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A NOTE ON ENDOMORPHISMS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let I denote an ideal of a Noetherian local ring $(R, \mathfrak{m})$. Let M denote a finitely generated R-module. We study the endomorphism ring of the local cohomology module $H^c_I(M), c = \operatorname{grade}(I, M)$. In particular there is a natural homomorphism

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M)),$$

where $\hat{\cdot}^I$ denotes the I-adic completion functor. We provide sufficient conditions such that it becomes an isomorphism. Moreover, we study a homomorphism of two such endomorphism rings of local cohomology modules for two ideals $J \subset I$ with the property $\operatorname{grade}(I, M) = \operatorname{grade}(J, M)$. Our results extends constructions known in the case of $M = R$ (see e.g. [8], [17], [18]).

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring. For an ideal $I \subset R$ and an R-module M we denote by $H^i_I(M)$, $i \in \mathbb{Z}$, the local cohomology modules of M with respect to I (see [2] and [5] for the definition). In recent papers (see e.g. [4], [7], [8], [17], [18], [19]) there is some interest in the study of the endomorphism ring $\operatorname{Hom}_R(H^i_I(R), H^i_I(R))$ for certain ideals I and several $i \in \mathbb{N}$. Note that the first results in this direction were obtained by M. Hochster and C. Huneke (see [10]) for the case of $I = \mathfrak{m}$ and $i = \dim R$.

In the case of $(R, \mathfrak{m})$ an n-dimensional Gorenstein ring and $I \subset R$ an ideal with $c = \operatorname{grade} I$ it was shown (see [19, Theorem 1.1]) that there is a natural homomorphism

$$\hat{R}^I \to \operatorname{Hom}_R(H^c_I(R), H^c_I(R)),$$

where $\hat{R}^I$ denotes the I-adic completion of R. Moreover there are results when this homomorphism is in fact an isomorphism (see also [8], [17]). The main subject of the present paper is the extension of some of these results to the

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case of a finitely generated R-module M. More precisely we shall prove the following result:

Theorem 1.1. *Let I denote an ideal of R. Let M denote a finitely generated R-module and $c = \operatorname{grade}(I, M)$. Then there is a natural homomorphism*

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M)).$$

Furthermore, if $H^i_I(M) = 0$ for all $i \neq c$, then it is an isomorphism. Moreover

$$\operatorname{Ext}^i_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \cong \operatorname{Ext}^{i+c}_R(H^c_I(M), M) \cong \operatorname{Ext}^i_R(H^c_I(M), H^c_I(M))$$

for all $i \in \mathbb{Z}$.

That is, the endomorphism rings of M and those of $H^c_I(M)$ are closely related by a natural map (see Theorem 3.3 and Corollary 3.4 for the proof). These investigations are related to the endomorphism ring of $D(H^c_I(M)) := \operatorname{Hom}_R(H^c_I(M), E), E$ is the injective hull of the residue field $k = R/\mathfrak{m}$. Let $(R, \mathfrak{m})$ denote a Noetherian complete local ring of dimension n. Then there is a natural homomorphism

$$\operatorname{Hom}_R(H^c_I(M), H^c_I(M)) \to \operatorname{Hom}_R(D(H^c_I(M)), D(H^c_I(M)))$$

that is an isomorphism (see Lemma 3.6). Moreover there are relative versions of the above homomorphisms relating two ideals $J \subset I$ of the same grade (see Theorem 4.1). Also there are some necessary conditions in order to prove the natural homomorphism $\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$ is an isomorphism. That is:

Theorem 1.2. *Let $J \subset I$ denote two ideals of a Noetherian complete local ring $(R, \mathfrak{m})$. Let M be a finitely generated R-module with $H^i_J(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M) = \operatorname{grade}(J, M)$. Suppose that $\operatorname{Rad} IR_\mathfrak{p} = \operatorname{Rad} JR_\mathfrak{p}$ for all $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ such that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. Then the natural homomorphism*

$$\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$$

is an isomorphism.

This is a generalization of [17, Theorem 1.2]) to the case of a finitely generated R-module M. In the case of an ideal I generated by an M-regular sequence of length c in a complete local ring R it follows that $\operatorname{Hom}_R(M, M) \cong \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$. So the above result gives some necessary conditions for the endomorphism ring of $H^c_I(M)$ to be isomorphic to that of M. The proof of Theorem 1.2 is shown in Corollary 4.4.

2. Preliminaries and auxiliary results

In this section we will fix some notation and summarize a few preliminaries and auxiliary results. For unexplained terminologies we refer to the textbooks [12] and [20]. Let $(R, \mathfrak{m})$ denote a commutative local Noetherian ring with

$\mathfrak{m}$ its maximal ideal and $k = R/\mathfrak{m}$ its residue field. Moreover we will denote the Matlis dual functor by $D(\cdot) := \operatorname{Hom}_R(\cdot, E)$ where $E = E_R(k)$ denotes the injective hull of k.

Lemma 2.1. *Let M, N be two arbitrary R-modules. Then for all $i \in \mathbb{Z}$:*

(1) $\operatorname{Ext}_R^i(N, D(M)) \cong D(\operatorname{Tor}_i^R(N, M))$.

(2) *If N is in addition finitely generated, then*

$$D(\operatorname{Ext}_R^i(N, M)) \cong \operatorname{Tor}_i^R(N, D(M)).$$

Proof. The proof is well known for details see e.g. [9, Example 3.6]. □

Let I be an ideal of R and let M denote an R-module. For the basics on local cohomology modules $H_I^i(M)$ we refer to the textbook [2]. Note that

$$\operatorname{grade}(I, M) = \inf\{i \in \mathbb{Z} : H_I^i(M) \neq 0\}$$

for a finitely generated R-module M. As a first result we will give a criterion of calculating $\operatorname{grade}(I, M)$ in terms of vanishing of the Tor-modules.

Proposition 2.2. *Let $I \subset R$ be an ideal. Let M denote a finitely generated R-module with $c = \operatorname{grade}(I, M)$. Let N be an R-module such that $\operatorname{Supp}_R(N) \subseteq V(I)$. Then the following holds*:

(a) $\operatorname{Ext}_R^i(N, M) = 0$ *for all $i < c$ and there is a natural isomorphism*

$$\operatorname{Ext}_R^c(N, M) \cong \operatorname{Hom}_R(N, H_I^c(M)).$$

(b) $\operatorname{Tor}_i^R(N, D(M)) = 0$ *for all $i < c$ and there is a natural isomorphism*

$$\operatorname{Tor}_c^R(N, D(M)) \cong N \otimes_R D(H_I^c(M)).$$

Proof. In the case of $M = R$ the result was shown by Schenzel (see [18, Thoerem 2.3]). For a finitely generated R-module M the statement follows by the same arguments. So, we skip the details. □

As an application of Proposition 2.2 we will give a characterization of $\operatorname{grade}(I, M)$. For $M = R$ this is shown in [18, Corollary 2.4].

Corollary 2.3. *With the notation of Proposition 2.2 it follows*

$$\operatorname{grade}(I, M) = \inf\{i \in \mathbb{Z} : \operatorname{Tor}_i^R(R/I, D(M)) \neq 0\}.$$

Proof. By Proposition 2.2 it will be enough to show that $\operatorname{Tor}_c^R(R/I, D(M)) \neq 0$. Since $\operatorname{grade}(I, M) = c$ it follows that (see Proposition 2.2)

$$\operatorname{Tor}_c^R(R/I, D(M)) \cong R/I \otimes_R D(H_I^c(M)).$$

This module is isomorphic to $D(\operatorname{Hom}_R(R/I, H_I^c(M)))$ (see Lemma 2.1). Therefore we conclude that (see Proposition 2.2)

$$\operatorname{Tor}_c^R(R/I, D(M)) \cong D(\operatorname{Ext}_R^c(R/I, M)) \neq 0.$$

Since $\operatorname{Ext}_R^c(R/I, M) \neq 0$ because of $c = \operatorname{grade}(I, M)$ (see [9, Remark 3.11]) it completes the proof. □

As a final point of this section let us recall the definition of the truncation complex as it was introduced in [16, Definition 4.1]. Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. Let $I \subset R$ be an ideal of R with $\operatorname{grade}(I, M) = c$. Suppose that $E_R^{\cdot}(M)$ is a minimal injective resolution of M. Then it follows (see Matlis' structure theory on injective modules [11]) that

$$E_R^{\cdot}(M)^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Supp}_R(M)} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)},$$

where $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})}(\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}))$ and $i \in \mathbb{Z}$. Let $\Gamma_I(\cdot)$ denote the section functor with support in I. Note that $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$ and $\Gamma_I(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$. In particular $\mu_i(\mathfrak{p}, M) = 0$ for all $i < c$ since $\operatorname{grade}(I, M) = c$. So it follows that

$$\Gamma_I(E_R^{\cdot}(M))^i \cong \bigoplus_{\mathfrak{p} \in V(I) \cap \operatorname{Supp} M} \Gamma_I(E_R(R/\mathfrak{p}))^{\mu_i(\mathfrak{p}, M)} = 0$$

for all $i < c$. Whence there is an exact sequence

$$0 \to H^c_I(M) \to \Gamma_I(E^c_R(M)) \to \Gamma_I(E^{c+1}_R(M)).$$

That is there is a natural morphism of complexes of R-modules $H^c_I(M)[-c] \to \Gamma_I(E_R^{\cdot}(M))$, where $H^c_I(M)$ is considered as a complex sitting in homological degree 0.

Definition 2.4. Define the complex $C_M^{\cdot}(I)$ as the cokernel of the above morphism of complexes. It is called the truncation complex of M with respect to the ideal I. There is a short exact sequence of complexes of R-modules

$$0 \to H^c_I(M)[-c] \to \Gamma_I(E_R^{\cdot}(M)) \to C_M^{\cdot}(I) \to 0.$$

Note that $H^i(C_M^{\cdot}(I)) = 0$ for all $i \leq c$ or $i > n$ and $H^i(C_M^{\cdot}(I)) \cong H^i_I(M)$ for all $c < i \leq n$. As a first application there is the following proposition.

Proposition 2.5. *Let $I \subset R$ denote an ideal. Let M denote a finitely generated R-module. Suppose that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$. Then there are natural isomorphisms $\operatorname{Ext}^i_R(N, M) \cong \operatorname{Ext}^{i-c}_R(N, H^c_I(M))$ for all $i \in \mathbb{Z}$ and any R-module N with $\operatorname{Supp}_R N \subseteq V(I)$.*

Proof. Let $E_R^{\cdot}(M)$ denote a minimal injective resolution of M. By the assumption it follows that $\Gamma_I(E_R^{\cdot}(M))$ is a minimal injective resolution of $H^c_I(M)[-c]$. Therefore there is an isomorphism

$$\operatorname{Ext}^{i-c}_R(N, H^c_I(M)) \cong H^i(\operatorname{Hom}_R(N, \Gamma_I(E_R^{\cdot}(M))))$$

for all $i \in \mathbb{Z}$. By an application of [18, Lemma 2.2] it follows that

$$\operatorname{Hom}_R(N, \Gamma_I(E_R^{\cdot}(M))) \cong \operatorname{Hom}_R(N, E_R^{\cdot}(M))$$

for an R-module N such that $\operatorname{Supp}_R N \subseteq V(I)$. Because of

$$\operatorname{Ext}^i_R(N, M) = H^i(\operatorname{Hom}_R(N, E_R^{\cdot}(M)))$$

the result follows. ∎

3. A natural map of endomorphism rings

In this section I denotes an ideal of a local ring $(R, \mathfrak{m})$. Let M be an R-module. We denote by $\hat{M}^I = \Lambda^I(M) = \varprojlim M/I^\alpha M$ the I-adic completion of M. In particular $\hat{R}^I$ denotes the I-adic completion of R. The natural homomorphism $R \to \hat{R}^I$ makes $\hat{R}^I$ into a faithfully flat R-module. We denote by $\Lambda^I(\cdot)$ the completion functor. Its left derived functors are denoted by $\mathrm{L}_i\Lambda^I(\cdot), i \in \mathbb{Z}$. Note that $\mathrm{L}_i\Lambda^I(M) = 0$ for all $i > 0$ and a finitely generated R-module M as follows since $\Lambda^I(\cdot)$ is exact on the category of finitely generated R-modules.

In the next remark we will recall some facts about the projective and injective resolutions of complexes.

Remark 3.1. Suppose that $X \to Y$ is a morphism of complexes such that it induces an isomorphism in cohomologies. Let $F_R^{\cdot}$ be a complex of flat R-modules bounded above. Then there is a quasi-isomorphism

$$F_R^{\cdot} \otimes_R X \xrightarrow{\sim} F_R^{\cdot} \otimes_R Y.$$

Similar results are true for $\mathrm{Hom}_R(\cdot, E_R^{\cdot})$ respectively $\mathrm{Hom}_R(P_R^{\cdot}, \cdot)$ for $E_R^{\cdot}$ a bounded below complex of injective R-modules respectively $P_R^{\cdot}$ a bounded above complex of projective R-modules. For the proof of these facts we refer to [6].

Before proving the main theorem we need another result on completion. To this end let $\underline{x} = x_1, \ldots, x_r \in I$ denote a system of elements such that $\mathrm{Rad}(\underline{x})R = \mathrm{Rad}\, I$. Then we consider the Čech complex $\check{C}_{\underline{x}}$ of the system $\underline{x}$ (see e.g. [15, Section 3]). Moreover, there is a bounded complex $L_{\underline{x}}$ of free R-modules (depending on $\underline{x} = x_1, \ldots, x_r$) and a natural homomorphism of complexes $L_{\underline{x}} \to \check{C}_{\underline{x}}$ that induces an isomorphism in cohomology (see [15, Section 4] for the details). That is, $L_{\underline{x}}$ provides a free resolution of $\check{C}_{\underline{x}}$, a complex of flat R-modules. In our context the importance of the Čech complex is the following result. It allows the expression of the left derived functors $\mathrm{L}_i\Lambda^I(\cdot)$ in different terms.

Theorem 3.2. *Let I denote an ideal of a ring $(R, \mathfrak{m})$. Let $X^{\cdot}$ denote a left bounded complex of R-modules. Then there are natural isomorphisms*

$$\mathrm{L}_i\Lambda^I(X^{\cdot}) \cong H_i(\mathrm{Hom}_R(L_{\underline{x}}, X^{\cdot})) \cong H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, E^{\cdot}))$$

for all $i \in \mathbb{Z}$, where $E^{\cdot}$ denotes an injective resolution of $X^{\cdot}$.

Proof. See [13, Theorem 0.2]. □

Note that the Errata of M. Porta, L. Shaul, A. Yekutieli to their paper [13] does not concern the correctness of [13, Theorem 0.2]. The last Theorem 3.2 is one of the main results of [15, Section 4] in the case of a bounded complex. The general version was proved by M. Porta, L. Shaul, A. Yekutieli (see [13]). In our formulation here we do not use the technique of derived functors and derived categories. (For a more advanced exposition based on derived categories and derived functors the interested reader might also consult [1] and [13].)

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *denote a local ring. Let* M *be a finitely generated* R*-module. Let* $I \subset R$ *be an ideal and* $c = \operatorname{grade}(I, M)$*. Then there is a natural homomorphism*

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Hom}_R(H_I^c(M), H_I^c(M)).$$

Proof. Let $E_R^{\cdot}(M)$ be a minimal injective resolution of M. Then we use the truncation complex as defined in Definition 2.4. We apply the functor $\operatorname{Hom}_R(\cdot, E_R^{\cdot}(M))$ to this short exact sequence of complexes. Since $E_R^{\cdot}(M)$ is a left bounded complex of injective R-modules it yields a short exact sequence of complexes of R-modules

$$0 \to \operatorname{Hom}_R(C_M^{\cdot}(I), E_R^{\cdot}(M)) \to \operatorname{Hom}_R(\Gamma_I(E_R^{\cdot}(M)), E_R^{\cdot}(M)) \\ \to \operatorname{Hom}_R(H_I^c(M), E_R^{\cdot}(M))[c] \to 0.$$

First we investigate the complex in the middle. There is a quasi-isomorphism of complexes $\Gamma_I(E_R^{\cdot}(M)) \xrightarrow{\sim} L_{\underline{x}} \otimes E_R^{\cdot}(M)$ (see [15, Theorem 1.1]). Let us assume that

$$X := \operatorname{Hom}_R(M, E_R^{\cdot}(M)) \text{ and } Y := \operatorname{Hom}_R(L_{\underline{x}} \otimes_R E_R^{\cdot}(M), E_R^{\cdot}(M)).$$

By Remark 3.1 the complex in the middle is quasi-isomorphic to Y since $E_R^{\cdot}(M)$ is a left bounded complex of injective R-modules. By adjunction (see Lemma 2.1) Y is isomorphic to the complex $\operatorname{Hom}_R(L_{\underline{x}}, \operatorname{Hom}_R(E_R^{\cdot}(M), E_R^{\cdot}(M)))$. Because $E_R^{\cdot}(M)$ is an injective resolution of M so there is a quasi-isomorphism of complexes

$$\operatorname{Hom}_R(E_R^{\cdot}(M), E_R^{\cdot}(M)) \xrightarrow{\sim} X.$$

Since $L_{\underline{x}}$ is a bounded complex of free R-modules it follows that Y is quasi-isomorphic to $\operatorname{Hom}_R(L_{\underline{x}}, X)$ (see Remark 3.1). That is, in order to compute the homology of Y it will be enough to compute the homology of $\operatorname{Hom}_R(L_{\underline{x}}, X)$. By virtue of Theorem 3.2 and [14, p. 352] there is the following spectral sequence

$$E_2^{i,j} = \mathrm{L}_i \Lambda^I(\operatorname{Ext}_R^j(M, M)) \Rightarrow E_\infty^{i+j} = H^{i+j}(\operatorname{Hom}_R(L_{\underline{x}}, X)).$$

It degenerates to isomorphisms $\mathrm{L}_0 \Lambda^I(\operatorname{Ext}_R^j(M, M)) \cong H^j(\operatorname{Hom}_R(L_{\underline{x}}, X))$ since $\operatorname{Ext}_R^j(M, M)$ is finitely generated for all $j \in \mathbb{Z}$. This finally proves that $H^j(Y) \cong \mathrm{L}_0 \Lambda^I(\operatorname{Ext}_R^j(M, M))$ for all $j \in \mathbb{Z}$.

The cohomology of the complex at the right of the above short exact sequence of complexes is $\operatorname{Ext}_R^{i+c}(H_I^c(M), M), i \in \mathbb{Z}$. Therefore the long exact cohomology sequence at degree 0 provides a natural homomorphism

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Ext}_R^c(H_I^c(M), M).$$

The second module is isomorphic to $\operatorname{Hom}_R(H_I^c(M), H_I^c(M))$ (see Proposition 2.2(a)). This completes the proof. $\square$

In the following we shall investigate the particular case when $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$.

Corollary 3.4. *With the previous notation suppose in addition that* $H^i_I(M) = 0$ *for all* $i \neq c$. *Then there are isomorphisms*

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \cong \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$$

and

$$\operatorname{Ext}^i_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \cong \operatorname{Ext}^{i+c}_R(H^c_I(M), M) \cong \operatorname{Ext}^i_R(H^c_I(M), H^c_I(M))$$

for all $i \in \mathbb{Z}$.

Proof. The assumption on the vanishing of $H^i_I(M)$ for all $i \neq c$ implies that the truncation complex $C^{\cdot}_M(I)$ is homologically trivial. Therefore

$$H^i(\operatorname{Hom}_R(C^{\cdot}_M(I), E^{\cdot}_R(M))) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

That is, the short exact sequence of complexes in the proof of Theorem 3.3 induces a quasi-isomorphism

$$\operatorname{Hom}_R(\Gamma_I(E^{\cdot}_R(M)), E^{\cdot}_R(M)) \xrightarrow{\sim} \operatorname{Hom}_R(H^c_I(M), E^{\cdot}_R(M))[c].$$

With the computations on the cohomology (done in the proof of Theorem 3.3) it provides isomorphisms

$$\operatorname{Ext}^i_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \cong \operatorname{Ext}^{i+c}_R(H^c_I(M), M)$$

for all $i \in \mathbb{Z}$. This finishes the proof (see Proposition 2.2(a) and Proposition 2.5). ☐

Remark 3.5. For an arbitrary R-module X there is a natural injective homomorphism $X \to D(D(X))$. By applying $\operatorname{Hom}_R(X, \cdot)$ and by adjunction (see Lemma 2.1) there is a natural injective homomorphism $\operatorname{Hom}_R(X, X) \to \operatorname{Hom}_R(D(X), D(X))$. Therefore by Theorem 3.3 the natural homomorphism $\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Hom}_R(D(H^c_I(M)), D(H^c_I(M)))$ induces a commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) & \to & \operatorname{Hom}_R(H^c_I(M), H^c_I(M)) \\ \| & & \downarrow \\ \operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) & \to & \operatorname{Hom}_R(D(H^c_I(M)), D(H^c_I(M))) \end{array}$$

The following lemma is a generalization of [8, Corollary 2.4 and Theorem 2.6] to a finitely generated module.

Lemma 3.6. *Let* R *be a complete local ring and let* $I \subset R$ *be an ideal. Let* M *denote a finitely generated* R*-module and* $\operatorname{grade}(I, M) = c$. *Then the following holds*:

(a) *The right vertical map in the preceding diagram is an isomorphism.*

(b) *The natural homomorphism*

$$\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(H_I^c(M), H_I^c(M))$$

is an isomorphism if and only if the natural homomorphism

$$\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(D(H_I^c(M)), D(H_I^c(M)))$$

is an isomorphism.

Proof. First of all recall the well known fact that R is I-adic complete provided it is complete (i.e., complete in the $\mathfrak{m}$-adic topology). By adjunction (see Lemma 2.1) there is an isomorphism

$$\operatorname{Hom}_R(D(H_I^c(M)), D(H_I^c(M))) \cong D(H_I^c(M) \otimes_R D(H_I^c(M))).$$

The module on the right side is isomorphic to

$$D(\operatorname{Tor}_c^R(H_I^c(M), D(M))) \cong \operatorname{Ext}_R^c(H_I^c(M), M).$$

This follows by view of Lemma 2.1 and Proposition 2.2 because R is complete and by Matlis duality the double Matlis dual of M is isomorphic to M since M is finitely generated. By virtue of Proposition 2.2 there is the following isomorphism

$$\operatorname{Ext}_R^c(H_I^c(M), M) \cong \operatorname{Hom}_R(H_I^c(M), H_I^c(M)).$$

By combining these isomorphisms together it proves (a). The statement (b) is clear from (a) and the commutative diagram in Remark 3.5. ☐

4. Applications

As before let $(R, \mathfrak{m})$ be a ring and let M denote a finitely generated R-module. The following theorem is actually proved by Schenzel (see [18, Theorem 1.3]) in case of an arbitrary Noetherian local ring. Here we will give a generalization to a finitely generated R-module M.

Theorem 4.1. *Let R be a local ring and let M be a finitely generated R-module. Let $J \subset I$ denote two ideals of R such that* $\operatorname{grade}(I, M) = c = \operatorname{grade}(J, M)$.

(a) *There is a natural homomorphism*

$$\operatorname{Hom}_R(H_J^c(M), H_J^c(M)) \to \operatorname{Hom}_R(H_I^c(M), H_I^c(M)).$$

(b) *Suppose that* $\operatorname{Rad} IR_\mathfrak{p} = \operatorname{Rad} JR_\mathfrak{p}$ *for all* $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ *such that* $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. *Then the above natural homomorphism is an isomorphism.*

(c) *Suppose in addition that R is complete. Then there is a natural homomorphism*

$$\operatorname{Hom}_R(D(H_J^c(M)), D(H_J^c(M))) \to \operatorname{Hom}_R(D(H_I^c(M)), D(H_I^c(M)))$$

and under the assumption of (b) *it is an isomorphism.*

Proof. Since $J \subset I$ it induces the following short exact sequence

$$0 \to I^\alpha/J^\alpha \to R/J^\alpha \to R/I^\alpha \to 0$$

for each integer $\alpha \geq 1$. Moreover note that $\operatorname{grade}(I^\alpha/J^\alpha, M) \geq c$ (see e.g. Proposition 2.2(a)). Then the application of the functor $\operatorname{Hom}_R(\cdot, M)$ to the last sequence yields the following exact sequence

$$0 \to \operatorname{Ext}^c_R(R/I^\alpha, M) \to \operatorname{Ext}^c_R(R/J^\alpha, M) \to \operatorname{Ext}^c_R(I^\alpha/J^\alpha, M).$$

Now the direct limit is an exact functor. So pass to the direct limit of this sequence we get that

$$0 \to H^c_I(M) \to H^c_J(M) \xrightarrow{f} \varinjlim \operatorname{Ext}^c_R(I^\alpha/J^\alpha, M). \tag{4.1}$$

Let $N := \operatorname{im} f$. Then this induces the short exact sequence

$$0 \to H^c_I(M) \to H^c_J(M) \to N \to 0.$$

Again by applying the functor $\operatorname{Hom}_R(\cdot, M)$ to this sequence we get the following homomorphism

$$\operatorname{Ext}^c_R(H^c_J(M), M) \to \operatorname{Ext}^c_R(H^c_I(M), M).$$

Then from Proposition 2.2(a) the statement (a) is shown.

For the proof of (b) note that the local cohomology is independent up to the radical. So without loss of generality we may assume that $IR_\mathfrak{p} = JR_\mathfrak{p}$ for all $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ such that

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c.$$

We claim that $\operatorname{Ext}^c_R(I^\alpha/J^\alpha, M) = 0$ for all $\alpha \geq 1$. Because $\operatorname{Ann}_R(I^\alpha/J^\alpha) = J^\alpha :_R I^\alpha$ it will be enough to prove that $\operatorname{grade}(J^\alpha :_R I^\alpha, M) \geq c + 1$. It is well-known that

$$\operatorname{grade}(J^\alpha :_R I^\alpha, M) = \inf\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in V(J^\alpha :_R I^\alpha) \cap \operatorname{Supp}_R(M)\}$$

(see e.g. [3, Proposition 1.2.10]). Suppose on contrary that the claim is not true. Then there exists a prime ideal

$$\mathfrak{p} \in V(J^\alpha :_R I^\alpha) \cap \operatorname{Supp}_R(M)$$

such that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. Moreover $\operatorname{Supp}_R(R/(J^\alpha :_R I^\alpha)) = V(J^\alpha :_R I^\alpha)$ is contained in $V(J)$. This implies that $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ and $R_\mathfrak{p}/(J^\alpha_\mathfrak{p} R_\mathfrak{p} :_{R_\mathfrak{p}} I^\alpha_\mathfrak{p} R_\mathfrak{p}) \neq 0$ which is a contradiction to our assumption $IR_\mathfrak{p} = JR_\mathfrak{p}$. So we have $\operatorname{Ext}^c_R(I^\alpha/J^\alpha, M) = 0$ which shows that $H^c_I(M) \cong H^c_J(M)$ (see (4.1)). This proves the claim in (b).

Finally the proof of (c) is a consequence of Lemma 3.6(b). □

Example 4.2. For instance, if $J = (x_1, \ldots, x_c)R$ is generated by an M-regular sequence it follows that $H^i_J(M) = 0$ for all $i \neq c$. Therefore the natural homomorphism

$$\operatorname{Hom}_{\hat{R}^J}(\hat{M}^J, \hat{M}^J) \to \operatorname{Hom}_R(H^c_J(M), H^c_J(M))$$

is an isomorphism (see Theorem 3.3). By view of Theorem 4.1 this could be a candidate for proving that

$$\operatorname{Hom}_{\hat{R}^I}(\hat{M}^I, \hat{M}^I) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$$

is an isomorphism where $J \subset I$. At least if R is I-adic complete.

Example 4.3. Let $R = A/J$ where $A = k[|x, y, z, w|]$ is the formal power series ring over a field k and $J = (xz, yw) = (x, y) \cap (y, z) \cap (z, w) \cap (x, w)$. Clearly R is a complete local Gorenstein ring of $\dim(R) = 2$. Suppose that $I = (x, y, z)R$. Let $M = R/\mathfrak{p}_1 \oplus R/\mathfrak{p}_2$ where $\mathfrak{p}_1 = (x, y)R$, and $\mathfrak{p}_2 = (y, z)R$. Then it follows by Independence of Base Ring Theorem (see [9, Proposition 2.14]) that for each $i \in \mathbb{Z}$

$$H^i_I(R/\mathfrak{p}_1) \cong H^i_{IR/\mathfrak{p}_1}(R/\mathfrak{p}_1) \text{ and } H^i_I(R/\mathfrak{p}_2) \cong H^i_{IR/\mathfrak{p}_2}(R/\mathfrak{p}_2).$$

Moreover $\operatorname{cd}(I, M) = \max\{\operatorname{cd}(I, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}_R(M)\}$ it implies that $\operatorname{cd}(I, M) = 1$. Here $\operatorname{cd}(I, M)$ denotes the cohomological dimension of M with respect to I. Also $x + z \notin \operatorname{Ass}_R(M)$ so $\operatorname{grade}(I, M) = 1$. That is $H^i_I(M) = 0$ for all $i \neq 1$. Then by Corollary 3.4 the natural homomorphism

$$\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$$

is an isomorphism.

Corollary 4.4. *Let* $(R, \mathfrak{m})$ *denote a complete local ring. Let* M *be a finitely generated* R*-module. Let* $J \subseteq I$ *be two ideals and* $H^i_J(M) = 0$ *for all* $i \neq c = \operatorname{grade}(J, M) = \operatorname{grade}(I, M)$. *Suppose that* $\operatorname{Rad} IR_\mathfrak{p} = \operatorname{Rad} JR_\mathfrak{p}$ *for all* $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ *such that* $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. *Then the natural homomorphism*

$$\operatorname{Hom}_R(M, M) \to \operatorname{Hom}_R(H^c_I(M), H^c_I(M))$$

is an isomorphism.

Proof. It follows from Theorem 4.1 and Corollary 3.4. □

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