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BERRY-ESSEEN BOUNDS OF RECURSIVE KERNEL ESTIMATOR OF DENSITY UNDER STRONG MIXING ASSUMPTIONS

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Abstract. Let $\{X_i\}$ be a sequence of stationary α-mixing random variables with probability density function $f(x)$. The recursive kernel estimators of $f(x)$ are defined by

$$\widehat{f}_n(x) = \frac{1}{n\sqrt{b_n}} \sum_{j=1}^{n} b_j^{-\frac{1}{2}} K\Big(\frac{x - X_j}{b_j}\Big) \text{ and } \widetilde{f}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{b_j} K\Big(\frac{x - X_j}{b_j}\Big),$$

where $0 < b_n \to 0$ is bandwith and K is some kernel function. Under appropriate conditions, we establish the Berry-Esseen bounds for these estimators of $f(x)$, which show the convergence rates of asymptotic normality of the estimators.

1. Introduction

A fundamental problem in statistics is estimating a probability density function. There have been many papers concerning the non-parametric density estimation. See the books by Silverman [17] and Scott [16] and the references therein for available methods and results.

In this paper, we focus on the density estimation of dependent sample. In particular, let $\{X_j\}$ be a sequence of random variables with the probability density function $f(x)$. Rosenblatt [13] and Parzen [10] introduced the following classical kernel estimator of $f(x)$:

$$f_n(x) = \frac{1}{nb_n} \sum_{j=1}^{n} K\Big(\frac{x - X_j}{b_n}\Big),$$

where $0 < b_n \to 0$ is bandwith and K is some kernel function. The estimator $f_n(x)$ has been discussed extensively under dependent case, such as Roussas [14], Tran [18] and Liebscher [6] studied strong convergence of $f_n(x)$, the asymptotic normality of $f_n(x)$ are derived by Robinson [12], Roussas [15] and Liebscher [7].

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In addition, $f(x)$ has the following two other recursive kernel estimators in literature:

$$\widehat{f}_n(x) = \frac{1}{n\sqrt{b_n}} \sum_{j=1}^{n} b_j^{-\frac{1}{2}} K\Big(\frac{x - X_j}{b_j}\Big), \quad \widetilde{f}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{b_j} K\Big(\frac{x - X_j}{b_j}\Big).$$

The estimator $\widehat{f}_n(x)$ was first introduced by Wegman and Davies [20] in the independent case, and $\widehat{f}_n(x)$ and $\widetilde{f}_n(x)$ had been thoroughly examined in Wegman and Davies [20]. It is easy to see

$$\begin{aligned}\widehat{f}_n(x) =& \frac{n-1}{n}\left(\frac{b_{n-1}}{b_n}\right)^{1/2} \hat{f}_{n-1}(x) + \frac{1}{nb_n} K\left(\frac{x - X_n}{b_n}\right), \\ \widetilde{f}_n(x) =& \frac{n-1}{n} \widetilde{f}_{n-1}(x) + \frac{1}{nb_n} K\left(\frac{x - X_n}{b_n}\right).\end{aligned}$$

These recursive properties are particularly useful in large sample sizes since $\widehat{f}_n(x)$ and $\widetilde{f}_n(x)$ can be easily updated with each additional observation, respectively. This is especially relevant in a time series context, where there has been an interest in the use of nonparametric estimates in very long financial time series. Also, under certain circumstances, the recursive estimators are more efficient than its nonrecursive counterpart $f_n(x)$ when efficiency is measured in terms of the variance of an appropriate asymptotic (normal) distribution. Therefore, the properties of $\widehat{f}_n(x)$ and $\widetilde{f}_n(x)$ are extensively discussed by some authors, for example, the quadratic mean convergence and asymptotic normality of these recursive estimators have been obtained by Masry [8] under various assumptions on the dependence of X_i; Strong pointwise consistency of $\widehat{f}_n(x)$ has been proved by Györfi [2]; Masry [9] established sharp rates of almost sure convergence of $\widehat{f}_n(x)$ to $f(x)$ for vector-valued stationary strong mixing processes under weak assumptions on the strong mixing condition, these rates were improved by Tran [18]; Tran [19] studied the uniform convergence and asymptotic normality of $\widehat{f}_n(x)$ under some dependent assumption defined in terms of joint densities.

It is well known that the accuracy of the confidence intervals depends on how fast the theoretical distributions of the estimators converge to their limits. As a result, Berry-Esseen type bounds can be used to assess the accuracy. Up to now, the Berry-Esseen bounds for the estimators of $f(x)$ have only a few results, for example, Yang and Hu [21] investigated the Berry-Esseen bounds of $f_n(x)$ with ϕ-mixing dependent sample; Liang and Baek [4] studied the Berry-Esseen bounds for density estimates $f_n(x)$, $\widehat{f}_n(x)$ and $\widetilde{f}_n(x)$ under negatively associated assumptions. As far as we know, the Berry-Esseen type bounds for the estimators of $\widehat{f}_n(x)$ and $\widetilde{f}_n(x)$ under α-mixing assumptions are not available in the literature. We discuss this topic in this paper.

Recall that a sequence $\{\zeta_k, k \geq 1\}$ is said to be α-mixing if the α-mixing coefficient

$$\alpha(n) \stackrel{\text{def}}{:=} \sup_{k\geq 1} \sup\{|P(AB) - P(A)P(B)| : A \in \mathcal{F}_{n+k}^{\infty}, B \in \mathcal{F}_1^k\}$$

converges to zero as $n \to \infty$, where $\mathcal{F}_l^m = \sigma\{\zeta_l, \zeta_{l+1}, \ldots, \zeta_m\}$ denotes the σ-algebra generated by $\zeta_l, \zeta_{l+1}, \ldots, \zeta_m$ with $l \leq m$. Among various mixing conditions used in the literature, the α-mixing is reasonably weak and is known to be fulfilled for many stochastic processes including many time series models. In fact, under very mild assumptions linear autoregressive and more generally bilinear time series models are strongly mixing with mixing coefficients decaying exponentially, i.e., $\alpha(k) = O(\rho^k)$ for some $0 < \rho < 1$. See Doukhan [1, page 99], for more details.

The paper is organized as follows. In next section, we list some assumption conditions and give main results. Some lemmas and proofs of the main results are provided in Section 3, and the proofs of the lemmas are put in Appendix (i.e., Section 4).

2. Main results

In the sequel, let C, c_0 and c denote generic finite positive constants, whose values are may change from line to line, and let $\Phi(\cdot)$ denote the standard normal distribution function. $c(f)$ stands for set of continuous points of function $f(\cdot)$ and $U(x)$ for a neighborhood of x. $A_n = O(B_n)$ means $|A_n| \leq C|B_n|$.

In order to formulate the main results, we need the following assumptions.

- (B1) The density function $f(u)$ satisfies that (i) $\sup_{u\in U(x)} |f'(x)| < \infty$; (ii) the second-order derivative $f''(u)$ of $f(u)$ exists and is bounded for $u \in U(x)$.
- (B2) For all integers $k \geq 1$, let $f(x, y, k)$ be joint density of (X_1, X_{1+k}) and $\sup_{x,y} |f(x, y, k) - f(x)f(y)| \leq c_0$ for $k \geq 1$.
- (B3) The kernel $K(\cdot)$ is a bounded function with bounded support, and satisfies that $\int_{\mathbb{R}} K(t)dt = 1$ and $\int_{\mathbb{R}} tK(t)dt = 0$.
- (B4) The bandwiths b_n satisfy that (i) b_n is monotonous non-increasing; (ii) $n^{-1}\sum_{j=1}^n b_n^{-1}b_j = O(1)$; (iii) $n^{-1}\sum_{j=1}^n b_j^{-1}b_n \to \theta_1$, where $0 < \theta_1 < \infty$.
- (B5) Let $p := p_n < n$ and $q := q_n < n$ be positive integers tending to ∞. Put $k := k_n = [\frac{n}{p+q}]$ and assume that (i) $p_n k_n/n \to 1$; (ii) $p_n b_n \to 0$, $p_n b_n^{1/2} \to \infty$; (iii) $b_n^{-1/3}u(q) \to 0$, $kb_n^{1/2}\alpha^{2/3}(q) \to 0$, where $u(q) = \sum_{j=q}^{\infty} \alpha^{1/3}(j)$.

Remark 2.1. (B5)(i) implies that $p_n k_n/n = O(1)$ and $q_n k_n/n \to 0$. Hence, we have $q_n/p_n \to 0$, so that $q_n < p_n$, eventually.

Put $\sigma_{1n}^2 = \text{Var}(\sqrt{nb_n}\widehat{f}_n(x))$, $\sigma_{2n}^2 = \text{Var}(\sqrt{nb_n}\widetilde{f}_n(x))$, $\sigma_1^2 = f(x)\int_{\mathbb{R}} K^2(u)du$, $\sigma_2^2 = \theta_1 f(x)\int_{\mathbb{R}} K^2(u)du$,

$$S_{1n} = \frac{\sqrt{nb_n}[\widehat{f}_n(x) - E\widehat{f}_n(x)]}{\sigma_{1n}},\ S_{2n} = \frac{\sqrt{nb_n}[\widetilde{f}_n(x) - E\widetilde{f}_n(x)]}{\sigma_{2n}},$$

$F_{ln}(u) = P(S_{ln} \le u)$ $(l = 1, 2)$, $\gamma_{1n} = \frac{kq}{n} + b_n^{-1/3}u(p)$ and $\gamma_{2n} = \frac{p}{nb_n^{3/2}}$.

Theorem 2.1. *Let $\{X_i\}$ be a sequence of stationary α-mixing random variables with $\alpha(n) = O(n^{-\tau})$ for some $\tau > 6$ and $f(x) > 0$ for $x \in c(f)$. Assume that* (B2), (B3), (B4)(i)(ii) *and* (B5) *are satisfied, then*

$$\begin{aligned}&\sup_u |F_{1n}(u) - \Phi(u)|\\ &\le C\big\{\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + (pb_n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\big\}.\end{aligned}$$

Corollary 2.1. *Set $S_{1n}^* = \sigma_1^{-1}\sqrt{nb_n}\{\widehat{f}_n(x) - E\widehat{f}_n(x)\}$. Under the assumptions of Theorem 2.1, if* (B1)(i) *holds, then*

$$\begin{aligned}&\sup_u |P(S_{1n}^* \le u) - \Phi(u)|\\ &\le C\big\{\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + (pb_n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\big\}.\end{aligned}$$

Theorem 2.2. *Let $\{X_i\}$ be a sequence of stationary α-mixing random variables with $\alpha(n) = O(n^{-\tau})$ for some $\tau > 6$ and $f(x) > 0$ for $x \in c(f)$. Assume that* (B2), (B3), (B4)(i)(iii) *and* (B5) *hold, then*

$$\sup_u |F_{2n}(u) - \Phi(u)| = O\Big(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\Big).$$

Corollary 2.2. *Set $S_{2n}^* = \sigma_{2n}^{-1}\sqrt{nb_n}\{\widetilde{f}_n(x) - f(x)\}$. Let* (B1)(ii) *hold, and that $\sum_{j=1}^n (b_j/b_n)^2 = O(1)\sqrt{nb_n^5} = o(1)$. Then, under the assumptions of Theorem 2.2 we have*

$$\begin{aligned}&\sup_u |P(S_{2n}^* \le u) - \Phi(u)|\\ &= O\Big(\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q) + \sqrt{nb_n^5}\Big).\end{aligned}$$

3. Proofs of main results

Let $\bar{K}(\cdot) = K(\cdot) - EK(\cdot)$, $\zeta_{nj}^{(1)} = \frac{1}{\sigma_{1n}\sqrt{nb_j}}\bar{K}(\frac{x-X_j}{b_j}) := \sigma_{1n}^{-1}\xi_{nj}^{(1)}$, $\zeta_{nj}^{(2)} = \sqrt{\frac{b_n}{n}}\frac{1}{\sigma_{2n}b_j}\bar{K}(\frac{x-X_j}{b_j}) := \sigma_{2n}^{-1}\xi_{nj}^{(2)}$. Then $S_{vn} = \sum_{j=1}^n \zeta_{nj}^{(v)} = \sigma_{vn}^{-1}\sum_{j=1}^n \xi_{nj}^{(v)}$ for $v = 1, 2$.

Under (B5), for $m = 1, 2, \ldots, k$, split the set $\{1, 2, \ldots, n\}$ into k(large) p-blocks, I_m, and k(small) q-blocks, J_m, as follows:

$$I_m = \{i : i = l_m, \ldots, l_m + p - 1\},\ J_m = \{j : j = l'_m + 1, \ldots, l'_m + q\},$$

where $l_m = (m-1)(p+q)+1$, $l'_m = (m-1)(p+q)+p$, the remaining points form the set $\{l : k(p+q)+1 \le l \le n\}$ (which may be $\emptyset$). Let

$$g_{vnm} = \sum_{i \in I_m} \xi_{nj}^{(v)},\ g'_{vnm} = \sum_{j \in J_m} \xi_{nj}^{(v)},\ g''_{vnk} = \sum_{h=k(p+q)+1}^{n} \xi_{nh}^{(v)},$$

$$S'_{vn} = \sigma_{vn}^{-1} \sum_{m=1}^{k} g_{vnm},\ S''_{vn} = \sigma_{vn}^{-1} \sum_{m=1}^{k} g'_{vnm},\ S'''_{vn} = \sigma_{vn}^{-1} g''_{vnk}.$$

Then $S_{vn} = S'_{vn} + S''_{vn} + S'''_{vn}$ for $v = 1, 2$. Set $s_{vn}^2 = \sigma_{vn}^{-2} \sum_{m=1}^{k} \mathrm{Var}(g_{vnm})$.

For $v = 1, 2$, let $\eta_{vnm}, m = 1, 2, \ldots, k$ be independent random variables and the distribution of η_{vnm} is the same as that of $z_{vnm} = \sigma_{vn}^{-1} g_{vnm}$ for $m = 1, 2, \ldots, k$. Put

$$H_{vn} = \sum_{m=1}^{k} \eta_{vnm},\ B_{vn} = \sum_{m=1}^{k} \mathrm{Var}(\eta_{vnm}),\ \widetilde{F}_{vn}(u) = P(S'_{vn} \le u)$$

and $G_{vn} = P\big(\frac{H_{vn}}{\sqrt{B_{vn}}} \le u\big)$. Then $B_{vn} = s_{vn}^2$ and $\widetilde{G}_{vn}(u) := P(H_{vn} \le u) = G_{vn}\big(\frac{u}{s_{vn}}\big)$.

In order to prove the main results, we give the following some lemmas, whose proofs are put in Appendix (i.e., Section 4).

Lemma 3.1.

(a) *Under the assumptions of Theorem 2.1 we have* $\sigma_{1n}^2 \to \sigma_1^2$. *Further, if* (B1)(i) *holds, then* $|\sigma_{1n}^2 - \sigma_1^2| = O\big(\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + b_n^{-1/3} u(q)\big)$.

(b) *Under the assumptions of Theorem 2.2 we have* $\sigma_{2n}^2 \to \sigma_2^2$.

Lemma 3.2.

(a) *Under the assumptions of Theorem 2.1 we have* $E(S''_{1n})^2 = O(\gamma_{1n} + qb_n)$, $E(S'''_{1n})^2 = O(p/n + pb_n)$ *and* $|s_{1n}^2 - 1| = O\big(\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + b_n^{-1/3} u(q)\big)$.

(b) *Under the assumptions of Theorem 2.2 we have* $E(S''_{2n})^2 = O\big(\gamma_{1n}\big)$, $E(S'''_{2n})^2 = O\big(\frac{p}{n}\big)$ *and* $|s_{2n}^2 - 1| \le C\{\gamma_{1n}^{1/2} + (p/n)^{1/2} + b_n^{-1/3} u(q)\}$.

Lemma 3.3.

(a) *Under the assumptions of Theorem 2.1, we have* $\sup_u |G_{1n}(u) - \Phi(u)| \le C\gamma_{2n}^{1/2}$.

(b) *Under the assumptions of Theorem 2.2, we have* $\sup_u |G_{2n}(u) - \Phi(u)| \le C\gamma_{2n}^{1/2}$.

Lemma 3.4.

(a) *Under the assumptions of Theorem 2.1, we have* $\sup_u |\widetilde{F}_{1n}(u) - \widetilde{G}_{1n}(u)| \le C\{kb_n^{1/2}\alpha^{2/3}(q) + \gamma_{2n}^{1/2}\}$.

(b) *Under the assumptions of Theorem 2.2, we have* $\sup_u |\widetilde{F}_{2n}(u) - \widetilde{G}_{2n}(u)| \le C\{kb_n^{1/2}\alpha^{2/3}(q) + \gamma_{2n}^{1/2}\}$.

Lemma 3.5. *Suppose that* (B1)(ii) *and* (B3) *hold. If* $n^{-1}\sum_{j=1}^n (\frac{b_j}{b_n})^2 = O(1)$, *then* $E\widetilde{f}_n(x) = f(x) + O(b_n^2)$.

Lemma 3.6 ([5, Lemma 3.1])**.** *Let* X *and* $Y_1, \dots, Y_m$ *be random variables. Then for positive numbers* $w_1, \dots, w_m$ *we have*

$$\begin{aligned}&\sup_u \Big|P\Big(X + \sum_{i=1}^m Y_i \le u\Big) - \Phi(u)\Big| \\ &\le \sup_u |P(X \le u) - \Phi(u)| + \sum_{i=1}^m \frac{w_i}{\sqrt{2\pi}} + \sum_{i=1}^m P(|Y_i| > w_i).\end{aligned}$$

Proof of Theorem 2.1. Using Lemma 3.6 we have

$$\begin{aligned}&\sup_u |F_{1n}(u) - \Phi(u)| \\ &= \sup_u |P(S'_{1n} + S'_{1n} + S'''_{1n} \le u) - \Phi(u)| \\ &\le \sup_u |P(S'_{1n} \le u) - \Phi(u)| + \frac{(\gamma_{1n} + qb_n)^{1/3} + (p/n + pb_n)^{1/3}}{\sqrt{2\pi}} \\ &\quad + P(|S''_{1n}| \ge (\gamma_{1n} + qb_n)^{1/3}) + P(|S'''_{1n}| \ge (p/n + pb_n)^{1/3}).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\sup_u |P(S'_{1n} \le u) - \Phi(u)| \le &\sup_u |\widetilde{F}_{1n}(u) - \widetilde{G}_{1n}(u)| + \sup_u \Big|\widetilde{G}_{1n}(u) - \Phi\Big(\frac{u}{\sqrt{B_{1n}}}\Big)\Big| \\ &+ \sup_u \Big|\Phi\Big(\frac{u}{\sqrt{B_{1n}}}\Big) - \Phi(u)\Big|.\end{aligned}$$

Lemma 3.4 gives $\sup_u |\widetilde{F}_{1n}(u) - \widetilde{G}_{1n}(u)| \le C\{kb_n^{1/2}\alpha^{2/3}(q) + \gamma_{2n}^{1/2}\}$. Applying Lemma 3.3(a) it follows that $\sup_u \big|\widetilde{G}_{1n}(u) - \Phi(\frac{u}{\sqrt{B_{1n}}})\big| = \sup_u |G_{1n}(u) - \Phi(u)| \le C\gamma_{2n}^{1/2}$. Note that

$$\begin{aligned}\sup_u \Big|\Phi\Big(\frac{u}{\sqrt{B_{1n}}}\Big) - \Phi(u)\Big| &\le C|B_{1n} - 1| = C|s_{1n}^2 - 1| \\ &\le C\big\{\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + b_n^{-1/3}u(q)\big\}.\end{aligned}$$

Then

$$\begin{aligned}&\sup_u |P(S'_{1n} \le u) - \Phi(u)| \\ &\le C\Big\{\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\Big\}.\end{aligned}$$

From Lemma 3.2 we have

$$P(|S''_{1n}| \ge (\gamma_{1n} + qb_n)^{1/3}) \le \frac{1}{(\gamma_{1n} + qb_n)^{2/3}} E|S''_{1n}|^2 \le C\{\gamma_{1n}^{1/3} + (qb_n)^{1/3}\}$$

and $P(|S'''_{1n}| \geq (p/n + pb_n)^{1/3}) \leq C\{(p/n)^{1/3} + (pb_n)^{1/3}\}$. Then

$$\begin{aligned} &\sup_u |F_{1n}(u) - \Phi(u)| \\ &\leq C\Big\{\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + (pb_n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\Big\}. \qquad \square \end{aligned}$$

Proof of Corollary 2.1. Note that

$$\begin{aligned} &\sup_u |P(S_{1n}^* \leq u) - \Phi(u)| \\ &= \sup_u \Big|P\Big(S_{1n} \leq \frac{\sigma_1}{\sigma_{1n}}u\Big) - \Phi(u)\Big| \\ &\leq \sup_u \Big|P\Big(S_{1n} \leq \frac{\sigma_1}{\sigma_{1n}}u\Big) - \Phi\Big(\frac{\sigma_1}{\sigma_{1n}}u\Big)\Big| + \sup_u \Big|\Phi\Big(\frac{\sigma_1}{\sigma_{1n}}u\Big) - \Phi(u)\Big| \\ &= \sup_u |F_{1n}(u) - \Phi(u)| + \sup_u \Big|\Phi\Big(\frac{\sigma_1}{\sigma_{1n}}u\Big) - \Phi(u)\Big|. \end{aligned}$$

Using Lemma 3.1 it follows that $\sup_u \big|\Phi\big(\frac{\sigma_1}{\sigma_{1n}}u\big) - \Phi(u)\big| \leq C|\sigma_{1n}^2 - \sigma_1^2| \leq C\big(\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + b_n^{-1/3}u(q)\big)$, which, together with Theorem 2.1, yields that

$$\begin{aligned} &\sup_u |P(S_{1n}^* \leq u) - \Phi(u)| \\ &\leq C\big\{\gamma_{1n}^{1/3} + \gamma_{2n}^{1/2} + (p/n)^{1/3} + (pb_n)^{1/3} + kb_n^{1/2}\alpha^{2/3}(q) + b_n^{-1/3}u(q)\big\}. \qquad \square \end{aligned}$$

Proof of Theorem 2.2. Applying (b) in Lemmas 3.2-3.4, following the arguments as for the proof of Theorem 2.1, one can verify Theorem 2.2. □

Proof of Corollary 2.2. Note that Lemma 3.1(b) shows that $\sigma_{2n}^2 \geq c_0 > 0$ for large n. Then using Lemmas 3.5 and 3.6, it follows that

$$\begin{aligned} &\sup_u (P(S_{2n}^* \leq u) - \Phi(u)) \\ &= \sup_u \Big|P\Big(S_{2n} + \frac{\sqrt{nb_n}\big(E\widetilde{f}_n(x)\big) - f(x)}{\sigma_{2n}} \leq u\Big) - \Phi(u)\Big| \\ &\leq \sup_u |P(S_{2n} \leq u) - \Phi(u)| + C\frac{\sqrt{nb_n}|E\widetilde{f}_n(x) - f(x)|}{\sigma_{2n}} \\ &\leq \sup_u |P(S_{2n} \leq u) - \Phi(u)| + C\sqrt{nb_n^5}. \end{aligned}$$

Then the conclusion is proved by using Theorem 2.2. □

4. Appendix

Lemma 4.1 (Toeplitz lemma, [3, page 31])**.** *Let a_{ni}, $1 \leq i \leq k_n$, $n \geq 1$, and x_i, $i \geq 1$, be real numbers such that for every fixed i, $a_{ni} \to 0$ and for all n, $\sum_i a_{ni} \leq C < \infty$. If $x_n \to 0$, then $\sum_i a_{ni}x_i \to 0$, and if $\sum_i a_{ni} \to 1$, then $x_n \to x$ ensures that $\sum_i a_{ni}x_i \to x$.*

Lemma 4.2 ([3, Corollary A.2, p. 278])**.** *Suppose that* X *and* Y *are random variables such that* $E|X|^p < \infty$, $E|Y|^q < \infty$, *where* p, $q > 1$, $p^{-1} + q^{-1} < 1$. *Then* $|EXY - EXEY| \leq 8\|X\|_p\|Y\|_q\big\{\sup_{A\in\sigma(X),B\in\sigma(Y)}|P(A\cap B) - P(A)P(B)|\big\}^{1-p^{-1}-q^{-1}}$.

Lemma 4.3 ([22, Theorem 2.2])**.** *Let* $r > 2$, $\delta > 0$. *Suppose that* $\{Z_i, i \geq 1\}$ *is a stationary* α*-mixing sequence of random variables with the mixing coefficients* $\{\alpha(n)\}$ *with* $EZ_n = 0$ *and* $\alpha(n) = O(n^{-\lambda})$ *for* $\lambda > r(r+\delta)/(2\delta)$. *If* $E|Z_i|^{r+\delta} < \infty$, *then, for any* $\varepsilon > 0$, *there exists a positive constant* $C = C(\varepsilon, r, \delta, \lambda)$ *such that* $E\max_{1\leq m\leq n}\big|\sum_{i=1}^m Z_i\big|^r \leq C\big\{n^\varepsilon\sum_{i=1}^n E|Z_i|^r + \big(\sum_{i=1}^n\|Z_i\|_{r+\delta}^2\big)^{r/2}\big\}$.

Lemma 4.4 ([23])**.** *Let* p *and* q *be positive integers. Suppose that* $\{Z_i, i \geq 1\}$ *is a stationary* α*-mixing sequence of random variables with the mixing coefficients* $\{\alpha(n)\}$. *Set* $\eta_r = \sum_{j=(r-1)(p+q)+1}^{(r-1)(p+q)+p} Z_j$ *for* $1 \leq r \leq w$. *If* $s > 0, r > 0$ *with* $1/s + 1/r = 1$, *then there exists constant* $C > 0$ *such that* $|E\exp(it\sum_{r=1}^w\eta_r) - \prod_{r=1}^w E\exp(it\eta_r)| \leq C|t|\alpha^{1/s}(q)\sum_{r=1}^w\|\eta_r\|_r$.

Proof of Lemma 3.1. We prove only (a), the proof of (b) is similar. Write

$$\begin{aligned}\sigma_{1n}^2 &= \mathrm{Var}\Big(\sum_{m=1}^k g_{1nm} + \sum_{m=1}^k g'_{1nm} + g''_{1nk}\Big)\\ &= \mathrm{Var}\Big(\sum_{m=1}^k g_{1nm}\Big) + \mathrm{Var}\Big(\sum_{m=1}^k g'_{1nm}\Big) + \mathrm{Var}\Big(g''_{1nk}\Big)\\ &\quad + \mathrm{Cov}\Big(\sum_{m=1}^k g_{1nm}, \sum_{m=1}^k g'_{1nm}\Big) + \mathrm{Cov}\Big(\sum_{m=1}^k g'_{1nm}, g''_{1nk}\Big)\\ &\quad + \mathrm{Cov}\Big(g''_{1nk}, \sum_{m=1}^k g_{1nm}\Big).\end{aligned}$$

Step 1. We prove $\sigma_{1n}^2 \to \sigma_1^2$.

First, we evaluate $\mathrm{Var}(\sum_{m=1}^k g_{1nm})$, $\mathrm{Var}(\sum_{m=1}^k g'_{1nm})$ and $\mathrm{Var}(g''_{1nk})$. Note that

$$\begin{aligned}\mathrm{Var}\Big(\sum_{m=1}^k g_{1nm}\Big) &= \sum_{m=1}^k\sum_{i=l_m}^{l_m+p-1}\mathrm{Var}(\xi_{ni}^{(1)}) + 2\sum_{m=1}^k\sum_{l_m\leq i<j\leq l_m+p-1}\mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})\\ &\quad + 2\sum_{1\leq i<j\leq k}\mathrm{Cov}(g_{1ni}, g_{1nj}),\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\Big(\sum_{m=1}^k g'_{1nm}\Big) &= \sum_{m=1}^k\sum_{i=l'_m+1}^{l'_m+q}\mathrm{Var}(\xi_{ni}^{(1)}) + 2\sum_{m=1}^k\sum_{l'_m+1\leq i<j\leq l'_m+q}\mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})\\ &\quad + 2\sum_{1\leq i<j\leq k}\mathrm{Cov}(g'_{1ni}, g'_{1nj}),\end{aligned}$$

$$\mathrm{Var}(g'''_{1nk}) = \sum_{i=k(p+q)+1}^{n} \mathrm{Var}(\xi_{ni}^{(1)}) + 2 \sum_{k(p+q)+1\le i<j\le n} \mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)}).$$

Since $x \in c(f)$, using (B3) it follows that

$$b_i^{-1}\mathrm{Var}\Big(K\Big(\frac{x-X_i}{b_i}\Big)\Big) = \int_{\mathbb{R}} K^2(u)f(x-b_iu)du - b_i\Big(\int_{\mathbb{R}} K(u)f(x-b_iu)\Big)du\Big)^2$$
$$\to f(x)\int_{\mathbb{R}} K^2(u)du = \sigma_1^2 \text{ as } i\to\infty,$$

which implies that $b_i^{-1}\mathrm{Var}(K(\frac{x-X_i}{b_i}) \le C$ for $i \ge 1$, and using Lemma 4.1 we have

$$\sum_{i=1}^{n} \mathrm{Var}(\xi_{ni}^{(1)}) = \frac{1}{n}\sum_{i=1}^{n} b_i^{-1}\mathrm{Var}\Big(K\Big(\frac{x-X_i}{b_i}\Big)\Big) \to \sigma_1^2.$$

Then from (B5) or Remark 2.1 we get

$$\sum_{m=1}^{k}\sum_{i=l'_m+1}^{l'_m+q} \mathrm{Var}(\xi_{ni}^{(1)}) \le C\frac{kq}{n} \to 0, \qquad \sum_{i=k(p+q)+1}^{n} \mathrm{Var}(\xi_{ni}^{(1)}) \le C\frac{p}{n} \to 0,$$

$$\sum_{m=1}^{k}\sum_{i=l_m}^{l_m+p-1} \mathrm{Var}(\xi_{ni}^{(1)}) + \sum_{m=1}^{k}\sum_{l'_m+1}^{i=l'_m+q} \mathrm{Var}(\xi_{ni}^{(1)}) + \sum_{i=k(p+q)+1}^{n} \mathrm{Var}(\xi_{ni}^{(1)}) \to \sigma_1^2.$$

In view of (B2), (B3) and (B4)(i), for $i < j$ we have

$$\begin{aligned}
&|\mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})| \\
&= \frac{1}{n\sqrt{b_ib_j}}\Big|\mathrm{Cov}\Big(K\Big(\frac{x-X_i}{b_i}\Big), K\Big(\frac{x-X_j}{b_j}\Big)\Big)\Big| \\
&= \frac{\sqrt{b_ib_j}}{n}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} K(s)K(t)\{f(x-sb_i, x-tb_j, j-i) - f(x-sb_i)f(x-tb_j)\}dsdt\Big| \\
&\le C\frac{b_i}{n}.
\end{aligned}$$

Therefore, from (B4)(ii) we have

$$\Big|\sum_{m=1}^{k}\sum_{l_m\le i<j\le l_m+p-1} \mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})\Big| \le C\frac{p}{n}\sum_{i=1}^{n} b_i = O(pb_n) \to 0,$$

$$\Big|\sum_{m=1}^{k}\sum_{l'_m+1\le i<j\le l'_m+q} \mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})\Big| \le C\frac{q}{n}\sum_{i=1}^{n} b_i = O(qb_n) \to 0,$$

$$\Big|\sum_{k(p+q)+1\le i<j\le n} \mathrm{Cov}(\xi_{ni}^{(1)}, \xi_{nj}^{(1)})\Big| \le C\frac{p}{n}\sum_{i=1}^{n} b_i = O(pb_n) \to 0.$$

Applying Lemma 4.2 we have

(4.1)

$$\begin{aligned}
&\Big|\sum_{1\le i<j\le k}\mathrm{Cov}(g_{1ni},g_{1nj})\Big|\\
\le&\sum_{1\le i<j\le k}\sum_{s\in I_i}\sum_{t\in I_j}\frac{C}{n\sqrt{b_sb_t}}\Big|\mathrm{Cov}\Big(K\Big(\tfrac{x-X_s}{b_s}\Big),K\Big(\tfrac{x-X_t}{b_t}\Big)\Big)\Big|\\
\le&\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\sum_{s=l_i}^{l_i+p-1}\sum_{t=l_j}^{l_j+p-1}\alpha^{1/3}(t-s)\frac{1}{n\sqrt{b_sb_t}}\Big[E\Big|K\Big(\tfrac{x-X_s}{b_s}\Big)\Big|^3E\Big|K\Big(\tfrac{x-X_t}{b_t}\Big)\Big|^3\Big]^{1/3}\\
\le&\, C\frac{1}{nb_n^{1/3}}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\sum_{s=l_i}^{l_i+p-1}\sum_{t=l_j}^{l_j+p-1}\alpha^{1/3}(t-s)\le Cb_n^{-1/3}u(q)\to 0.
\end{aligned}$$

Similarly $|\sum_{1\le i<j\le k}\mathrm{Cov}(g'_{1ni},g'_{1nj})|\le Cb_n^{-1/3}u(p)\to 0$.
Therefore $\mathrm{Var}(\sum_{m=1}^k g_{1nm})\to\sigma_1^2$, $\mathrm{Var}(\sum_{m=1}^k g'_{1nm})=O(\frac{kq}{n}+qb_n+b_n^{-1/3}u(p))\to 0$, $\mathrm{Var}(g''_{1nk})=O(\frac{p}{n}+pb_n)\to 0$. Further, applying the Cauchy-Schwarz inequality, one obtains that

$$\Big|\mathrm{Cov}\Big(\sum_{m=1}^k g_{1nm},\sum_{m=1}^k g'_{1nm}\Big)\Big|\le\sqrt{\mathrm{Var}\Big(\sum_{m=1}^k g_{1nm}\Big)\mathrm{Var}\Big(\sum_{m=1}^k g'_{1nm}\Big)}\to 0,$$

$$\Big|\mathrm{Cov}\Big(\sum_{m=1}^k g'_{1nm},g''_{1nk}\Big)\Big|\le\sqrt{\mathrm{Var}\Big(\sum_{m=1}^k g'_{1nm}\Big)\mathrm{Var}(g''_{1nk})}\to 0,$$

$$\Big|\mathrm{Cov}\Big(g''_{1nk},\sum_{m=1}^k g_{1nm}\Big)\Big|\le\sqrt{\mathrm{Var}\Big(\sum_{m=1}^k g_{1nm}\Big)\mathrm{Var}(g''_{1nk})}\to 0.$$

Therefore $\sigma_{1n}^2\to\sigma_1^2$.

Step 2. We verify $|\sigma_{1n}^2-\sigma_1^2|=O\big(\gamma_{1n}^{1/2}+(p/n)^{1/2}+(pb_n)^{1/2}+b_n^{-1/3}u(q)\big)$. Note that

$$\begin{aligned}
|\sigma_{1n}^2-\sigma_1^2|\le&\Big|\sum_{i=1}^n\mathrm{Var}(\xi_{ni}^{(1)})-\sigma_1^2\Big|+\Big|\sum_{1\le i<j\le k}\mathrm{Cov}(g_{1ni},g_{1nj})\Big|\\
&+\Big|\sum_{1\le i<j\le k}\mathrm{Cov}(g'_{1ni},g'_{1nj})\Big|+\Big|\sum_{k(p+q)+1\le i<j\le n}\mathrm{Cov}(\xi_{ni}^{(1)},\xi_{nj}^{(1)})\Big|\\
&+\Big|\sum_{m=1}^k\sum_{l_m\le i<j\le l_m+p-1}\mathrm{Cov}(\xi_{ni}^{(1)},\xi_{nj}^{(1)})\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\sum_{m=1}^{k}\sum_{l'_m+1\le i<j\le l'_m+q}\mathrm{Cov}(\xi_{ni}^{(1)},\xi_{nj}^{(1)})\Big| \\
&+\Big|\mathrm{Cov}\Big(\sum_{m=1}^{k} g_{1nm},\sum_{m=1}^{k} g'_{1nm}\Big)\Big|+\Big|\mathrm{Cov}\Big(\sum_{m=1}^{k} g'_{1nm},g''_{1nk}\Big)\Big| \\
&+\Big|\mathrm{Cov}\Big(g''_{1nk},\sum_{m=1}^{k} g_{1nm}\Big)\Big| := \Big|\sum_{i=1}^{n}\mathrm{Var}(\xi_{ni}^{(1)})-\sigma_1^2\Big|+H_5.
\end{aligned}
$$

According to (B)(i), (B3) and (B4)(ii) we have

$$
\begin{aligned}
\Big|\sum_{i=1}^{n}\mathrm{Var}(\xi_{ni}^{(1)})-\sigma_1^2\Big| &\le \Big|\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}}K^2(u)[f(x-b_iu)-f(x)]du\Big| \\
&\quad+\frac{1}{n}\sum_{i=1}^{n}b_i\Big(\int_{\mathbb{R}}K(u)f(x-b_iu)du\Big)^2 \\
&\le \frac{1}{n}\sum_{i=1}^{n}b_i=O(b_n).
\end{aligned}
$$

From the proof in *Step 1* we have

$$
\begin{aligned}
H_5 &= O\Big(b_n^{-1/3}u(q)+b_n^{-1/3}u(p)+pb_n+qb_n+\Big[\frac{kq}{n}+qb_n+b_n^{-1/3}u(p)\Big]^{1/2} \\
&\qquad+\Big[\frac{p}{n}+pb_n\Big]^{1/2}\Big) \\
&= O\Big(\gamma_{1n}^{1/2}+(p/n)^{1/2}+(pb_n)^{1/2}+b_n^{-1/3}u(q)\Big).
\end{aligned}
$$

Therefore $|\sigma_{1n}^2-\sigma_1^2|=O\big(\gamma_{1n}^{1/2}+(p/n)^{1/2}+(pb_n)^{1/2}+b_n^{-1/3}u(q)\big)$. □

Proof of Lemma 3.2. (a) Lemma 3.1(a) shows that $\sigma_{1n}^2\ge c_0>0$ for large n, and from the proof of (a) in Lemma 3.1, we have

$$
\begin{aligned}
\mathrm{Var}\Big(\sum_{m=1}^{k} g'_{1nm}\Big) &= O\Big(\frac{kq}{n}+qb_n+b_n^{-1/3}u(p)\Big) \\
&= O(\gamma_{1n}+qb_n), \\
\mathrm{Var}(g''_{1nk}) &= O\Big(\frac{p}{n}+pb_n\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E(S''_{1n})^2 &= \sigma_{1n}^{-2}\mathrm{Var}\big(\sum_{m=1}^{k} g'_{1nm}\big)=O(\gamma_{1n}+qb_n), \\
E(S'''_{1n})^2 &= \sigma_{1n}^{-2}\mathrm{Var}(g''_{1nk}))=O(p/n+pb_n).
\end{aligned}
$$

From definition of s_{1n}^2 we have

$$s_{1n}^2 = E(S'_{1n})^2 - 2\Gamma_{1n} \text{ with } \Gamma_{1n} = \sigma_{1n}^{-2} \sum_{1\le i\le j\le k} \mathrm{Cov}\big(g_{1ni}, g_{1nj}\big).$$

(4.1) gives $|\Gamma_{1n}| = O(b_n^{-1/3}u(q))$. Note that $ES_{1n}^2 = 1$ and

$$\begin{aligned} E(S'_{1n})^2 &= E\{S_{1n} - (S''_{1n} + S'''_{1n})\}^2 \\ &= 1 + E(S''_{1n} + S'''_{1n})^2 - 2E\{S_{1n}(S''_{1n} + S'''_{1n})\}. \end{aligned}$$

Hence

$$\begin{aligned} &|E(S'_{1n})^2 - 1| \\ &= |E(S''_{1n} + S'''_{1n})^2 - 2E[S_{1n}(S''_{1n} + S'''_{1n})]| \\ &\le C\{E(S''_{1n})^2 + E(S'''_{1n})^2 + (ES_{1n}^2)^{1/2}(E(S''_{1n})^2)^{1/2} + (ES_{1n}^2)^{1/2}(E(S'''_{1n})^2)^{1/2}\} \\ &\le C\{\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2}\}. \end{aligned}$$

Therefore $|s_{1n}^2 - 1| = O\big(\gamma_{1n}^{1/2} + (p/n)^{1/2} + (pb_n)^{1/2} + b_n^{-1/3}u(q)\big)$.

(b) Note that Lemma 3.1(b) shows that $\sigma_{2n}^2 \ge c_0 > 0$ for large n. Then

$$E(S''_{2n})^2 \le C\mathrm{Var}\Big(\sum_{m=1}^{k} g'_{2nm}\Big),\ E(S'''_{2n})^2 \le C\mathrm{Var}(g''_{2nk}).$$

Similarly to the arguments as in *Step 1* of the proof in Lemma 3.1, we have

$$\begin{aligned} \mathrm{Var}\Big(\sum_{m=1}^{k} g'_{2nm}\Big) =& \sum_{m=1}^{k}\sum_{i=l'_m+1}^{l'_m+q} \mathrm{Var}(\xi_{ni}^{(2)}) + 2\sum_{m=1}^{k}\sum_{l'_m+1\le i<j\le l'_m+q} \mathrm{Cov}(\xi_{ni}^{(2)}, \xi_{nj}^{(2)}) \\ &+ 2\sum_{1\le i<j\le k} \mathrm{Cov}(g'_{2ni}, g'_{2nj}), \\ \mathrm{Var}(g''_{2nk}) =& \sum_{i=k(p+q)+1}^{n} \mathrm{Var}(\xi_{ni}^{(2)}) + 2\sum_{k(p+q)+1\le i<j\le n} \mathrm{Cov}(\xi_{ni}^{(2)}, \xi_{nj}^{(2)}). \end{aligned}$$

From $b_i^{-1}\mathrm{Var}(K(\frac{x-X_i}{b_i}) \le C$ for $i \ge 1$, using (B4)(i), it follows that

$$\begin{aligned} &\sum_{m=1}^{k}\sum_{i=l'_m+1}^{l'_m+q} \mathrm{Var}(\xi_{ni}^{(2)}) \le C\sum_{m=1}^{k}\sum_{i=l'_m+1}^{l'_m+q} \frac{b_n}{nb_i} = O\Big(\frac{kq}{n}\Big), \\ &\sum_{i=k(p+q)+1}^{n} \mathrm{Var}(\xi_{ni}^{(2)}) = O\Big(\frac{p}{n}\Big). \end{aligned}$$

In view of (B2) and (B3) one can verify $|\mathrm{Cov}(\xi_{ni}^{(2)}, \xi_{nj}^{(2)})| \le Cb_n/n$. Similarly to the proof for (4.1) we can obtain that $\big|\sum_{1\le i<j\le k} \mathrm{Cov}(g'_{2ni}, g'_{2nj})\big| =$

$O(b_n^{-1/3}u(p))$. Therefore

$$\operatorname{Var}\Big(\sum_{m=1}^{k} g'_{2nm}\Big) \le C\Big\{\frac{kq}{n} + \frac{b_n k q^2}{n} + b_n^{-1/3}u(p)\Big\}$$
$$= O\Big(\frac{kq}{n} + b_n^{-1/3}u(p)\Big)$$

and $\operatorname{Var}(g''_{2nk}) = O\big(\frac{p}{n} + \frac{p^2 b_n}{n}\big) = O\big(\frac{p}{n}\big)$. Thus

$$E(S''_{2n})^2 = O\Big(\frac{kq}{n} + b_n^{-1/3}u(p)\Big),\ E(S'''_{2n})^2 = O\Big(\frac{p}{n}\Big).$$

Similarly to the proof in (a), it follows that

$$\begin{aligned}
&|s_{2n}^2 - 1| \\
&= \Big|E(S'_{2n})^2 - 2\sigma_{2n}^{-2}\sum_{1\le i\le j\le k}\operatorname{Cov}\big(g_{2ni}, g_{2nj}\big)\Big| \\
&\le C\{E(S''_{2n})^2 + E(S'''_{2n})^2 + (ES_{2n}^2)^{1/2}(E(S''_{2n})^2)^{1/2} + (ES_{2n}^2)^{1/2}(E(S'''_{2n})^2)^{1/2}\} \\
&\quad + Cb_n^{-1/3}u(q) \\
&\le C\{\gamma_{1n}^{1/2} + (p/n)^{1/2} + b_n^{-1/3}u(q)\}. \qquad \square
\end{aligned}$$

Proof of Lemma 3.3. (a) Since $s_{1n}^2 \to 1$ and $\sigma_{1n}^2 \to \sigma_1^2$, by Berry-Esseen inequality (see [11, page 154, Theorem 5.7]), there exists some constant $C > 0$ such that

$$\sup_u |G_{1n}(u) - \Phi(u)| \le C\frac{\sum_{m=1}^k E|\eta_{1nm}|^3}{s_{1n}^3} \le C\sum_{m=1}^{k} E|g_{1nm}|^3.$$

Applying Lemma 4.3, for any $\epsilon > 0$ we have $E|g_{1nm}|^3 \le C\big\{p^\epsilon \sum_{i\in I_m} E|\xi_{ni}^{(1)}|^3 + \big(\sum_{i\in I_m}\|\xi_{ni}^{(1)}\|_4^2\big)^{3/2}\big\}$.

From (B3) and (B4)(i), it follows that

$$\sum_{i\in I_m} E|\xi_{ni}^{(1)}|^3 = \sum_{i\in I_m} E\Big|\frac{1}{\sqrt{nb_j}}\Big\{K\Big(\frac{x-X_j}{b_j}\Big) - EK\Big(\frac{x-X_j}{b_j}\Big)\Big\}\Big|^3$$
$$= O\big(n^{-3/2}b_n^{-1/2}p\big),$$
$$\Big\{\sum_{i\in I_m}\|\xi_{ni}^{(1)}\|_4^2\Big\}^{3/2} \le C\Big\{\sum_{i\in I_m}\Big(\frac{1}{n^2 b_j}\int_{\mathbb{R}}\frac{1}{b_j}\Big|K\Big(\frac{x-u}{b_j}\Big)\Big|^4 f(u)du\Big)^{1/2}\Big\}^{3/2}$$
$$= O\Big(\Big(\frac{p}{n\sqrt{b_n}}\Big)^{3/2}\Big).$$

Then, by arbitrariness of $\epsilon > 0$ and (B5)(i) we get

$$\sup_u |G_{1n}(u) - \Phi(u)| \le C\sum_{m=1}^{k} E|g_{1nm}|^3 \le C\frac{kp}{n}\Big(\frac{p}{nb_n^{3/2}}\Big)^{1/2} \le C\gamma_{2n}^{1/2}.$$

(b) Using (B3) and (B4)(i) we have

$$\begin{aligned}\sum_{i\in I_m} E|\xi_{ni}^{(2)}|^3 &= \sum_{i\in I_m} E\Big|\sqrt{\frac{b_n}{n}}\frac{1}{b_i}\Big(K\Big(\frac{x-X_i}{b_i}\Big)-EK\Big(\frac{x-X_j}{b_j}\Big)\Big)|^3\\ &\le \Big(\frac{b_n}{n}\Big)^{3/2}\sum_{i\in I_m}\int_{\mathbb{R}}\Big|\frac{1}{b_i}K\Big(\frac{x-u}{b_i}\Big)\Big|^3 f(u)du \le \Big(\frac{1}{n}\Big)^{3/2}\frac{p}{\sqrt{b_n}},\end{aligned}$$

$$\begin{aligned}\Big\{\sum_{i\in I_m}\|\xi_{ni}^{(2)}\|_4^2\Big\}^{3/2} &= \Big\{\sum_{i\in I_m}\Big(E\Big|\sqrt{\frac{b_n}{n}}\frac{1}{b_i}\bar{K}\Big(\frac{x-X_i}{b_i}\Big)\Big|^4\Big)^{1/2}\Big\}^{3/2}\\ &\le C\Big\{\sum_{i\in I_m}\Big(\frac{b_n^2}{n^2b_i^3}\int_{\mathbb{R}}\frac{1}{b_i}\Big|K\Big(\frac{x-u}{b_i}\Big)\Big|^4 f(u)du\Big)^{1/2}\Big\}^{3/2}\\ &\le C\Big(\frac{p}{nb_n^{1/2}}\Big)^{3/2}.\end{aligned}$$

Since $s_{2n}^2 \to 1$ and $\sigma_{2n}^2 \to \sigma_2^2$, similarly to the arguments as in (a) we get $\sup_u |G_{2n}-\Phi(u)| \le C\gamma_{2n}^{1/2}$. □

Proof of Lemma 3.4. (a) Assume that $\varphi(t)$ and $\psi(t)$ are the characteristic functions of S_{1n}' and H_{1n}, respectively. By Esseen inequality (see [11, page 146, Theorem 5.3]), for any $T>0$

$$\begin{aligned}&\sup_u |\widetilde{F}_{1n}(u)-\widetilde{G}_{1n}(u)|\\ &\le \int_{-T}^{T}\Big|\frac{\varphi(t)-\psi(t)}{t}\Big|dt + T\sup_u\int_{|y|\le c/T}|\widetilde{G}_{1n}(u+y)-\widetilde{G}_{1n}(u)|dy\\ &:= H_7+H_8.\end{aligned}$$

The proof in Lemma 3.3(a) shows that $E|g_{1nm}|^3 \le C\big(\frac{p}{nb_n^{1/2}}\big)^{3/2}$. Then from Lemma 4.4, it follows that

$$\begin{aligned}|\varphi(t)-\psi(t)| &= \Big|Ee^{it\sum_{m=1}^k z_{1nm}} - \prod_{m=1}^k Ee^{itz_{1nm}}\Big|\\ &\le C|t|\alpha^{2/3}(q)\sum_{m=1}^k ||z_{1nm}||_3 \le C|t|\alpha^{2/3}(q)\sum_{m=1}^k ||\sigma_{1n}^{-1}g_{1nm}||_3\\ &\le C|t|\alpha^{2/3}(q)\sum_{m=1}^k\Big(\frac{p}{nb_n^{1/2}}\Big)^{(3/2)\times(1/3)} \le C|t|\frac{k^{1/2}\alpha^{2/3}(q)}{b_n^{1/4}}.\end{aligned}$$

Therefore $H_7 \le CT\frac{k^{1/2}\alpha^{2/3}(q)}{b_n^{1/4}}$.

Since $s_{1n}^2 \to 1$, by Lemma 3.3(a), we have

$$\sup_u |\widetilde{G}_{1n}(u+y)-\widetilde{G}_{1n}(u)| = \sup_u\Big|G_{1n}\Big(\frac{u+y}{s_{1n}}\Big)-G_{1n}\Big(\frac{u}{s_{1n}}\Big)\Big|$$

$$\begin{aligned}
&\leq \sup_u \Big|G_{1n}\Big(\frac{u+y}{s_{1n}}\Big) - \Phi\Big(\frac{u+y}{s_{1n}}\Big)\Big| + \Big|G_{1n}\Big(\frac{u}{s_{1n}}\Big) - \Phi\Big(\frac{u}{s_{1n}}\Big)\Big| \\
&\quad + \Big|\Phi\Big(\frac{u+y}{s_{1n}}\Big) - \Phi\Big(\frac{u}{s_{1n}}\Big)\Big| \\
&\leq 2\sup_u \Big|G_{1n}(u) - \Phi(u)\Big| + \Big|\Phi\Big(\frac{u+y}{s_{1n}}\Big) - \Phi\Big(\frac{u}{s_{1n}}\Big)\Big| \\
&\leq C\Big\{\gamma_{2n}^{1/2} + \frac{|y|}{s_{1n}}\Big\} \leq C\{\gamma_{2n}^{1/2} + |y|\}.
\end{aligned}$$

Choose $T = \gamma_{2n}^{-1/2}$. Then $H_7 \leq Ckb_n^{1/2}\alpha^{2/3}(q)$ and $H_8 \leq CT\int_{|y|\leq c/T}\{\gamma_{2n}^{1/2} + |y|\}dy \leq C\{\gamma_{2n}^{1/2} + T^{-1}\} \leq C\gamma_{2n}^{1/2}$. Therefore

$$\sup_u |\widetilde{F}_{1n}(u) - \widetilde{G}_{1n}(u)| \leq C\{kb_n^{1/2}\alpha^{2/3}(q) + \gamma_{2n}^{1/2}\}.$$

(b) Following the line in the proof in (a), one can verify that $\sup_u |\widetilde{F}_{2n}(u) - \widetilde{G}_{2n}(u)| \leq C\{kb_n^{1/2}\alpha^{2/3}(q) + \gamma_{2n}^{1/2}\}$. □

Proof of Lemma 3.5. From (B3), it is easy to see that

$$\begin{aligned}
E\widetilde{f}_n(x) &= \frac{1}{n}\sum_{j=1}^n \frac{1}{b_j} EK\Big(\frac{x-X_j}{b_j}\Big) = \frac{1}{n}\sum_{j=1}^n \int_{\mathbb{R}} K(u)f(x-b_ju)du \\
&= \frac{1}{n}\sum_{j=1}^n \int_{\mathbb{R}} K(u)\Big(f(x) - f'(x)b_ju + \frac{1}{2}f''(x^*)b_j^2u^2\Big)du \\
&= f(x) + \frac{b_n^2}{2n}\sum_{j=1}^n \Big(\frac{b_j}{b_n}\Big)^2 \int_{\mathbb{R}} u^2K(u)f''(x^*)du,
\end{aligned}$$

where x^* is between $x - b_ju$ and x. Note that

$$\Big|\int_{\mathbb{R}} u^2K(u)f''(x^*)du\Big| \leq \sup_{u\in U(x)} |f''(u)| \int_{\mathbb{R}} u^2|K(u)|du < \infty$$

and $n^{-1}\sum_{j=1}^n(\frac{b_j}{b_n})^2 = O(1)$. Then $E\widetilde{f}_n(x) = f(x) + O(b_n^2)$. □

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References

[1] P. Doukhan, *Mixing: Properties and Examples*, Lecture Notes in Statistics, Vol. 85. Springer, Berlin, 1994.

[2] L. Györfi, *Strong consistent density estimate from ergodic sample*, J. Multivariate Anal. **11** (1981), no. 1, 81–84.

[3] P. Hall and C. C. Heyde, *Martingale Limit Theory and its Application*, Academic Press, New York, 1980.

[4] H. Y. Liang and J. I. Baek, *Berry-Esseen bounds for density estimates under NA assumption*, Metrika **68** (2008), no. 3, 305–322.

[5] H. Y. Liang and G. L. Fan, *Berry-Esseen type bounds of estimators in a semiparametric model with linear process errors*, J. Multivariate Anal. **100** (2009), no. 1, 1–15.
[6] E. Liebscher, *Strong convergence of sums of α-mixing random variables with applications to density estimation*, Stochastic Process. Appl. **65** (1996), no. 1, 69–80.
[7] ______, *Estimating the density of the residuals in autoregressive models*, Stat. Inference Stoch. Process. **2** (1999), no. 2, 105–117.
[8] E. Masry, *Recursive probability density estimation for weakly dependent stationary processes*, IEEE. Trans. Inform. Theory **32** (1986), no. 2, 254–267.
[9] ______, *Almost sure convergence of recursive density estimators for stationary mixing processes*, Statist. Probab. Lett. **5** (1987), no. 4, 249–254.
[10] E. Parzen, *On estimation of a probability density function and mode*, Ann. Math. Statist. **33** (1962), 1065–1076.
[11] V. V. Petrov, *Limit Theorems of Probability Theory*, Oxford Univ. Press Inc., New York, 1995.
[12] P. M. Robinson, *Nonparametric estimators for time series*, J. Time Ser. Anal. **4** (1983), no. 3, 185–207.
[13] M. Rosenblatt, *A central limit theorem and strong mixing conditions*, Proc. Nat. Acad. Sci. **4** (1956), 43–47.
[14] G. G. Roussas, *Nonparametric estimation in mixing sequences of random variables*, J. Statist. Plann. Inference **18** (1988), no. 2, 135–149.
[15] ______, *Asymptotic normality of the kernel estimate under dependence conditions: application to hazard rate*, J. Statist. Plann. Inference **25** (1990), no. 1, 81–104.
[16] D. W. Scott, *Multivariate Density Estimation: Theory, Practice, and Visualization*, Wiley, New York, 1992.
[17] B. W. Silverman, *Density Estimation for Statistics and Data Analysis*, Chapman & Hall, New York, 1986.
[18] L. T. Tran, *Recursive density estimation under dependence*, IEEE. Trans. Inform. Theory **35** (1989), no. 5, 1103–1108.
[19] ______, *Recursive kernel density estimation under a weak dependence condition*, Ann. Inst. Statist. Math. **42** (1990), no. 2, 305–329.
[20] E. J. Wegman and H. I. Davies, *Remarks on some recursive estimators of a probability density*, Ann. Statist. **7** (1979), no. 2, 316–327.
[21] W. Yang and S. Hu, *The Berry-Esséen bounds for kernel density estimator under dependent sample*, J. Inequal. Appl. **2012** (2012), 287, 13 pp.
[22] S. C. Yang, *Maximal moment inequality for partial sums of strong mixing sequences and application*, Act. Math. Sinica (Engl. Ser.) **23** (2007), no. 6, 1013–1024.
[23] S. C. Yang and Y. M. Li, *Uniformly asymptotic normality of the regression weighted estimator for strong mixing samples*, Acta Math. Sinica (Chin. Ser.) **49** (2006), no. 5, 1163–1170.

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