Bull. Korean Math. Soc. **54** (2017), No. 1, pp. 331–342
https://doi.org/10.4134/BKMS.b160135
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE ANNIHILATOR GRAPH OF GROUP RINGS

Mojgan Afkhami, Kazem Khashyarmanesh, and Sepideh Salehifar

Abstract. Let R be a commutative ring with nonzero identity and G be a nontrivial finite group. Also, let $Z(R)$ be the set of zero-divisors of R and, for $a \in Z(R)$, let $\mathrm{ann}(a) = \{r \in R \mid ra = 0\}$. The annihilator graph of the group ring RG is defined as the graph $AG(RG)$, whose vertex set consists of the set of nonzero zero-divisors, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$. In this paper, we study the annihilator graph associated to a group ring RG.

1. Introduction

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be the set of zero-divisors of R. If X is a subset of R, then the annihilator of X is the ideal $\mathrm{ann}(X) = \{r \in R \mid rX = 0\}$. The *Jacobson radical* of R is denoted by $J(R)$. For any subset Y of R, the cardinality of Y is denoted by $|Y|$. Put $Y^* = Y\backslash\{0\}$. Let G be a finite group that is defined multiplicatively. Also we denote the cyclic group of order n by C_n, and a finite field with q elements by $\mathbb{F}_q$.

The concept of the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, was introduced by Beck in [12], who let all elements of R be vertices and was mainly interested in colorings. The work of Beck is further continued by Anderson and Naseer in [6] and, for other graph theoretical aspects, by Anderson and Livingston in [5]. While they focus just on the zero-divisors of the rings, there are many other kinds of graphs associated to ring, some of which are extensively studied, see for example [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 20].

In [9], Badawi introduced the concept of the annihilator graph for a commutative ring R, which is denoted by $AG(R)$. The annihilator graph $AG(R)$ is an undirected graph whose vertex set is the set of all nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$. Also, the annihilator graph of a commutative semigroup is studied in [1].

Received February 17, 2016; Revised May 15, 2016.
2010 *Mathematics Subject Classification.* 05C69, 05C75, 13A15.
Key words and phrases. zero-divisor graph, annihilator graph, bipartite graph, planar graph, line graph.

Let RG be a commutative group ring and $Z(RG)$ be its set of zero-divisors. In this paper, we study the annihilator graph of the group ring RG, which is denoted by $AG(RG)$. Also, we examine the planarity, outerplanarity of $AG(RG)$ and some properties of the line graph of $AG(RG)$.

Let G be a graph with vertex set $V(G)$. For distinct vertices $x,\ y \in V(G)$, we use the notation $x \sim y$ to say that x and y are adjacent. The *distance* between two distinct vertices x and y in G is the number of edges in a shortest path connecting them and it is denoted by $\mathrm{d}(x, y)$. The *diameter* of a connected graph G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of the vertices of G. The degree of a vertex v of G, denoted by $\deg(v)$, is the number of edges of G incident with v such that the maximum degree of a graph G, denoted by $\triangle(G)$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G. If G does not contain a cycle, then $\mathrm{gr}(G)$ is defined to be infinity. The complete graph is a graph in which any two distinct vertices are adjacent. A *complete* graph with n vertices is denoted by K_n. A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is adjacent to every vertex of the other part. If the size of one of the parts in a complete bipartite graph is 1, then the complete bipartite graph is said to be a *star graph*. The *line graph* $L(G)$ of G is the graph whose vertices correspond to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

2. Preliminaries

Throughout the paper, R is a nontrivial commutative ring and G is a nontrivial Abelian group. A group ring RG is a construction which involves a group G and a ring R. The group ring is a ring and the underlying set consists of formal sums

$$\textstyle\sum_{g\in G} a_g g \ (a_g \in R, g \in G)$$

for which all but finitely many coefficients a_g are zero. The addition of two elements of RG is defined point-wise

$$\textstyle\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g,$$

and the multiplication is defined by

$$\textstyle(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = (\sum_{g\in G} c_g g),$$

where

$$\textstyle c_g = \sum_{e\in G} a_g b_e.$$

If this multiplication seems strange, it will surely help to notice that this is exactly what we would get by requiring that $(a_g g)(b_h h) = (a_g b_h)gh$ and that the multiplication map $RG \times RG \longrightarrow RG$ is additive in both arguments. The above definitions make RG into a commutative ring with nonzero identity $1_R.1_G$.

Clearly, if R and G are commutative, then RG is commutative. We can define an action of the ring R on RG by

$$r.\sum_{g\in G} a_g g = \sum_{g\in G}(ra_g)g.$$

This definition makes RG into a left R-module. The group ring is then a free R-module with basis consisting (of copies) of elements of G, and it is of rank $|G|$. Indeed, $\{1_R g : g \in G\}$ is a basis for RG. So if R and G are finite, then $|RG| = |R|^{|G|}$.

If RG is a group ring and X is a finite subset of G, then $\hat{X} := \Sigma_{x\in X} x$. In particular, if $X = G$ such that $|G| < \infty$, then $\hat{G} = \Sigma_{g\in G} g$. $\hat{G} \in Z(RG)$, since $\hat{G}(1-g) = 0$.

The following lemmas are needed for the rest of the paper.

Lemma 2.1. *The following statements hold.*

(i) $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
(ii) $RG \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. If RG is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$, then $|RG| = 4$. So $R \cong \mathbb{Z}_2$ and $G \cong C_2$ such that $G = \{1, g\}$. It is easy to see that $Z(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{1}), (\overline{1}, \overline{0})\}$ and $Z(\mathbb{Z}_2C_2) = \{0, \hat{G}\}$. Hence $\mathbb{Z}_2C_2 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, if RG is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2$ and $G \cong C_3$ such that $G = \{1, g, g^2\}$. $RG \not\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, since $Z(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2) = \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2 \setminus \{(\overline{1}, \overline{1}, \overline{1})\}$ and $Z(\mathbb{Z}_2C_3) = \{0, \hat{G}, 1-g, 1-g^2, g-g^2\}$. So the proof is completed. □

Lemma 2.2. *If* $|RG| = 9$*, then* $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Proof. Suppose that $|RG| = 9$. Then $R \cong \mathbb{Z}_3$ and $G = \{1, g\} \cong C_2$. So $Z(RG) = \{\overline{0}, \hat{G}, \overline{2}\hat{G}, \overline{1} + \overline{2}g, \overline{2} + g\}$. Thus RG is a nonlocal ring. We know that RG is a finite commutative ring. So RG is a direct product of at least two local rings. On the other hand, $|RG| = 9$. So $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. □

Lemma 2.3. *If* $AG(RG)$ *is a complete graph, then* R *is a local ring,* G *is a* p*-group, and* $p \in J(R)$.

Proof. Since RG is a finite ring, $RG \cong R_1 \times \cdots \times R_n$ such that R_i is a local ring for $1 \le i \le n$. If $n \ge 3$, then, by [9, Theorem 2.2], $\mathrm{d}((0, 1, 0, \ldots, 0), (1, 1, 0, \ldots, 0)) = 2$ in $AG(RG)$. So $n \le 2$. Suppose that $RG \cong R_1 \times R_2$ with $|R_2| \ge 3$. Then $\mathrm{d}((0, 1), (0, r)) = 2$ in $AG(RG)$, where $1 \ne r \in R_2^*$. If $|R_1| = |R_2| = 2$, then $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, which is impossible, by Lemma 2.1. Hence RG is a local ring. So, by [19], the proof is completed. □

3. Some properties of $AG(RG)$

We begin this section with the following proposition which is obtained from [9, Theorem 2.2] and [9, Corollary 2.11].

Proposition 3.1. *The following statements hold.*

(i) $AG(RG)$ *is connected and* $\mathrm{diam}(AG(RG)) \le 2$;

(ii) $\mathrm{gr}(AG(RG)) \in \{3, 4, \infty\}$.

Theorem 3.2. $\mathrm{gr}(AG(RG)) = \mathrm{gr}(\Gamma(RG))$.

Proof. Clearly, $\Gamma(RG)$ is a spanning subgraph of $AG(RG)$. By [2, Proposition 2.8], $\mathrm{gr}(\Gamma(RG)) = 3$ if and only if RG is neither $\mathbb{Z}_2C_2$ nor $\mathbb{F}_{p^r}C_q$ such that $\mathbb{F}_{p^r}C_q \cong \mathbb{F}_{q_1} \times \mathbb{F}_{q_2}$. First, we show that $\mathrm{gr}(\Gamma(RG)) = 3$ if and only if $\mathrm{gr}(AG(RG)) = 3$. Suppose that $\mathrm{gr}(\Gamma(RG)) = 3$, which implies that $\mathrm{gr}(AG(RG)) = 3$. If $\mathrm{gr}(AG(RG)) = 3$, then, by [9, Corollary 2.11], $\mathrm{gr}(\Gamma(RG)) \in \{3, 4, \infty\}$. Let $\mathrm{gr}(\Gamma(RG)) = \infty$. Then, by [2, Proposition 2.8.1], $RG \cong \mathbb{Z}_2C_2$. Hence $AG(RG) \cong K_1$. Thus $\mathrm{gr}(AG(RG)) = \infty$, which is impossible. Now, suppose that $\mathrm{gr}(\Gamma(RG)) = 4$. Then, by [2, Proposition 2.8.2] and Lemmas 2.1 and 2.2, $RG \cong \mathbb{F}_1 \times \mathbb{F}_2$ such that $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields with at least three elements. In this situation, $AG(\mathbb{F}_1 \times \mathbb{F}_2)$ is a complete bipartite graph. Hence $\mathrm{gr}(AG(RG)) = 4$, which is impossible. So $\mathrm{gr}(\Gamma(RG)) = 3$.

Now, we show that $\mathrm{gr}(AG(RG)) = 4$ if and only if $\mathrm{gr}(\Gamma(RG)) = 4$. Let $\mathrm{gr}(AG(RG)) = 4$. Then, by [2, Proposition 2.8], $\mathrm{gr}(\Gamma(RG)) \in \{4, \infty\}$. If $\mathrm{gr}(\Gamma(RG)) = \infty$, then, by [2, Proposition 2.8], $RG \cong \mathbb{Z}_2C_2$. Thus $AG(RG) \cong K_1$, and so $\mathrm{gr}(AG(RG)) = \infty$, which is impossible. Thus $\mathrm{gr}(\Gamma(RG)) = 4$. Now, if $\mathrm{gr}(\Gamma(RG)) = 4$, then, by Proposition 3.1, $\mathrm{gr}(AG(RG)) \in \{3, 4\}$. By the above argument, $\mathrm{gr}(AG(RG)) = 3$ if and only if $\mathrm{gr}(\Gamma(RG)) = 3$. So we conclude that $\mathrm{gr}(AG(RG)) = 4$.

Finally, let $\mathrm{gr}(AG(RG)) = \infty$. Then clearly, $\mathrm{gr}(\Gamma(RG)) = \infty$. If $\mathrm{gr}(\Gamma(RG)) = \infty$, then, by [2, Proposition 2.8], $RG \cong \mathbb{Z}_2C_2$. Hence $AG(RG) \cong K_1$. Therefore $\mathrm{gr}(AG(RG)) = \infty$. □

4. Planarity of $AG(RG)$

In this section, we investigate when $AG(RG)$ is planar, outerplanar or ring graph whenever RG is a finite ring.

Recall that a graph is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. We say that C is a *primitive cycle* if it has no chord. Also, a graph G has a primitive cycle property (PCP) if any two primitive cycles intersect in at most one edge. The number $\mathrm{frank}(G)$ is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number $\mathrm{rank}(G) = q - n + r$ is called the *cycle rank* of G, where r is the number of connected components of G. By [16, Proposition 2.2], we have $\mathrm{rank}(G) \leq \mathrm{frank}(G)$. A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [16]).

(i) $\mathrm{rank}(G) = \mathrm{frank}(G)$;
(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Also, an undirected graph is *outerplanar* if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

We begin this section with the following theorem.

Theorem 4.1. *$AG(RG)$ is planar if and only if RG is isomorphic to one the following group rings.*
(i) $\mathbb{Z}_2C_2$;
(ii) $\mathbb{Z}_2C_3$;
(iii) $\mathbb{Z}_3C_2$;
(iv) $\mathbb{F}_4C_2$.

Proof. First, suppose that $AG(RG)$ is planar. Then we have the following cases.

Case 1. $|Z(R)| \geq 3$. Then there exist distinct nonzero zero-divisors r and s such that $rs = 0$, since $\Gamma(RG)$ is a connected spanning subgraph of $AG(RG)$. If $1 \neq g \in G$, then $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{r, rg, r\hat{G}\} \cup \{s, sg, s\hat{G}\}.$$

Case 2. $|Z(R)| = 2$. Since $|R| \leq |Z(R)|^2$ and R is not a field, $|R| = 4$. Let $Z(R) = \{0, a\}$. Then $a^2 = 0$. Now, suppose that $|G| \geq 5$. Then there exist distinct nonidentity elements g_1, g_2, g_3 and g_4 in G. Hence $AG(RG)$ has a copy of K_5 with vertex set

$$\{\hat{G}, a(1-g_1), a(1-g_2), a(1-g_3), a(1-g_4)\}.$$

So we conclude that $|G| < 5$.

First, suppose that $|G| = 2$. Then $\mathrm{Char}(R) = 4$ or $\mathrm{Char}(R) = 2$, since $|R| = 4$. If $\mathrm{Char}(R) = 4$, then $R \cong \mathbb{Z}_4$ and $Z(R) = \{\overline{0}, \overline{2}\}$. Clearly, we have

$$\begin{aligned}
\overline{2} &\in \mathrm{ann}((-1+g)(1-g)) \backslash (\mathrm{ann}(-1+g) \cup \mathrm{ann}(1-g));\\
\overline{1} - g &\in \mathrm{ann}((\overline{2}g)(1-g)) \backslash (\mathrm{ann}(\overline{2}g) \cup ann(1-g));\\
\hat{G} &\in \mathrm{ann}((\overline{2}g)(\overline{3}\hat{G})) \backslash (ann(\overline{2}g) \cup ann(\overline{3}\hat{G})), \text{ and}\\
\overline{2} &\in \mathrm{ann}((\hat{G})(\overline{3}\hat{G})) \backslash (\mathrm{ann}(\hat{G}) \cup ann(\overline{3}\hat{G})).
\end{aligned}$$

Hence $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{\overline{2}g, -1+g, \hat{G}\} \cup \{1-g, \overline{3}\hat{G}, \overline{2}\hat{G}\}.$$

If $\mathrm{Char}(R) = 2$, then $R \cong \frac{\mathbb{Z}_2[x]}{(x^2)} = \{\overline{0}, \overline{1}, \overline{x}, \overline{1}+\overline{x}\}$ such that $Z(R) = \{\overline{0}, \overline{x}\}$. Now, it is easy to see that

$$\overline{x} \in \mathrm{ann}(((\overline{1}+\overline{x})+g)(\overline{1}+g)) \backslash (\mathrm{ann}((\overline{1}+\overline{x})+g) \cup \mathrm{ann}(\overline{1}+g));$$

$$\begin{aligned}
\overline{x} &\in \operatorname{ann}((\overline{1}+g)(\overline{1}+(\overline{1}+\overline{x})g))\backslash(\operatorname{ann}(\overline{1}+g)\cup\operatorname{ann}(\overline{1}+(\overline{1}+\overline{x})g));\\
(\overline{1}+\overline{x})+g &\in \operatorname{ann}((\overline{x}g)((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g))\backslash(\operatorname{ann}(\overline{x}g)\cup\\
&\qquad \operatorname{ann}((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g));\\
\overline{x} &\in \operatorname{ann}(((\overline{1}+\overline{x})+g)((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g))\backslash(\operatorname{ann}((\overline{1}+\overline{x})+g)\cup\\
&\qquad \operatorname{ann}((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g));\\
\overline{x} &\in \operatorname{ann}((\overline{1}+(\overline{1}+\overline{x})g)((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g))\backslash(\operatorname{ann}(\overline{1}+(\overline{1}+\overline{x})g)\cup\\
&\qquad \operatorname{ann}((\overline{1}+\overline{x})+(\overline{1}+\overline{x})g)), \text{and}\\
(\overline{1}+\overline{x})+g &\in \operatorname{ann}((\overline{1}+g)(\overline{x}g))\backslash(\operatorname{ann}(\overline{1}+g)\cup\operatorname{ann}(\overline{x}g)).
\end{aligned}$$

Thus $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{\overline{x}g, (\overline{1}+\overline{x})+g, \overline{1}+(\overline{1}+\overline{x})g\}\cup\{\overline{x}+\overline{x}g, \overline{1}+g, (\overline{1}+\overline{x})+(\overline{1}+\overline{x})g\}.$$

Now, suppose that $|G|=3$. Then $G=\{1,g,g^2\}$. In this situation, if $\mathrm{Char}(R)=4$, then $R\cong\mathbb{Z}_4$ such that $Z(R)=\{\overline{0},\overline{2}\}$. It is easy to see that $(\overline{2}+\overline{2}g)(\overline{1}+g-g^2)=0$. Hence $\overline{1}+g-g^2\in Z(RG)$. Also, we have

$$\begin{aligned}
&\overline{2}\in\operatorname{ann}((\overline{3}-\overline{3}g^2)(\overline{1}+g-g^2))\backslash(\operatorname{ann}(\overline{3}-\overline{3}g^2)\cup\operatorname{ann}(\overline{1}+g-g^2));\\
&\overline{2}\in\operatorname{ann}((\overline{3}-\overline{3}g)(\overline{1}+g-g^2))\backslash(\operatorname{ann}(\overline{3}-\overline{3}g)\cup\operatorname{ann}(\overline{1}+g-g^2)), \text{and}\\
&\overline{2}\in\operatorname{ann}((\overline{1}-g^2)(\overline{1}+g-g^2))\backslash(\operatorname{ann}(\overline{1}-g^2)\cup\operatorname{ann}(\overline{1}+g-g^2)).
\end{aligned}$$

So $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{\overline{3}-\overline{3}g, \overline{3}-\overline{3}g^2, \overline{1}-g^2\}\cup\{\hat{G}, \overline{2}\hat{G}, \overline{1}+g-g^2\}.$$

If $\mathrm{Char}(R)=2$, then $R\cong\frac{\mathbb{Z}_2[x]}{(x^2)}$ such that $Z(R)=\{\overline{0},\overline{x}\}$. Hence $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{\overline{1}-g, \overline{1}-g^2, \overline{x}-\overline{x}g\}\cup\{\hat{G}, \overline{x}\hat{G}, (\overline{1}+\overline{x})\hat{G}\}.$$

Finally, in this case, suppose that $|G|=4$. Then there exist nonidentity distinct elements g_1, g_2 and g_3 such that $G=\{1,g_1,g_2,g_3\}$. Let r,s be nonzero and nonidentity distinct elements in R. Thus $AG(RG)$ contains a copy of $K_{3,3}$ with vertex set

$$\{1-g_1, 1-g_2, 1-g_3\}\cup\{\hat{G}, r\hat{G}, s\hat{G}\}.$$

Case 3. $|Z(R)|=1$. Since R is finite and $Z(R)=\{0\}$, we conclude that R is a field. So we have the following subcases.

Subcase 1. $\mathrm{Char}(R)\mid|G|$ and $|R|\geq 6$. Then $\hat{G}^2=0$ and there exist distinct nonzero elements r_1, r_2, r_3, r_4 and r_5 in R. Thus $AG(RG)$ contains a copy of K_5 with vertex set

$$\{r_1\hat{G}, r_2\hat{G}, r_3\hat{G}, r_4\hat{G}, r_5\hat{G}\}.$$

Subcase 2. $\text{Char}(R) \mid |G|$ and $|R| = 5$. So $\hat{G}^2 = 0$ and $R \cong \mathbb{Z}_5$. Let $1 \neq g \in G$. Then $AG(RG)$ contains a copy of K_5 with vertex set

$$\{\hat{G}, 2\hat{G}, 3\hat{G}, 4\hat{G}, 1-g\}.$$

Subcase 3. $\text{Char}(R) \mid |G|$ and $|R| = 4$ such that $R = \{0, 1, r, s\}$. Then $\hat{G}^2 = 0$. We know that R is a field. Hence $\text{Char}(R) = 2$. If $|G| \geq 4$, then there exists proper subgroup H of G such that $|H| = 2$. Hence there exists $g_1 \in G\backslash H$. We have $\text{Char}(R)|\ |H|$. Hence $\hat{H}^2 = 0$. Also $\hat{H}\hat{G} = 2\hat{G} = 0$ and $g_1\hat{H} \neq \hat{H}$. Thus $AG(RG)$ contains a copy of K_5 with vertex set

$$\{\hat{G}, \hat{H}, r\hat{G}, s\hat{G}, g_1\hat{H}\}.$$

If $\text{Char}(R) = 2$ and $|G| < 4$, then $|G| = 2$. In this subcase, R is a local ring, G is a 2-group and $2 \in J(R)$. So, by [19], RG is a local ring. By [2, Definition 2.3], $Z(RG) = \langle 1 - g; g \in G\rangle$. Since $\text{Char}(R) \mid |G|$, $AG(RG) \cong K_3$. Thus $RG \cong \mathbb{F}_4C_2$.

Subcase 4. $\text{Char}(R) \mid |G|$ and $|R| = 3$. Then $\text{Char}(R) = 3$, $R \cong \mathbb{Z}_3$ and $|G| = 3k$, for some positive integer k.

Suppose that $|G| > 3$. Then G has a proper subgroup H such that $|H| = 3$. In this situation, $\text{Char}(R) = 3$. Hence $\hat{H}^2 = 0$. So $\hat{H}\hat{G} = 0$. Let $1 \neq g \in G\backslash H$ and $1 \neq h \in H$. Then $AG(RG)$ contains a copy of K_5 with vertex set

$$\{\hat{G}, \hat{H}, \overline{1} - h, \hat{G} - g\hat{H}, g\hat{H}\}.$$

Let $|G| = 3$. Then $G = \{1, g, g^2\}$. Then we have

$$\overline{1} - g \in \text{ann}((\overline{1} - g^2)(g - g^2))\backslash(\text{ann}(\overline{1} - g^2) \cup \text{ann}(g - g^2));$$
$$\overline{1} - g \in \text{ann}((\overline{1} - g)(\overline{1} - g^2))\backslash(\text{ann}(\overline{1} - g) \cup \text{ann}(\overline{1} - g^2)), \text{and}$$
$$\overline{1} - g \in \text{ann}((\overline{1} - g)(g - g^2))\backslash(\text{ann}(\overline{1} - g) \cup \text{ann}(g - g^2)).$$

Hence $AG(RG)$ contains a copy of K_5 with vertex set

$$\{\hat{G}, \overline{2}\hat{G}, g - g^2, \overline{1} - g, \overline{1} - g^2\}.$$

Subcase 5. $\text{Char}(R) \mid |G|$ and $|R| = 2$. Then $R \cong \mathbb{Z}_2$ and $G \cong C_2$. It is easy to see that $Z(RG) = \{0, \hat{G}\}$. So $AG(RG) \cong K_1$, which is planar.

Subcase 6. $\text{Char}(R) \nmid |G|$. Then, by Perlis-Walker Theorem [18, Theorem 3.5.4], RG is a direct product of copies of at least two fields. First, if RG is a direct product of copies of two fields $\mathbb{F}_1$ and $\mathbb{F}_2$, then $Z^*(\mathbb{F}_1 \times \mathbb{F}_2) = \{(r, 0) \mid r \in \mathbb{F}_1^*\} \cup \{(0, r) \mid r \in \mathbb{F}_2^*\}$. Hence $AG(RG)$ is a complete bipartite graph with parts $\{(r, 0) \mid r \in \mathbb{F}_1^*\}$ and $\{(0, r) \mid r \in \mathbb{F}_2^*\}$. If $|\mathbb{F}_1^*|, |\mathbb{F}_2^*| \geq 3$, then $AG(RG)$ contains a copy of $K_{3,3}$. So, without loss of generality, we may assume that $|\mathbb{F}_1^*| \leq 2$ and $|\mathbb{F}_2^*| \leq 3$. Hence $\mathbb{F}_1 \times \mathbb{F}_2$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{F}_4$ such that $\mathbb{F}_4$ is a field with four elements. By the definition of RG, $|\mathbb{F}_1 \times \mathbb{F}_2|$ can not be 6 and 12, so $\mathbb{F}_1 \times \mathbb{F}_2$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{F}_4$. By Lemma 2.1, RG is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. If

$|RG| = 9$, then, by Lemma 2.2, $RG \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In this situation, $AG(RG)$ is a cycle with length four such that $\hat{G} \sim -\overline{1} + g \sim \overline{2}\hat{G} \sim \overline{1} - g \sim \hat{G}$, where $1 \neq g \in G$. Hence $RG \cong \mathbb{Z}_3 C_2$. Suppose that $|RG| = 8$. Then we show that $RG \cong \mathbb{Z}_2 \times \mathbb{F}_4$. In this situation, $R \cong \mathbb{Z}_2$ and $G \cong C_3$. Let $G = \{1, g, g^2\}$. Then $Z(RG) = \{0, 1+g, 1+g^2, g+g^2, \hat{G}\}$. $Z(RG) \ntrianglelefteq RG$, since $|Z(RG)| = 5$. Thus RG is nonlocal. On the other hand, RG is finite. Hence RG is a direct product of at least two local rings. We show that RG is not isomorphic to three local rings. By the way of contradiction, assume that RG is isomorphic to three local rings. Since $|RG| = 8$, $RG \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which is impossible, by Lemma 2.1. Thus $RG \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence RG is a direct product of copies of two local rings. So RG is isomorphic to one the following rings.

$$\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \frac{\mathbb{Z}_2[x]}{(x^2)} \times \mathbb{Z}_2 \text{ and } \mathbb{F}_4 \times \mathbb{Z}_2.$$

We consider the rings $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ and $\mathbb{Z}_2 \times \mathbb{F}_4$. $\mathrm{char}(RG) = 2$, since $R \cong \mathbb{Z}_2$. On the other hand, $\mathrm{char}(\mathbb{Z}_2 \times \mathbb{Z}_4) = 4$. Thus $RG \ncong \mathbb{Z}_2 \times \mathbb{Z}_4$. Also, we know that the nonzero element $(\overline{0}, \overline{x})^2 = (\overline{0}, \overline{0})$ in $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ and RG does not have such as element. Hence $RG \ncong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$. Now, we show that RG is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}_4$ which is planar. If φ is a homomorphism from RG to $\mathbb{Z}_2 \times \mathbb{F}_4$ given by

$$\begin{array}{llll} \varphi(0) = (0,0), & \varphi(1) = (1,1), & \varphi(\hat{G}) = (1,0), & \varphi(g+g^2) = (0,1), \\ \varphi(g) = (1,a), & \varphi(g^2) = (1,a^2), & \varphi(1+g) = (0,a^2), & \varphi(1+g^2) = (0,a), \end{array}$$

where $\mathbb{F}_4 = \{0, 1, a, a^2 : a^3 = 1\}$, then φ is a ring isomorphism. So $R \cong \mathbb{Z}_2$ and $G \cong C_3$. In this situation, $Z^*(RG) = \{\hat{G}, \overline{1} + g, \overline{1} + g^2, g + g^2\}$ such that $AG(\mathbb{Z}_2 C_3) \cong K_{1,3}$, which is planar. Now, suppose that RG is a direct product of three fields, say $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$. Let $|\mathbb{F}_1|$, $|\mathbb{F}_2|$ and $|\mathbb{F}_3| \geq 3$ such that $\{0, 1, r_i\} \subseteq \mathbb{F}_i$ for $1 \leq i \leq 3$. Then since

$$(1,1,0) \in \mathrm{ann}((0, r_2, 1)(1, 0, r_3)) \backslash (\mathrm{ann}(0, r_2, 1) \cup \mathrm{ann}(1, 0, r_3)),$$

$AG(RG)$ has a copy of $K_{3,3}$ with vertex set

$$\{(0, r_2, 1), (0, r_2, 0), (0, 1, 0)\} \cup \{(1, 0, r_3), (1, 0, 0), (r_1, 0, 0)\}.$$

Without loss of generality, we may assume that $|\mathbb{F}_1| = 2$, $|\mathbb{F}_2| \leq 3$ and $|\mathbb{F}_3| \leq 3$. Then RG can be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We know that $|RG|$ is neither 12 nor 18. Also, by Lemma 2.1, RG is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus RG is not isomorphic to direct product of copies of three fields. Finally, suppose that RG is isomorphic to direct product of copies of at least four fields. Then the element $(1, 1, 0, 1, \ldots, 1)$ belongs to

$$\mathrm{ann}((1,0,1,0,\ldots,0)(0,0,1,1,0,\ldots,0)) \backslash (\mathrm{ann}(1,0,1,0,\ldots,0) \cup \\ \mathrm{ann}(0,0,1,1,0,\ldots,0)),$$

$(1,1,0,1,\ldots,1)$ belongs to

$$\mathrm{ann}((1,0,1,0,\ldots,0)(0,1,1,0,\ldots,0))\backslash(\mathrm{ann}(1,0,1,0,\ldots,0)\cup \mathrm{ann}(0,1,1,0,\ldots,0)),$$

$(1,0,1,1,\ldots,1)$ belongs to

$$\mathrm{ann}((1,1,0,\ldots,0)(0,1,1,0,\ldots,0))\backslash(\mathrm{ann}(1,1,0,\ldots,0)\cup \mathrm{ann}(0,1,1,0,\ldots,0)),$$

$(1,0,1,1,\ldots,1)$ belongs to

$$\mathrm{ann}((1,1,0,\ldots,0)(0,1,0,1,0,\ldots,0))\backslash(\mathrm{ann}(1,1,0,\ldots,0)\cup \mathrm{ann}(0,1,0,1,0,\ldots,0)),$$

$(1,1,1,0,\ldots,0)$ belongs to

$$\mathrm{ann}((1,0,0,1,0,\ldots,0)(0,0,1,1,0,\ldots,0))\backslash(\mathrm{ann}(1,0,0,1,\ldots,0)\cup \mathrm{ann}(0,0,1,1,0,\ldots,0)),$$

and $(1,1,1,0,\ldots,0)$ belongs to

$$\mathrm{ann}((1,0,0,1,\ldots,0)(0,1,0,1,0,\ldots,0))\backslash(\mathrm{ann}(1,0,0,1,\ldots,0)\cup \mathrm{ann}(0,1,0,1,0,\ldots,0)),$$

$AG(RG)$ has a copy of $K_{3,3}$ with vertex sets

$$\{(0,0,1,1,0,\ldots,0),(0,1,0,1,0,..,0),(0,1,1,0,\ldots,0)\},$$

and

$$\{(1,0,0,1,0,\ldots,0),(1,0,1,0,\ldots,0),(1,1,0,\ldots,0)\}.$$

Thus RG is not isomorphic to at least four fields.

The converse statement is clear. □

Now, the following corollaries are obtained from Theorem 4.1.

Corollary 4.2. *$AG(RG)$ is a ring graph if and only if $AG(RG)$ is planar.*

Corollary 4.3. *$AG(RG)$ is outerplanar if and only if $AG(RG)$ is planar.*

5. Line graph of $AG(RG)$

We begin this section with the following lemma.

Lemma 5.1 ([15, Lemma 2.1])**.** *If G is a graph, then $\mathrm{diam}(L(G))=1$ if and only if G is isomorphic to K_3 or $K_{1,n}$.*

In the following lemma, which is from [21], the planarity of a line graph $L(G)$ is characterized by using the planarity of G and its vertex degrees.

Lemma 5.2. *A nonempty graph G has a planar line graph $L(G)$ if and only if*

(i) *G is planar;*
(ii) *$\triangle(G)\leq 4$, and*

(iii) *if* $\deg(v) = 4$*, then* v *is a cut-vertex in the graph* G*.*

Proposition 5.3. *The graph* $AG(RG)$ *is isomorphic to* K_3 *if and only if* RG *is isomorphic to* $\mathbb{F}_4C_2$*.*

Proof. Suppose that $|G| \geq 3$. Then there exist distinct elements g_1 and g_2 in G such that $\{1, g_1, g_2\} \subseteq G$. Thus $\{\hat{G}, 1-g_1, 1-g_2, g_1-g_2\} \subseteq Z^*(RG)$, which is impossible. So $G \cong C_2$.

Now, consider the case that $|R| \geq 5$. Then there exist distinct elements r_1, r_2 and r_3 such that $\{0, 1, r_1, r_2, r_3\} \subseteq R$. Thus $\{\hat{G}, r_1\hat{G}, r_2\hat{G}, r_3\hat{G}\} \subseteq Z^*(RG)$. Thus $|R| \leq 4$. If $|R| = 2$, then $Z(RG) = \{0, \hat{G}\}$. So $|R| \neq 2$. If $|R| = 3$, then $\{\hat{G}, 2\hat{G}, 1-g, 2(1-g)\} \subseteq Z^*(RG)$. Hence $|R| = 4$. If $\text{char}(R) = 4$, then $R \cong \mathbb{Z}_4$. So $\{\hat{G}, 2\hat{G}, 3\hat{G}, 2\} \subseteq Z^*(RG)$, which is impossible. Let $|R| = 4$ and $\text{char}(R) = 2$ such that R be not a field. Then there exist distinct elements r_1 and r_2 in R. We know that R is not a field. Thus without loss of generality, let $r_1 \in Z^*(R)$. Hence $\{r_1, \hat{G}, r_2\hat{G}, r_1\hat{G}\} \subseteq Z^*(RG)$, which is impossible.

Finally, suppose that R is a field with four elements, say $R = \{0, 1, a, a^2\}$. If $RG \cong \mathbb{F}_4C_2$, then $Z(RG) = \{0, \hat{G}, a\hat{G}, a^2\hat{G}\}$ such that $\hat{G}^2 = 0$, since $\text{char}(R) \mid |G|$. Hence the proof is completed. □

Lemma 5.4. $\text{diam}(L(AG(RG))) \leq 3$*.*

Proof. Suppose that uv and xy are nonadjacent vertices in $L(AG(RG))$. Then u, v, x and y are distinct vertices in $AG(RG)$. Since $AG(RG)$ is connected with $\text{diam}(AG(RG)) \leq 2$, by [9, Theorem 2.2], there exists a path P from x to u with length at most two, say $P : x \sim w \sim u$ such that $w \notin \{x, y, u, v\}$ and $w \in V(AG(RG))$. So there is a path with length at most three from uv to xy and the proof is completed. □

Theorem 5.5. $\text{gr}(L(AG(RG))) = 3$ *or* RG *is isomorphic to one of the following rings.*

(i) $\mathbb{Z}_2C_2$;

(ii) $\mathbb{Z}_3C_2$.

Proof. By Theorem 3.2, we know that $\text{gr}(AG(RG)) = 3$ if and only if $\text{gr}(\Gamma(RG)) = 3$. So, by [2, Proposition 2.8], $\text{gr}(AG(RG)) = 3$ if and only if $RG \ncong \mathbb{Z}_2C_2$ and $RG \ncong \mathbb{F}_{p^r}C_q$ such that p and q are distinct prime numbers, p is a generator for $(\frac{\mathbb{Z}}{q\mathbb{Z}})^*$ and $\gcd(r, q-1) = 1$ such that $\mathbb{F}_{p^r}C_q$ is isomorphic to the direct product of two fields. Suppose that $RG \cong \mathbb{Z}_2C_2$. Then $AG(RG) \cong K_1$. So $L(AG(RG))$ is a null graph. So $L(AG(RG))$ does not contain any cycle. If $RG \cong \mathbb{F}_1 \times \mathbb{F}_2$ such that $|\mathbb{F}_1|, |\mathbb{F}_2| \geq 4$, then $AG(RG) \cong K_{|\mathbb{F}_1|,|\mathbb{F}_2|}$ such that $AG(RG)$ contains a copy of $K_{3,3}$. So $\text{gr}(L(AG(RG))) = 3$. Without loss of generality, we may assume that $|\mathbb{F}_1| \leq 3$ and $|\mathbb{F}_2| \leq 4$. In this situation, by Lemmas 2.1 and 2.2, $RG \cong \mathbb{Z}_3C_2$. Hence $AG(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a cycle of length four. Hence $\text{gr}(L(AG(RG))) = 4$. Thus the proof is completed. □

Theorem 5.6. $\operatorname{diam}(L(AG(RG))) = 1$ *if and only if* RG *is isomorphic to* $\mathbb{F}_4C_2$ *or* $\mathbb{Z}_2C_3$.

Proof. First, suppose that $\operatorname{diam}(L(AG(RG))) = 1$. Then, by Theorem 4.1, if $RG \notin \{\mathbb{Z}_2C_2, \mathbb{Z}_2C_3, \mathbb{Z}_3C_2, \mathbb{F}_4C_2\}$, then RG contains a copy of $K_{3,3}$ or K_5, which is not a star graph. Now, suppose that $RG \in \{\mathbb{Z}_2C_2, \mathbb{Z}_2C_3, \mathbb{Z}_3C_2, \mathbb{F}_4C_2\}$. Then $AG(\mathbb{Z}_2C_2) \cong K_1$, $AG(\mathbb{Z}_2C_3) \cong K_{1,3}$, $AG(\mathbb{F}_4C_2) \cong K_3$ and $AG(\mathbb{Z}_3C_2)$ is a cycle. Also, by Lemma 5.1 and Proposition 5.3, RG is isomorphic to $\mathbb{F}_4C_2$ or $\mathbb{Z}_2C_3$.

The converse statement is clear. □

Theorem 5.7. $L(AG(RG))$ *is planar if and only if* RG *is isomorphic to one of the following rings.*

(i) $\mathbb{Z}_2C_2$;
(ii) $\mathbb{Z}_3C_2$;
(iii) $\mathbb{Z}_2C_3$;
(iv) $\mathbb{F}_4C_2$.

Proof. First, suppose that $L(AG(RG))$ is planar. Then, by Theorem 4.1, RG is one of the rings $\mathbb{Z}_2C_2$, $\mathbb{Z}_3C_2$, $\mathbb{Z}_2C_3$ or $\mathbb{F}_4C_2$. We know that $|V(AG(\mathbb{Z}_2C_2))| = 1$. Hence $L(AG(RG))$ is a null graph, which is planar. Now, suppose that RG is isomorphic to $\mathbb{Z}_3C_2$. Then $Z(\mathbb{Z}_3C_2) = \{0, \hat{G}, \overline{2}\hat{G}, \overline{1} + \overline{2}g, \overline{2} + g\}$. Hence $L(AG(RG))$ is isomorphic to the cycle of length four, which is planar. If RG is isomorphic to $\mathbb{Z}_2C_3$, then $Z(\mathbb{Z}_2C_3) = \{0, \hat{G}, 1 - g, 1 - g^2, g - g^2\}$. In this situation, $L(AG(RG))$ is isomorphic to K_3, which is planar. Finally, suppose that RG is isomorphic to $\mathbb{F}_4C_2$. Then $L(AG(RG)) \cong K_3$, which is planar.

The converse statement is clear. □

The following corollaries are obtained from Theorem 5.7.

Corollary 5.8. $L(AG(RG))$ *is outerplnar if and only if* $L(AG(RG))$ *is planar.*

Corollary 5.9. $L(AG(RG))$ *is ring graph if and only if* $L(AG(RG))$ *is planar.*

Acknowledgments. The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

References

[1] M. Afkhami, K. Khashyarmanesh, and S. M. Shakhdari, *The annihilator graph of a commutative semigroup*, J. Algebra Appl. **14** (2015), 1550015, 14 pp.
[2] F. Aliniaeifard and Y. Li, *Zero-divisor graphs for group rings*, J. Algebra **42** (2014), no. 11, 4790–4800.
[3] D. F. Anderson and A. Badawi, *The total graph of a commutative ring*, J. Algebra **320** (2008), no. 7, 2706–2719.
[4] ———, *The generalized total graph of a commutative ring* J. Algebra Appl. **12** (2013), 1250212, 18 pp.
[5] D. F. Anderson and P. S. Livignston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[6] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), no. 2, 500–514.
[7] A. Ashrafi, H. R. Maimani, M. R. Pournaki, and S. Yassemi, *Unit graphs associated with rings*, Comm. Algebra **38** (2010), no. 8, 2851–2871.
[8] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addision-Wesley Publishing Company London, 1969.
[9] A. Badawi, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), no. 1, 108–121.
[10] ______, *On the dot product graph of a commutative ring*, Comm. Algebra **43** (2015), no. 1, 43–50.
[11] Z. Barati, K. Khashyarmanesh, F. Mohammadi, and K. Nafar, *On the associated graphs to a commutative ring*, J. Algebra Appl. **11** (2012), no. 2, 1250037, 17 pp.
[12] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.
[13] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008.
[14] F. R. DeMeyer, T. McKenzie, and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2002), no. 2, 206–214.
[15] A. B. Ganagi, I. Gutman, and H. S. Ramane, *On diameter of line graphs*, Iran. J. Math. Sci. Inform. **8** (2013), no. 1, 105–109.
[16] I. Gilter, E. Reyes, and R. H. Villareal, *Ring graphs and complete intersection toric ideals*, Discrete Math. **310** (2010), no. 3, 430–441.
[17] K. R. McLean, *Commutative Artinian principal ideal rings*, Proc. Lond. Math. Soc. **26** (1973), 249–272.
[18] C. P. Milies and S. K. Sehgal, *An Introduction to Group Rings*, Dordrecht Kluwer Academic Publisher, 2002.
[19] W. K. Nicholson, *Local group rings*, Canad. Math. Bull. **15** (1972), 137–138.
[20] S. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math. **307** (2007), no. 9-10, 1155–1166.
[21] J. Sedlaĉek, *Some properties of interchange graphs*, Theory of Graphs and its Applications (Proceedings of the Symposium, Smolenice, 1963), 145–150, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964.

Mojgan Afkhami
Department of Mathematics
University of Neyshabur
P.O.Box 91136-899, Neyshabur, Iran
E-mail address: mojgan.afkhami@yahoo.com

Kazem Khashyarmanesh
Department of Pure Mathematics
Ferdowsi University of Mashhad
P.O.Box 1159-91775, Mashhad, Iran
E-mail address: khashyar@ipm.ir

Sepideh Salehifar
Department of Pure Mathematics
Ferdowsi University of Mashhad
P.O.Box 1159-91775, Mashhad, Iran
E-mail address: sepidehsalehifar@gmail.com