max\{k_q^{q^+}, k_q^{q^-}\} > 0$ such that

$$\int_\Omega |u(x)|^{q(x)}\, dx \leq \max\left\{ \|u\|^{q^+}_{L^{q(x)}(\Omega)}, \|u\|^{q^-}_{L^{q(x)}(\Omega)} \right\} \leq C' \|u\|^{q^+},$$

where k_q is the best constant of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see [11]). Then

$$I(u) \leq a_1 k_1 \|u\| + \frac{a_2}{q^+} C' \|u\|^{q^+}$$

for all $u \in X$ such that $\|u\| \geq 1$, where k_1 is the best constant of the embedding $W_0^{1,p(x)} \hookrightarrow L^1(\Omega)$. Take (7) into account, one has

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)} \Big(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \Big)\, dx \\ &\geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-}, \end{aligned}$$

for every $\lambda > 0$, we have

$$T_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \lambda a_1 k_1 \|u\| - \frac{\lambda a_2 C'}{q^-} \|u\|^{q^+}.$$

Since $p^- > q^+$, the functional T_λ is coercive. Then also condition (ii) holds. So, for each $\lambda \in \Lambda$, the functional T_λ admits at least three distinct critical points that are weak solutions of problem (1). $\square$

Remark 3.1. If we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a non-negative L^1-Carathéodory function, then the previous theorems guarantee the existence of non-negative weak solutions. In fact, let $\bar{u} \in X$ be one (non-trivial) weak solution of problem (1). Arguing by contradiction, if we assume that $\bar{u}$ is negative at a point of Ω, the set

$$\Omega^- := \{x \in \overline{\Omega} : \bar{u}(x) < 0\},$$

is non-empty and open. Moreover, let us consider $\bar{v} := \min\{\bar{u}, 0\}$, one has $\bar{v} \in X$. Taking into account that $\bar{u}$ is a weak solution and by choosing $v = \bar{v}$, we deduce

$$\int_{\Omega^-} \left(|\nabla \bar{u}|^{p(x)} + \frac{|\nabla \bar{u}|^{2p(x)}}{\sqrt{1 + |\nabla \bar{u}|^{2p(x)}}} \right) dx = \lambda \int_{\Omega^-} f(x, \bar{u}(x))\bar{u}(x)\, dx \leq 0,$$

that is, $\|u\|_{W^{1,p(x)}(\Omega^-)} = 0$ which is absurd. Hence, our claim is proved.

As an example, we state here the following special case of our results.

Theorem 3.5. *Let $p(x) = p > N$ for every $x \in \Omega$ and $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(t) := \int_0^t f(\xi)\, d\xi$ for each $t \in \mathbb{R}$. Assume that $F(d) > 0$ for some $d \geq \gamma$ and, moreover,*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = \limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, there is $\lambda^\star > 0$ such that for each $\lambda > \lambda^\star$ the problem

$$\begin{cases} -\operatorname{div}\left(\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}} \right) |\nabla u|^{p-2} \nabla u \right) = \lambda f(u), & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega \end{cases}$$

admits at least three non-negative weak solutions.

Proof. Fix $\lambda > \lambda^* := \dfrac{\beta^\circ d^p}{pF(d)|\Omega|}$ for some $d \geq \gamma$ such that $F(d) > 0$. Since

$$\liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{c_n\} \subset]0, +\infty[$ such that $\lim_{n \to +\infty} c_n = 0$ and

$$\lim_{n \to +\infty} \frac{F(c_n)}{c_n^p} = 0.$$

Therefore, there exists $\overline{c} \geq m$ such that

$$\frac{F(\overline{c})}{\overline{c}^p} < \min\left\{ \frac{F(d)}{(md)^p \beta^\circ}, \frac{1}{p|\Omega| m^p \lambda} \right\}$$

and $\overline{c} < md(\beta^\circ)^{1/p}$. Also, by the assumption

$$\limsup_{|\xi| \to +\infty} \frac{F(\xi)}{\xi^p} = 0,$$

the functional T_λ is coercive. Hence, by taking Remark 3.1 into account, the conclusion follows from Theorem 3.4. □

Acknowledgments. The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

References

1. G.A. Afrouzi, N.T. Chung and Z. Naghizadeh, *Existence and nonexistence of nontrivial weak solution for a class of general capillarity systems*, Acta Math. Appl. Sinica, English Series **30**(4) (2014), 1121-1130.
2. G. Barletta and A. Chinnì, *Existence of solutions for Neumann problem involvingthe $p(x)$-Laplacian*, Electron. J. Diff. Equa. **33** (2010), 1-11.
3. G. Bin, *On superlinear p(x)-Laplacian-like problem without Ambrosetti and Rabinowitz condition*, Bull. Korean Math. Soc. **51**(2) (2014), 409-421.
4. G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012), 2992-3007.
5. G. Bonanno, *Relations between the mountain pass theorem and minima*, Adv. Nonlinear Anal. **1** (2012), 205-220.
6. G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Diff. Eqs. **244** (2008), 3031-3059.
7. G. Bonanno and A. Chinnì, *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, J. Math. Anal. Appl. **418** (2014), 812-827.
8. G. Bonanno and S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. **89** (2010), 1-10.
9. K.C. Chang, *Critical Point Theory and Applications*, Shanghai Scientific and Technology Press, Shanghai, 1986.
10. P. Concus and P. Finn, *A singular solution of the capillary equation I, II*, Invent. Math. **29** (1975), 143-148, 149-159.
11. X.L. Fan and Q.H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), 1843-1852.
12. X.L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), 424-446.
13. R. Finn, *On the behavior of a capillary surface near a singular point*, J. Anal. Math. **30** (1976), 156-163.
14. L. Gasiński and N.S. Papageorgiou, *Anisotropic nonlinear Neumann problems*, J. Glob. Optim. **56** (2013), 1347-1360.
15. L. Gasiński and N.S. Papageorgiou, *A pair of positive solutions for the Dirichlet $p(z)$-Laplacian with concave and convex nonlinearities*, J. Glob. Optim. **56** (2013), 1347-1360.
16. O. Kováčik and J. Rákosík, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, Czechoslovak Math. J. **41** (1991), 592-618.
17. Y. Li and L. Li, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Bull. Malays. Math. Sci. Soc. to appear.
18. W.M. Ni and J. Serrin, *Non-existence theorems for quasilinear partial differential equations*, Rend. Circ. Math. Palermo (suppl.) **8** (1985), 171-185.
19. W.M. Ni and J. Serrin, *Existence and non-existence theorems for ground states quasilinear partial differential equations*, Att. Conveg. Lincei **77** (1985), 231-257.
20. F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Differential Equations **249** (2010), 1674-1725.
21. P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., Vol. **65**, Amer. Math. Soc., Providence, RI, 1986.
22. Q.M. Zhou, *Multiple solutions to a class of inclusion problems with operator involving $p(x)$-Laplacian*, Electronic J. Qualitative Theory of Diff. Equ. Vol. **2013**(63) (2013), 1-12.
23. M.M. Rodrigues, *Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators*, Mediterr. J. Math. **9** (2012), 211-223.
24. M. Willem, *Minimax Theorems*, Birkhauser, Basel, 1996.

G.A. Afrouzi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
e-mail: afrouzi@umz.ac.ir

S. Shokooh

Department of Mathematics, Faculty of Sciences, Gonbad Kavous University, Gonbad Kavous, Iran.
e-mail: shokooh@gonbad.ac.ir

N.T. Chung

Department of Mathematics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Viet Nam.
e-mail: ntchung82@yahoo.com