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SEXTIC MOMENT PROBLEMS ON 3 PARALLEL LINES

Seonguk Yoo

Abstract. Sextic moment problems with an infinite algebraic variety are still widely open. We study the problem with a single cubic column relation associated to 3 parallel lines in which the variety is infinite. It turns out that this specific column relation has a strong connection with moment problems that have a symmetric algebraic variety. We present more concrete solutions to some sextic moment problems with a symmetric variety.

1. Introduction

Let $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{m,0}, \beta_{m-1,1}, \ldots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$ denote a real 2-dimensional multisequence of order m. The *truncated real moment problem* (TRMP) entails finding necessary and sufficient conditions for the existence of a positive Borel measure μ supported in the real plane $\mathbb{R}^2$ such that

$$\beta_{ij} = \int x^i y^j \, d\mu \;\; (i, j \in \mathbb{Z}_+,\ 0 \le i + j \le m).$$

We call μ a *representing measure* for β; if a moment sequence has such a measure, then we say the problem is *soluble* and the necessary and sufficient conditions are said to be a solution.

There is also a complex version of the problem defined as follows: given a collection of complex numbers $\gamma \equiv \gamma^{(m)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{0,m}, \gamma_{1,m-1}, \ldots, \gamma_{m-1,1}, \gamma_{m,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* (TCMP) concerns under what conditions there is a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \quad (0 \le i+j \le m)$. It is well-known that TRMP and TCMP are equivalent for an even m [3], and hence all the techniques developed for a solution to TCMP are valid for TRMP

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as well. Both problems are simply referred to as the *truncated moment problem* (TMP).

When $m = 2n$, we define the *moment matrix* $M(n) \equiv M(n)(\beta^{(2n)})$ of β as

$$M(n)(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\,\mathbf{j}\in\mathbb{Z}_+^2:|\mathbf{i}|,\,|\mathbf{j}|\leq n}.$$

R. Curto and L. Fialkow have provided elegant results for various moment problems in a series of papers. Some of their work was established with matrix positivity and an extension of $M(n)$; new techniques were developed through the functional calculus of the columns in $\mathcal{C}_{M(n)}$, the column space of $M(n)$. For the application of functional calculus, we label the columns in $M(n)$ with the following degree lexicographical order: $\mathit{1}, X, Y, X^2, XY, Y^2, \ldots, X^n, \ldots, Y^n$. We note that each block with the moments of the same order in $M(n)$ is Hankel and that $M(n)$ is symmetric.

When $m = 2n + 1$, partial solutions can be seen in [14] and [16] as well as a solution to the *truncated matrix moment problem*; in particular, a solution to the *cubic complex moment problem* (when $n = 1$) was given in [15]. However, the problem is still often for $n \geq 2$.

Including important applications of TMP in various areas, we should remark that a solution of the *full* moment problem can be obtained from a complete solution of TCMP, via a weak-$*$ convergence argument, as shown by J. Stochel [18]; thus it seems essential to find a solution of TMP for all orders to cover the full moment problem.

We know moment problems with a single cubic column relation or with an invertible moment matrices are much more difficult to solve since these cases naturally satisfy all the necessary conditions and require some other properties; for $m = 6$, some specific cases are resolved as in [9, 10] but most cases remain open. In this note we focus on sextic moment problems with a single cubic column relation of 3 parallel lines, that is, rank $M(n) = 9$. We will also see this specific relations is deeply connected to moment problems that have a symmetric algebraic variety (see the definition in (1)). We later find more concrete solution for the sextic moment problems with a symmetric variety. Beyond these arguments, the technique used for the main results is again valid for sextic moment problems with reducible conic column relations which are much larger class and the topic will be seen in the forthcoming paper.

2. Preliminaries

2.1. Necessary conditions

Let μ be a representing measure of β. Then we first compute that

$$0 \leq \int p(x,y)^2\,d\mu = \sum_{i,j,k,l} a_{ij}a_{kl}\int x^{i+l}y^{j+k}\,d\mu = \sum_{i,j,k,l} a_{ij}a_{kl}\beta_{i+l}\beta_{j+k},$$

which is equivalent to $M(n) \geq 0$, that is, the most basic necessary condition for the existence of a measure is the positive semidefiniteness (or positivity) of

$M(n)$. Next define an assignment from $\mathcal{P}_n$, the set of all polynomials of degree $\leq n$, to $\mathcal{C}_{M(n)}$; given a polynomial $p(x,y) \equiv \sum_{ij} a_{ij} x^i y^j$, we take $p(X,Y) := \sum_{ij} a_{ij} X^i Y^j$, which is the so-called functional calculus. We also let $\mathcal{Z}(p)$ denote the zero set of p and define the *algebraic variety* of β by

$$\mathcal{V} \equiv \mathcal{V}(\beta) \equiv \mathcal{V}(M(n)) := \bigcap\nolimits_{p(X,Y)=\mathbf{0},\, \deg\, p \leq n} \mathcal{Z}(p). \tag{1}$$

If $\widehat{p}$ denotes the column vector of coefficients of p, then we know $p(X,Y) = M(n)\widehat{p}$, that is, $p(X,Y) = \mathbf{0}$ if and only if $\widehat{p} \in \ker M(n)$. It is also known that the existence of a measure requires the properties, supp $\mu \subseteq \mathcal{V}(\beta)$ and $r :=$ rank $M(n) \leq$ card supp $\mu \leq v :=$ card $\mathcal{V}$, which is the *variety condition* [1].

We need another assignment to discuss additional necessary conditions. The *Riesz functional* is a map from $\mathcal{P}$, the set of all polynomials, to $\mathbb{R}$ defined by $\Lambda\left(\sum_{ij} a_{ij} x^i y^j\right) = \sum_{ij} a_{ij} \beta_{ij}$. If p is any polynomial of degree at most $2n$ such that $p|_{\mathcal{V}} \equiv 0$, then the Riesz functional Λ must satisfy $\Lambda(p) = \int p\, d\mu = 0$, which is referred to as *consistency* of the moment sequence. If $r = v$, then $M(n)$ is said to be *extremal*; the consistency is a key solution to the extremal problems [5]. Also note that consistency cannot be replaced by the weaker condition that $M(n)$ is *recursively generated*, that is,

$$\begin{aligned} &\text{if } p(X,Y) = \mathbf{0}, \text{ then } (p\,q)(X,Y) = \mathbf{0} \\ &\text{for each polynomial } q \text{ with } \deg(p\,q) \leq n. \end{aligned} \tag{RG}$$

For solutions of the quadratic and quartic moment problems, positive semi-definiteness, (RG), and the variety condition were sufficient (see [1], [3], [12]). Beyond this order, the situation gets more complicated; many instances show a solution must include numerical conditions involving moments (see [6], [7], and [9]).

2.2. Flat extension

The Flat Extension Theorem [1] says that if $M(n)$ admits a rank-preserving positive extension $M(n+1)$, then β has a rank $M(n)$-atomic measure. The extension $M(n+1)$ is called a *flat extension.* This result is probably the most general solution to TMP to date even though the construction of an extension is not feasible for many cases when $n \geq 3$. We briefly summarize how to build a flat extension. Observe that each rectangular block with the same order moments of $M(n)$ is Hankel, and that an extension $M(n+1)$ can be written as $M(n+1) = \left(\begin{smallmatrix} M(n) & B \\ B^* & C \end{smallmatrix}\right)$ for some matrices B and C. To make sure a prospective moment matrix $M(n+1)$ is positive, we use the following classical result:

Theorem 2.1 (Smul'jan's Theorem [17]). *Let A, B, C be matrices of complex numbers, with A and C square matrices. Then*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0 \\ B = AW \text{ (for some } W) \\ C \geq W^* AW. \end{cases}$$

Moreover, rank $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) = \text{rank } A \iff C = W^* AW$.

Remark 2.2. If the equality involving the rank occurs in Theorem 2.1, we write $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} A & AW \\ W^*A & W^*AW \end{smallmatrix}\right)$, which is a flat extension of A. Construction of a flat extension seems to be easy in principle but satisfying another requirement, Hankelicity of C-block, is an extremely nontrivial process. In other words, it is quite difficult to maintain positivity and the moment matrix structure of $M(n+1)$ at the same time.

We now discuss a method for finding the explicit formula of a representing measure. An extended version of the Flat Extension Theorem says if $M(n)$ admits a positive extension $M(n+k)$ for some $k \in \mathbb{Z}_+$ that has a flat extension $M(n+k+1)$, then β has an rank $M(n+k)$-atomic measure μ. Let $\tau :=$ rank $M(n+k)$. According to this flat extension theorem, the algebraic variety $\mathcal{V}(M(n+k))$ consists of exactly τ points, and hence we may write $\mathcal{V}(M(n+k)) = \{(x_1, y_1), \ldots, (x_\tau, y_\tau)\}$. Now construct the Vandermonde matrix V as

$$V = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 & \cdots & x_1^n & \cdots & y_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_\tau & y_\tau & x_\tau^2 & x_\tau y_\tau & y_\tau^2 & \cdots & x_\tau^n & \cdots & y_\tau^n \end{pmatrix}. \tag{2}$$

Suppose $\mathcal{B} := \{\mathbf{t}_1, \ldots, \mathbf{t}_\tau\}$ is the basis for the column space of $M(n+k)$. If $V_\mathcal{B}$ is the submatrix of V with columns selected from $\mathcal{B}$, then the densities can be evaluated by solving the matrix equation:

$$V_\mathcal{B}^T \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_\tau \end{pmatrix}^T = \begin{pmatrix} \Lambda(\mathbf{t}_1) & \Lambda(\mathbf{t}_2) & \cdots & \Lambda(\mathbf{t}_\tau) \end{pmatrix}^T. \tag{3}$$

Thus, we have $\mu = \sum_{k=1}^{\tau} \rho_k \delta_{(x_k, y_k)}$, where δ denotes the point mass.

2.3. Auxiliary results

We collect old results to prove the main theorems. The rank-one decomposition method was adopted to TMP for the first time in [7] and the following theorem was essential to establish the method; it shows that the relationship between the eigenvalues of the matrix and its perturbation by a rank-one matrix. Let $\lambda_k(A)$ denote the k-th greatest eigenvalue of a matrix A.

Theorem 2.3 ([13]). *Let A be an $n \times n$ Hermitian matrix and let $z \in \mathbb{C}^n$ be a given vector. If the eigenvalues of A and $A \pm zz^*$ are arranged in increasing order as above, we have for $k = 1, 2, \ldots, n-2$,*

(i) $\lambda_k(A \pm zz^*) \leq \lambda_{k+1}(A) \leq \lambda_{k+2}(A \pm zz^*)$,
(ii) $\lambda_k(A) \leq \lambda_{k+1}(A \pm zz^*) \leq \lambda_{k+2}(A)$.

There is a more general version of the preceding result.

Theorem 2.4 ([13]). *Let $A, B \in M_n$ be Hermitian and suppose that B has rank at most r. Then*

(i) $\lambda_k(A+B) \le \lambda_{k+r}(A) \le \lambda_{k+2r}(A+B)$ *for* $k = 1, 2, \ldots, n-2r$;
(ii) $\lambda_k(A) \le \lambda_{k+r}(A+B) \le \lambda_{k+2r}(A)$ *for* $k = 1, 2, \ldots, n-2r$;
(iii) *If* $A = U\Lambda U^*$ *with* $U = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n\end{pmatrix} \in M_n$ *unitary and* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, *and if*

$$B = \lambda_n \mathbf{u}_n \mathbf{u}_n^* + \lambda_{n-1}\mathbf{u}_{n-1}\mathbf{u}_{n-1}^* + \cdots + \lambda_{n-r+1}\mathbf{u}_{n-r+1}\mathbf{u}_{n-r+1}^*,$$

then $\lambda_{\max}(A-B) = \lambda_{n-r}(A)$.

The proposition below is very useful to predict column relations in $M(n)$ for a given algebraic variety and will be applied frequently.

Proposition 2.5 ([1, Proposition 3.1]). *Suppose μ is a representing measure for β. For $p \in \mathcal{P}_n$,*

$$\operatorname{supp}\ \mu \subseteq \mathcal{Z}(p) \iff p(X,Y) = \mathbf{0}.$$

The following theorem says that once we have a linear column relation, positivity and (RG) solve the problem.

Theorem 2.6 ([2, cf. Theorem 2.1]). *Assume that $M(n) \ge 0$ satisfies* (RG) *and that $M(n)$ has a linear column relation. Then $M(n)$ admits a flat extension $M(n+1)$, so $M(n)$ has a* rank $M(n)$*-atomic representing measure.*

To drive results about symmetric variety in Section 4, we need to understand a special class of TMP. Assume the two columns X^n and Y^n in $M(n)$ are linearly dependent. Then X^{n+1} and Y^{n+1} are necessarily to be fixed to satisfy the property (RG), and hence all the new moments of the extension $M(n+1)$ are determined by the structure of the moment matrix. This rigidity enables to solve this type of problems. We begin with the formal definition:

Definition 2.7. $M(n)$ is said to be *recursively determined*, if $M(n)$ has only column dependence relations of the form

(4) $\qquad X^n = p(X,Y) \qquad (p \in \mathcal{P}_{n-1});$

(5) $\qquad Y^m = q(X,Y) \qquad (q \in \mathcal{P}_m,\ q \text{ has no } y^m \text{ term},\ m \le n).$

Theorem 2.8 ([4, Theorem 2.3.]). *If $M(n)$ is positive, with column relations generated entirely by* (4) *and* (5) *via recursiveness and linearity, then $M(n)$ admits a unique* (RG) *extension $M(n+1)$.*

Corollary 2.9 ([4, Corollary 2.4.]). *If $M(n)$ satisfies the hypotheses of Theorem 2.8 and $n = m-2$, then $M(n)$ admits a flat moment matrix extension $M(n+1)$ and β admits a rank $M(n)$-atomic representing measure.*

L. Fialkow recently provided a solution to $M(n)$ whose algebraic variety consists of two parallel lines. This result gives an inspiration to consider main results in this note.

Table 1. Classification of sextic moment problems in terms of r and v

rank $M(3)$	card $\mathcal{V}$	What to Check	Solution Presented in
7	7	numerical conditions	Section 6.3 [19]
7	8	consistency	Theorem 3.2 in [7]
7	9	consistency	Theorem 3.2 in [7]
7	∞	consistency	Theorem 3.2 in [7]
8	8	numerical conditions	Section 6.4 [19]
8	9	consistency	Algorithm 4.2 in [7]
8	∞	consistency	Algorithm 5.5 in [7]
9	∞		[9, 10] (particular cases)
10	∞		unknown

Theorem 2.10 ([10, Theorem 1.2.]). *Let $n \geq 2$. Suppose $\deg p(x,y) = 2$ and $\mathcal{Z}(p)$ consists of 2 parallel lines. Then $\beta \equiv \beta^{(2n)}$ has a representing measure supported in $\mathcal{Z}(p)$ if and only if $M(n)$ is positive semidefinite, recursively generated, satisfies the variety condition, and $p(X,Y) = \mathbf{0}$ in the column space of $M(n)$.*

Reading [10] thoroughly, we readily know this case admits a rank $M(n)$-atomic representing measure.

3. Sextic moment problems on 3 parallel lines

A complete solution to quartic moment problems is found; we may focus on the cases where the submatrix $M(2)$ in $M(3)$ is nonsingular, that is, $M(2)$ needs to be positive definite. We collect all the possible sextic moment problems in Table 1 and list information about each case. As Table 1 shows, $M(3)$ with an infinite variety remains unsolved. Building an extension of the cases requires too many new moments (parameters), even a computer algebra could not execute the calculation. Some instances further show that solutions to problems of order ≥ 6 involve numerical conditions with given moments whose origin is vague. This might be the reason why sometimes the best solutions are just algorithms.

Recall that truncated moment problems are equivalent under a degree-one transformation [3], which allows us to consider a new moment matrix $M(3)$ with a simpler column relation. Let us begin with noting that if a set of points is on 3 parallel lines, then we can use a shift and a dilation to make 2 of 3 lines as $y = 0$ and $y = 1$. Consequently, if $\widehat{M(3)}$ has a column relation of 3 parallel lines, we could transform the given column relation to one that corresponds to $y(y-1)(y-k) = 0$ for some $k \in \mathbb{R}$. If $k = 0$ or 1, then $\mathcal{V}(M(3))$ is contained in $y = 0$ and $y = 1$, which fact forces for $M(3)$ to have a conic column relation $Y^2 = Y$; we can exclude these possibilities. Thus, it is always possible to find a matrix of degree-one transformation J such that $M(3) = J^T \widehat{M(3)} J$, where $M(3)$ has a column relation $Y^3 = -kY + (1+k)Y^2$ for some $k \in \mathbb{R}$ ($k \neq 0, 1$). From now on, we focus on $M(3)$ with the column relation $Y^3 = -kY + (1+k)Y^2$.

When $M(3)$ has another cubic column relation, we may solve the problem using the results in Table 1. Hence, our focus is on $M(3)$ with a single dependence relation $Y^3 = -kY + (1+k)Y^2$. While this specific cubic column relation is investigated, it is found that this column relation has a strong connection to the sextic moment problems with a "symmetric" variety. We will improve previous results in Table 1 for $M(3)$ whose algebraic variety is symmetric in Section 4.

Before the main results are introduced, we need the following argument: Assume that $M \equiv M(3)$ has a representing measure μ. Then we may write $\mu = \mu_0 + \mu_1 + \mu_k$, where supp $\mu_0 \subseteq \{(x,y) : y = 0\}$, supp $\mu_1 \subseteq \{(x,y) : y = 1\}$, and supp $\mu_k \subseteq \{(x,y) : y = k\}$. We can also write $M = M[\mu_0] + M[\mu_1] + M[\mu_k]$, where each summand is the moment matrix generated by the associated measure. Since $M - M[\mu_0] = M[\mu_1] + M[\mu_k]$, Proposition 2.5 says $M - M[\mu_0]$ must have the column relation $Y^2 = -k\mathit{1} + (1+k)Y$ due to the fact that the support of a representing measure for $M - M[\mu_0]$ is contained the pair of lines $y^2 = (1+k)y - k$. The moment matrix $M - M[\mu_0]$ needs to be recursively generated so that the columns XY^2 and Y^3 must be linearly dependent in its column space, which tells us that rank $(M - M[\mu_0]) \leq 7$.

We next get a crucial observation that plays a key role for our results: Since $\int x^i y^j d\mu_0 = 0$ for $0 \leq i \leq 6$ and $1 \leq j \leq 6$, the moment matrix $M[\mu_0]$ must be written in the form of:

$$M^{\natural} := \begin{pmatrix} a_1 & a_2 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & a_4 & 0 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & a_5 & 0 & 0 & a_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_6 & 0 & 0 & a_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

for some $a_i \in \mathbb{R}$, $1 \leq i \leq 7$. For the existence of a measure, we may determine $a_1, \ldots, a_5$ concretely; indeed, the column relation $Y^2 = (1+k)Y - k\mathit{1}$ must appear in

$$\widetilde{M} := M - M^{\natural}, \tag{7}$$

and hence we have the linear system of equations:

$$
\begin{cases}
k(-a_1+\beta_{00}-\beta_{01})-\beta_{01}+\beta_{02}=0\\
k(-a_2+\beta_{10}-\beta_{11})-\beta_{11}+\beta_{12}=0\\
k(-a_3+\beta_{20}-\beta_{21})-\beta_{21}+\beta_{22}=0\\
k(-a_4+\beta_{30}-\beta_{31})-\beta_{31}+\beta_{32}=0\\
k(-a_5+\beta_{40}-\beta_{41})-\beta_{41}+\beta_{42}=0
\end{cases} \tag{8}
$$

Because $k \neq 0$, the system has an obvious solution. The moment matrices $M^{\natural}$ and $\widetilde{M}$ are consequently dependent on only two variables a_6 and a_7. Note that all the entries in $M^{\natural}(2)$ and $\widetilde{M}(2)$ are completely determined. The nested determinants of the two matrices will be an important discriminants for the main results; let $A_{\{i_1,\ldots,i_k\}}$ denote the compression of an $m \times m$ matrix A to the columns and the rows indexed by $\{i_1,\ldots,i_k\}$ and let us define:

$$\Delta_1^{\natural} := a_1,\ \Delta_2^{\natural} := \det M^{\natural}_{\{1,2\}},\ \Delta_3^{\natural} := \det M^{\natural}_{\{1,2,4\}},\ \text{and } \Delta_4^{\natural} := \det M^{\natural}_{\{1,2,4,7\}};$$

$$\widetilde{\Delta}_1 := \det \widetilde{M}_{\{1,2,3,4,5,8\}},\ \widetilde{\Delta}_2 := \det \widetilde{M}_{\{1,2,3,4,5,7\}},\ \text{and } \widetilde{\Delta}_3 := \det \widetilde{M}_{\{1,2,3,4,5,7,8\}}.$$

Note also from the multilinearity of the determinant that $\Delta_4^{\natural}$, $\widetilde{\Delta}_2$, and $\widetilde{\Delta}_3$ are quadratic polynomials of a_6 and a_7 and that the only second-degree term with nonzero coefficients in these three determinants is a_6^2; thus, they are fairly easy to handle. Obviously, if $\Delta_1^{\natural} < 0$, $\Delta_2^{\natural} < 0$, $\Delta_3^{\natural} < 0$, or $\widetilde{\Delta}_1 < 0$, then we conclude that M does not admit a representing measure. From now on, we may assume $M^{\natural}(2) \geq 0$ and $\widetilde{M}(2) \geq 0$.

Through this section, we assume $M \equiv M(3)$ is positive semidefinite with the only column relation $Y^3 = -kY + (1+k)Y^2$. We also define $M^{\natural}$ and $\widetilde{M}$ as in (6) and (7).

Remark 3.1. It is known that the maximal cardinality of the support of a representing for $\beta^{(2n)}$ is the same as $\dim \mathcal{P}_{2n}$. For $n = 3$, it would be 28. However, we have a much better bound for this case. Note that since $M^{\natural}$ has a linear column relation, it has a minimal representing measure which is at most 4-atomic. The fact that the two parallel lines contain the support of a measure for $\widetilde{M}$ implies a minimal measure would be 7-atomic. Thus a minimal representing measure for M may have 11 atoms at most.

We now observe that since rank $M^{\natural} \leq 4$ and rank $M \leq$ rank $M^{\natural}+$rank $\widetilde{M}$, we know rank $\widetilde{M} \geq 5$. Besides, $\widetilde{M}$ must have at least 3 column relations, which shows $5 \leq$ rank $\widetilde{M} \leq 7$.

Since $\widetilde{M}$ is a moment matrix with a column relation of two parallel lines, we can apply Theorem 2.10 to check the existence of a measure for $\widetilde{M}$. Because $M^{\natural}$ has a linear column relation, Theorem 2.6 is to be used; for both, we just need to check positivity, the variety condition, and (RG).

Before going further, we have to notice that $\Delta_2^{\natural}$ and $\widetilde{\Delta}_1$ are free of a_6 and a_7; then, according to the values of $\Delta_i^{\natural}$'s, we may have the following subcases:

First, note that if $\Delta_1^\natural = 0$, then $M^\natural$ needs to be the zero matrix but this is not feasible. Second, if $\Delta_1^\natural > 0$ and $\Delta_2^\natural = 0$, then all of X, X^2, and X^3 in $\mathcal{C}_{M^\natural}$ must be linearly dependent, equivalently, rank $M^\natural = 1$, so that a representing measure for $M^\natural$ is 1-atomic; however, if we say $(\alpha, 0)$ is the atom, then M must have another column relation associated to $(x-\alpha)(y-1)(y-k) = 0$, which cannot happen under the condition rank $M = 9$.

Thus, in the rest of the section, we may naturally assume $\Delta_1^\natural > 0$ and $\Delta_2^\natural > 0$. Now, in terms of the values of $\Delta_3^\natural$ and $\widetilde{\Delta}_1$, we may have the following 4 cases:

Case I. ${\Delta^\natural}_3 = 0$ and $\widetilde{\Delta}_1 = 0$;
Case II. ${\Delta^\natural}_3 = 0$ and $\widetilde{\Delta}_1 > 0$;
Case III. ${\Delta^\natural}_3 > 0$ and $\widetilde{\Delta}_1 = 0$;
Case IV. ${\Delta^\natural}_3 > 0$ and $\widetilde{\Delta}_1 > 0$.

However, Case I cannot happen; for, if ${\Delta^\natural}_3 = 0$, then since $X^2 \in \mathcal{C}_{M^\natural}$ is linearly dependent and $M^\natural$ must be (RG), we know rank $M^\natural = 2$. On the other hand, if $\widetilde{\Delta}_1 = 0$, then by the Extension Principle in [1], $X^2Y \in \mathcal{C}_{M^\natural}$ is linearly dependent, so rank $\widetilde{M} \le 6$. Nonetheless the inequality, $9 = \text{rank } M \le \text{rank } M^\natural + \text{rank } \widetilde{M} \le 2 + 6$, shows this is not a feasible case.

We are now ready to have our main results. The next covers Case II:

Theorem 3.2. *Suppose $M(3)$ is positive semidefinite with the only cubic column relation of 3 parallel lines, $Y^3 = -kY + (1+k)Y^2$. Suppose ${\Delta^\natural}_3 = 0$ and $\widetilde{\Delta}_1 > 0$. If $\{\mathit{1}, X\}$ in $\mathcal{C}_{M^\natural}$ forms a basis, then a_6 and a_7 are fixed. With such a_6 and a_7, if $\widetilde{M} \ge 0$, then $M(3)$ admits a 9-atomic representing measure.*

Proof. If $\Delta_3^\natural = 0$ (recall $\Delta_3^\natural$ is free of a_6 and a_7), then X^2, and X^3 in $\mathcal{C}_{M^\natural}$ must be both linearly dependent in order to implement the (RG)-condition in $M^\natural$. By the Extension Principle, the row reduction of $M^\natural_{\{1,2,4\}}$ helps to identify the column relation $X^2 = \alpha_1 \mathit{1} + \alpha_2 X$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$; then through the required column relation $X^3 = \alpha_1 X + \alpha_2 X^2$ in $M^\natural$. An easy calculation shows that there are always such a_6 and a_7. We next check whether $\widetilde{M} \ge 0$ or not. If so, $\widetilde{M}$ has a 7-atomic measure because rank $\widetilde{M} = 7$ (for, $9 = \text{rank } M \le \text{rank } M^\natural + \text{rank } \widetilde{M} \le 2 + \text{rank } \widetilde{M}$ and $\widetilde{M}$ has 3 column relations, that is, rank $\widetilde{M} \le 7$). Since $M^\natural$ has a 2-atomic measure, we conclude that M admits a minimal 9-atomic measure. $\square$

Example 3.3. Consider

$$M \equiv M(3) := \begin{pmatrix} 10 & 0 & 12 & 94 & 0 & 20 & 0 & 174 & 0 & 36 \\ 0 & 94 & 0 & 0 & 174 & 0 & 1798 & 0 & 338 & 0 \\ 12 & 0 & 20 & 174 & 0 & 36 & 0 & 338 & 0 & 68 \\ 94 & 0 & 174 & 1798 & 0 & 338 & 0 & 3558 & 0 & 666 \\ 0 & 174 & 0 & 0 & 338 & 0 & 3558 & 0 & 666 & 0 \\ 20 & 0 & 36 & 338 & 0 & 68 & 0 & 666 & 0 & 132 \\ 0 & 1798 & 0 & 0 & 3558 & 0 & 39574 & 0 & 7082 & 0 \\ 174 & 0 & 338 & 3558 & 0 & 666 & 0 & 7082 & 0 & 1322 \\ 0 & 338 & 0 & 0 & 666 & 0 & 7082 & 0 & 1322 & 0 \\ 36 & 0 & 68 & 666 & 0 & 132 & 0 & 1322 & 0 & 260 \end{pmatrix}$$

with the unique column relation $Y^3 = -2Y + 3Y^2$ $(k = 2)$. Solve (8) and we have

$$a_1 = 2,\ a_2 = 0,\ a_3 = 2,\ a_4 = 0,\ a_5 = 2$$

and we compute $\Delta^\natural{}_3 = 0$ and $\widetilde{\Delta}_1 = 9564480$. We then write $M = M^\natural + \widetilde{M}$ as in (7); for $M^\natural \geq 0$, $M^\natural$ must have the column relation $X^2 = \mathit{1}$ which was obtained from row reduction of $M^\natural{}_{\{1,2,4\}}$. This relation sets $a_6 = 0$ and $a_7 = 2$; thus now $M^\natural \geq 0$ and rank $M^\natural = 2$. With the fixed a_6 and a_7, it is straightforward to see $\widetilde{M} \geq 0$ and rank $\widetilde{M} = 7$. Therefore, M admits 9-atomic representing measure.

The next theorem is for Case III; before we run into the result, we should exclude the case where $\Delta^\natural{}_3 > 0$, $\Delta^\natural{}_4 = 0$, $\widetilde{\Delta}_1 = 0$, and $\widetilde{\Delta}_2 = 0$. For, if so, $\widetilde{M}$ is flat with rank $\widetilde{M} = 5$ and $\widetilde{M}$ admits a unique 5-atomic measure. Since $M^\natural$ has 3-atomic measure, M has a minimal 8-atomic measure; but, it is absurd since the cardinality of a measure sets a bound for the rank of the moment matrix.

Notice that the condition $\widetilde{\Delta}_1 = 0$ makes the last column in $\widetilde{M}_{\{1,2,3,4,5,8\}}$ to be linearly dependent, and so we may identify the column relation as $X^2Y = \alpha_1 \mathit{1} + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 XY$. The Extension Principle tells us that

$$X^2Y = \alpha_1 \mathit{1} + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 XY \tag{9}$$

(denote the corresponding polynomial of this relation as $p(x, y)$) must hold in $\mathcal{C}_{\widetilde{M}}$, which enables to fix a_6. Assume a_6 is determined in such a manner in Theorem 3.4, so $M^\natural$ and $\widetilde{M}$ only depend on a_7.

Theorem 3.4. *Suppose $M(3)$ is positive semidefinite with only a cubic column relation of 3 parallel lines, $Y^3 = -kY + (1 + k)Y^2$. Assume that $\Delta^\natural{}_3 > 0$ and $\widetilde{\Delta}_1 = 0$.*

(i) *If there are a_6 and a_7 such that $\widetilde{\Delta}_2 = 0$ and $\Delta^\natural{}_4 > 0$, then $M(3)$ admits a 9-atomic representing measure.*

(ii) *If there are a_6 and a_7 such that $\widetilde{\Delta}_2 > 0$, $\Delta^\natural{}_4 = 0$, and $|\mathcal{Z}(p) \cap \mathcal{Z}((y - 1)(y - k))| \geq 6$ (p as in (9)), then $M(3)$ admits a 9-atomic representing measure.*

(iii) *If there are a_6 and a_7 such that $\widetilde{\Delta}_2 > 0$, $\Delta^\natural{}_4 > 0$, and $|\mathcal{Z}(p) \cap \mathcal{Z}((y - 1)(y - k))| \geq 6$ (p as in (9)), then $M(3)$ admits a 10-atomic representing measure.*

Proof. (i) If $\widetilde{\Delta}_2 = 0$, then a_7 is fixed and $\widetilde{M}$ is flat with rank $\widetilde{M} = 5$. Thus, $\widetilde{M}$ has a unique 5-atomic measure. With such a_7, if $\Delta^\natural{}_4 > 0$, then $M^\natural \geq 0$ and $M^\natural$ has a 4-atomic representing measure. Thus, M admits a 9-atomic measure.

(ii) Solving $\Delta^\natural_4 = 0$, a_7 is to be fixed and we see that $M^\natural$ with a 3-atomic measure. With such a_7, if $\widetilde{M} \geq 0$ and $\widetilde{M}$ satisfies the variety condition, equivalently, $|\mathcal{Z}(p) \cap \mathcal{Z}((y - 1)(y - k))| \geq 6$, it follows from Theorem 2.10 that $\widetilde{M}$ has a 6-atomic measure; so, M admits a 9-atomic measure.

(iii) Solve the system of inequalities $\Delta_4^\natural > 0$ and $\widetilde{\Delta}_2 > 0$. If there is a solution, we know $M^\natural \geq 0$ and $\widetilde{M} \geq 0$ simultaneously. $M^\natural$ obviously has a 4-atomic measure. Among such a_7, if $\widetilde{M}$ satisfies the variety condition, again equivalently, $|\mathcal{Z}(p) \cap \mathcal{Z}((y-1)(y-k))| \geq 6$, we conclude that $\widetilde{M}$ has a 6-atomic measure; so, M admits a 10-atomic measure. $\square$

Example 3.5. Consider

$$M \equiv M(3) := \begin{pmatrix} 9 & 0 & 8 & 44 & 0 & 12 & 0 & 74 & 0 & 20 \\ 0 & 44 & 0 & 0 & 74 & 0 & 548 & 0 & 138 & 0 \\ 8 & 0 & 12 & 74 & 0 & 20 & 0 & 138 & 0 & 36 \\ 44 & 0 & 74 & 548 & 0 & 138 & 0 & 1058 & 0 & 266 \\ 0 & 74 & 0 & 0 & 138 & 0 & 1058 & 0 & 266 & 0 \\ 12 & 0 & 20 & 138 & 0 & 36 & 0 & 266 & 0 & 68 \\ 0 & 548 & 0 & 0 & 1058 & 0 & 8324 & 0 & 2082 & 0 \\ 74 & 0 & 138 & 1058 & 0 & 266 & 0 & 2082 & 0 & 522 \\ 0 & 138 & 0 & 0 & 266 & 0 & 2082 & 0 & 522 & 0 \\ 20 & 0 & 36 & 266 & 0 & 68 & 0 & 522 & 0 & 132 \end{pmatrix}$$

with the single cubic column relation $Y^3 = -2Y + 3Y^2$ ($k = 2$). Solve (8) and we have

$$a_1 = 3,\ a_2 = 0,\ a_3 = 2,\ a_4 = 0,\ a_5 = 2$$

and we compute

$$\begin{array}{ll} \Delta^\natural{}_3 = 4, & \Delta^\natural{}_4 = -2(4 + 3a_6^2 - 2a_7), \\ \widetilde{\Delta}_1 \ = 0, & \widetilde{\Delta}_2 \ = -512(-738 + 5a_6^2 + 45a_7). \end{array}$$

We next write $M = M^\natural + \widetilde{M}$ as in (7); for $\widetilde{M} \geq 0$, since $\widetilde{M}$ should have the column relation $X^3 = -16\mathit{1} + 16Y + X^2$ (obtained from row reduction of $\widetilde{M}_{\{1,2,3,4,5,8\}}$), we take $a_6 = 0$; thus, now $\Delta^\natural{}_4 = -2(4 - 2a_7)$ and $\widetilde{\Delta}_2 = -512(-738 + 45a_7)$. It is straightforward to see $\mathcal{V}(\widetilde{M}) = \{(\pm 4, 2)\} \cup \{(x, 1) : x \in \mathbb{R}\}$ which is an infinite set, and hence the variety conditions is confirmed. For both $M^\natural \geq 0$ and $\widetilde{M} \geq 0$, it is necessary $\Delta^\natural{}_4 \geq 0$ and $\widetilde{\Delta}_2 \geq 0$, that is, $2 \leq a_7 \leq 82/5$. If $a_7 = 2$ ($\widetilde{\Delta}_2 > 0$ and $\Delta^\natural{}_4 = 0$), then rank $M^\natural = 3$ and rank $\widetilde{M} = 6$; thus M has a 9-atomic representing measure; if $2 < a_7 < 82/5$ ($\widetilde{\Delta}_2 > 0$ and $\Delta^\natural{}_4 > 0$), then rank $M^\natural = 4$ and rank $\widetilde{M} = 6$; thus M has a 10-atomic representing measure. Finally, if $a_7 = 82/5$ ($\widetilde{\Delta}_2 = 0$ and $\Delta^\natural{}_4 > 0$), then rank $M^\natural = 4$, rank $\widetilde{M} = 5$, and $\widetilde{M}$ satisfies the variety condition; thus M has a 9-atomic representing measure.

It remains to cover Case IV:

Theorem 3.6. *Suppose $M(3)$ is positive semidefinite with only a cubic column relation of 3 parallel lines, $Y^3 = -kY + (1+k)Y^2$. Assume that $\Delta^\natural{}_3 > 0$ and $\widetilde{\Delta}_1 > 0$.*

(i) *If there are a_6 and a_7 such that $\Delta^\natural{}_4 = 0$ (resp. $\Delta^\natural{}_4 > 0$), $\widetilde{\Delta}_2 = 0$, and $\widetilde{M}$ satisfies the variety condition, then $M(3)$ admits a 9-(resp. 10-)atomic representing measure.*

(ii) *If there are a_6 and a_7 such that $\Delta^{\natural}{}_4 = 0$ (resp. $\Delta^{\natural}{}_4 > 0$) and $\widetilde{\Delta}_2 > 0$, then $M(3)$ admits a 10-(resp. 11-)atomic representing measure.*

Proof. (i) Observe that if $\widetilde{\Delta}_2 = 0$ for some a_6 and a_7, then X^3 in $\mathcal{C}_{\widetilde{M}}$ must be linearly dependent to get $\widetilde{M} \geq 0$ because of the Extension Principle. Row reduction of $\widetilde{M}_{\{1,2,3,4,5,7\}}$ reveals $X^3 = \alpha_1 \mathit{1} + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 XY$ as the column relation in the last column for some $\alpha_1, \ldots, \alpha_5 \in \mathbb{R}$; then through the required column relation $X^3 = \alpha_1 \mathit{1} + \alpha_2 X + \alpha_3 Y + \alpha_4 X^2 + \alpha_5 XY$ must appear in $\mathcal{C}_{\widetilde{M}}$. Let us denote the corresponding polynomial of the relation as $q(x, y)$. It is easy to see there are always such a_6 and a_7. We know that $\widetilde{M}$ is naturally positive semidefinite; then we proceed to check if $\widetilde{M}$ satisfies the variety condition, equivalently, check if $|\mathcal{Z}(q) \cap \mathcal{Z}((y-1)(y-k))| \geq 6$ or not. When the inequality holds, we end with that $\widetilde{M}$ has 6-atomic measure by Theorem 2.10. Moreover, if the fixed a_6 and a_7 make $\Delta^{\natural}{}_4 = 0$ (resp. $\Delta^{\natural}{}_4 > 0$), then rank $M^{\natural} = 3$ (resp. rank $M^{\natural} = 4$) and $M^{\natural} \geq 0$; thus, $M^{\natural}$ has a 3-(resp. 4-)atomic representing measure.

(ii) If the system of inequalities $\widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_2 > 0$ has some solution of a_6 and a_7, then $\widetilde{M} \geq 0$ and rank $\widetilde{M} = 7$; thus, $\widetilde{M}$ has a 7-atomic representing measure. With some a_6 and a_7 satisfying $\widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_2 > 0$, if $\Delta^{\natural}{}_4 = 0$ (resp. $\Delta^{\natural}{}_4 > 0$), then $M^{\natural} \geq 0$ and rank $M^{\natural} = 3$ (resp. rank $M^{\natural} = 4$); thus, $M^{\natural}$ has a 3-(resp. 4-)atomic representing measure. Combining the two observations, we proved the desired result. □

The example below shows how the previous results work:

Example 3.7. Consider

$$M \equiv M(3) := \begin{pmatrix} \beta_{00} & 0 & 14 & 94 & 0 & 24 & 0 & 174 & 0 & 44 \\ 0 & 94 & 0 & 0 & 174 & 0 & 1798 & 0 & 338 & 0 \\ 14 & 0 & 24 & 174 & 0 & 44 & 0 & 338 & 0 & 84 \\ 94 & 0 & 174 & 1798 & 0 & 338 & 0 & 3558 & 0 & 666 \\ 0 & 174 & 0 & 0 & 338 & 0 & 3558 & 0 & 666 & 0 \\ 24 & 0 & 44 & 338 & 0 & 84 & 0 & 666 & 0 & 164 \\ 0 & 1798 & 0 & 0 & 3558 & 0 & 39574 & 0 & 7082 & 0 \\ 174 & 0 & 338 & 3558 & 0 & 666 & 0 & 7082 & 0 & 1322 \\ 0 & 338 & 0 & 0 & 666 & 0 & 7082 & 0 & 1322 & 0 \\ 44 & 0 & 84 & 666 & 0 & 164 & 0 & 1322 & 0 & 324 \end{pmatrix}.$$

If $\beta_{00} = 10$ or 12, then $M(3) \geq 0$ with a single cubic column relation $Y^3 = -2Y + 3Y^2$ ($k = 2$). With $\beta_{00} = 10$, we solve the linear system (8) and obtain

$$a_1 = 1,\ a_2 = 0,\ a_3 = 2,\ a_4 = 0,\ a_5 = 2.$$

Since $\Delta^{\natural}{}_3 = -4$, the moment matrix $M^{\natural}$ cannot have a possibility to be positive, and hence $M(3)$ cannot have a representing measure. On the other hand, solving (8) with $\beta_{00} = 12$, we get

$$a_1 = 3,\ a_2 = 0,\ a_3 = 2,\ a_4 = 0,\ a_5 = 2$$

and we compute

$$\begin{aligned} \Delta^{\natural}{}_3 &= 4, & \Delta^{\natural}{}_4 &= -2(4 + 3a_6^2 - 2a_7), \\ \widetilde{\Delta}_1 &= 61578720, & \widetilde{\Delta}_2 &= -16(-697608422 + 1025a_6^2 + 436855a_7). \end{aligned}$$

We then write $M = M^{\natural} + \widetilde{M}(\tilde{\beta})$ as in (7). To see if $\widetilde{M}$ satisfies the conditions in Theorem 3.6 (i), solve $\widetilde{\Delta}_2 = 0$ and set $a_7 := (697608422 - 1025a_6^2)/436855$. Next row reduction of $\widetilde{M}_{\{1,2,3,4,5,7\}}$ brings the column relation $X^3 = (-23370a_6 \mathit{1} - 12832882X + 28495a_6 Y - 2050a_6 X^2 + 15803496XY)/873710$, which should appear in $\mathcal{C}_{\widetilde{M}}$. This fact leads to $a_6 = 0$; thus, $M^{\natural} \geq 0$, $\widetilde{M} \geq 0$, rank $M^{\natural} = 4$, rank $\widetilde{M} = 6$, and $\widetilde{M}$ satisfies the variety condition because $\mathcal{V}(\widetilde{M}) = \left\{ \left(\pm\sqrt{17/5}, 1\right), \left(\pm\sqrt{881/41}, 2\right), (0,1), (0,2) \right\}$. Therefore, we know that M has 10-atomic representing measure.

On the other hand, it is possible to find a_6 and a_7 such that $\Delta^{\natural}{}_4 = 0$ and $\widetilde{\Delta}_2 > 0$. Indeed, we set $a_7 = (4 + 3a_6^2)/2$ to have $\Delta^{\natural}{}_4 = 0$. For $\widetilde{M} \geq 0$, we need

$$-\frac{18}{5}\sqrt{\frac{9472526}{149855}} \leq a_6 \leq \frac{18}{5}\sqrt{\frac{9472526}{149855}}.$$

For concreteness, if we take $a_6 = 0$, then we can calculate $\mu_0 = \delta_{(-1,0)} + \delta_{(0,0)} + \delta_{(1,0)}$; for $\widetilde{M} \equiv \widetilde{M}(\tilde{\beta})$, we build a flat extension by keeping recursiveness and, in particular, with new moments $\tilde{\beta}_{70} = 0$ and $\tilde{\beta}_{61} = 395142/5$. A measure for $\widetilde{M}$ is $\sum_{k=1}^{7} \rho_i \delta_{(x_k,y_k)}$, where

$$\begin{aligned}
\rho_1 &= 18/17, & (x_1,y_1) &= (0,1), \\
\rho_2 &= 25/17, & (x_2,y_2) &= \left(-\sqrt{17/5}, 1\right), \\
\rho_3 &= 25/17, & (x_3,y_3) &= \left(\sqrt{17/5}, 1\right), \\
\rho_4 &= 5/4 + \sqrt{7627246245/33927876176}, & (x_4,y_4) &= \left(-\sqrt{(131995 + \sqrt{10602461305})/10430}, 2\right), \\
\rho_5 &= 5/4 + \sqrt{7627246245/33927876176}, & (x_5,y_5) &= \left(\sqrt{(131995 + \sqrt{10602461305})/10430}, 2\right), \\
\rho_6 &= 5/4 - \sqrt{7627246245/33927876176}, & (x_6,y_6) &= \left(-\sqrt{(131995 - \sqrt{10602461305})/10430}, 2\right), \\
\rho_7 &= 5/4 - \sqrt{7627246245/33927876176}, & (x_7,y_7) &= \left(\sqrt{(131995 - \sqrt{10602461305})/10430}, 2\right).
\end{aligned}$$

We just now obtained a minimal 10-atomic representing measure for $M(3)$.

Finally, the system of inequalities $\Delta^{\natural}{}_3 > 0$, $\Delta^{\natural}{}_4 > 0$, $\widetilde{\Delta}_1 > 0$, and $\widetilde{\Delta}_2 > 0$ has the following solution:

$$\begin{aligned}
&2 < a_7 \leq \tfrac{110148914}{69085}, -\tfrac{\sqrt{2a_7-4}}{\sqrt{3}} < a_6 < \tfrac{\sqrt{2a_7-4}}{\sqrt{3}} \qquad \text{or} \\
&\tfrac{110148914}{69085} < a_7 < \tfrac{327362}{205}, -\tfrac{\sqrt{697608422-436855a_7}}{5\sqrt{41}} < a_6 < \tfrac{\sqrt{697608422-436855a_7}}{5\sqrt{41}}.
\end{aligned}$$

If we select a_6 and a_7 from this solution, then both $M^{\natural}$ and $\widetilde{M}$ are positive, rank $M^{\natural} = 4$, and rank $\widetilde{M} = 7$. Thus, M may have an 11-atomic measure as well.

4. A symmetric variety

We now would like to discuss sextic moment problems with a symmetric variety that is on 3 parallel lines. If an algebraic variety is symmetric with respect to a line, then there are bundles of lines passing through the symmetric pairs of points. By Proposition 2.5, the existence of a representing measure for a singular $M(3)$ imposes a column relation whose associated polynomial contains all the points in the algebraic variety, and hence we may assume the symmetric points lie on a cubic planar curve; such a concrete cubic is a combination of 3 parallel lines. The most general cases with a symmetric variety whose generators are of degree ≤ 3 is a product of a line and a conic; this topic might be more difficult and may be handled in the future.

Recall again that truncated moment problems are equivalent under a degree-one transformation, which allows us to take $M(3)$ with a simpler column relation. Let us begin with noting that if a set of points in the plane is symmetric about a line, then using a linear transformation we may rearrange the points being symmetric to the y-axis. A proper rotation enables us to have the axis of symmetry of $\mathcal{V}$ as $x = 0$. Thus we may think that all points in $\mathcal{V}$ are contained in 3 horizontal lines. We then use a shift and a dilation to make 2 of lines as $y = 0$ and $y = 1$. We finally made all the points in $\mathcal{V}$ lie on $y(y-1)(y-k) = 0$ for some $k \in \mathbb{R}$. As discussed earlier, k needs to be different from 0 or 1. From now on, we may focus on a positive semidefinite $M(3)$ with the column relation $Y^3 = -kY + (1+k)Y^2$ $(k \neq 0, 1)$. We also assume $M(3)$ has a symmetric algebraic variety $\mathcal{V}$; we classify $M(3)$ according to its rank and present the solutions.

4.1. The case of rank 6 or less.

This category is covered by a recent result proved by L. Fialkow; it says that if a positive semidefinite $M(3)(\beta)$ has rank 6 or less, then β admits a sequence of *approximate* representing measure (that is, even though $M(3)$ may not admit positive measure, it can be represented by a measure which is a limit of a sequence of a positive representing measures), and the subsequence $\beta^{(5)}$ has a representing measure [11]. If there is a linear column relation in $M(3)$, then the required property, (RG), forces rank $M(3)$ to be at most 4 and this group falls into the above category.

The next argument is about how we have the case of rank $M(3) \geq 7$: Suppose $M(3)$ has no linear column relation but a unique conic relation; say, X^2 is linearly dependent in $M(3)$. Then X^3 and X^2Y must be dependent as well. Together with the column relation $Y^3 = -kY + (1+k)Y^2$, we know rank $M(3) = 6$. A similar argument can be obtained when XY is linearly dependent. Both cases are under the preceding class. The last possibility is with the basis of $\mathcal{C}_{M(3)}$ being $\left\{\mathit{1}, X, Y, X^2, XY, X^3, X^2Y\right\}$. This is the moment problem on two parallel lines and can be easily handled by Theorem 2.10.

Finally, if there is neither linear nor conic column relation, then the submatrix $M(2)$ is positive definite and rank $M(3) \geq 6$. If $M(3)$ is flat, that is, rank $M(3) = 6$, then it obviously has a unique 6-atomic measure; otherwise, the rank of $M(3)$ is at least 7. In the sequel we assume the submatrix $M(2)$ in $M(3)$ is positive definite and rank $M(3) \geq 7$.

4.2. The case of rank 7.

If rank $M(3) = 7$, we can find two more cubic column relations besides $Y^3 = -kY + (1+k)Y^2$. Since the associated polynomial of $Y^3 = -kY + (1+k)Y^2$ is simply a product of a 3 lines, it is easy to calculate the algebraic variety $\mathcal{V}$ and then the subcases are distinguished by $v = \text{card } \mathcal{V}$. $M(3)$ needs to satisfy the variety condition, so we must have $v \geq 7$.

Suppose $M(3)$ has the column relation $Y^3 = -kY + (1+k)Y^2$ for some $k \in \mathbb{R}$ and has rank 7. Then the possible bases of $\mathcal{C}_{M(3)}$ are:

$$\begin{aligned} \mathcal{B}_1 &:= \left\{ \mathit{1}, X, Y, X^2, XY, Y^2, X^3 \right\}, \\ \mathcal{B}_2 &:= \left\{ \mathit{1}, X, Y, X^2, XY, Y^2, X^2Y \right\}, \\ \mathcal{B}_3 &:= \left\{ \mathit{1}, X, Y, X^2, XY, Y^2, XY^2 \right\}. \end{aligned}$$

In order to have a symmetric algebraic variety $\mathcal{V}$, additional column relations should be like one of the following (because $\mathcal{V}$ is symmetric with respect to $x = 0$),

$$\begin{aligned} X^3 &= A_1X + A_2XY, \\ X^2Y &= B_1\mathit{1} + B_2Y + B_3X^2 + B_4Y^2, \\ XY^2 &= C_1X + C_2XY + C_3X^3, \end{aligned}$$

for some $A_i, B_i, C_i \in \mathbb{R}$.

We can compute the intersection of each case and observe first that: If X^2Y is linearly dependent, then the intersection of $x^2y = B_1 + B_2y + B_3x^2 + B_4y^2$ and $y^3 = -ky + (1+k)y^2$ contains at most 5 points, which fact blocks to satisfy the variety condition. We thus ignore $M(3)$ with the bases $\mathcal{B}_1$ or $\mathcal{B}_3$. Assume now that the basis for $\mathcal{C}_{M(3)}$ is $\mathcal{B}_2$ and let

$$\begin{aligned} p_1(x,y) &:= y^3 - (1+k)y^2 - ky = y(y-1)(y-k), \\ p_2(x,y) &:= x^3 - A_1x - A_2xy, \\ p_3(x,y) &:= xy^2 - C_1x - C_2xy, \end{aligned} \tag{10}$$

which will appear through this section. We point out that since X^3 is linearly dependent, we may drop C_3 from the column relation $XY^2 = C_1X + C_2XY + C_3X^3$ and define p_3 without the term of x^3. The column relations in XY^2 and Y^3 make $\mathcal{Z}(p_1) \cap \mathcal{Z}(p_3) = \{(0,0), (0,1), (0,k)\}$ unless the polynomial p_3 has two factors among y, $y-1$ or $y-k$; the intersection needs to be the origin and two lines. Without loss of generality, assume first that p_1 and p_3 have a single common factor, say $y-1$. Then

$$p_3(x,y) = x(y-1)(y-\alpha) \quad \text{for some } \alpha \neq 0, k.$$

We see $\mathcal{Z}(p_1) \cap \mathcal{Z}(p_3) = \{(0,0)\} \cup \{(x,1) : x \in \mathbb{R}\}$. By Proposition 2.5, $XY = X$ must be a column relation for the existence of a measure. Since we assumed $M(2) > 0$, this case is beyond our interest. The next possibility is that p_1 and

p_3 have two common factors. We may choose p_3 among

(11) $p_3(x,y) = xy(y-1),\ p_3(x,y) = xy(y-k),\ \text{or}\ p_3(x,y) = x(y-1)(y-k).$

The argument about all 3 p_3's are similar and we focus on the first choice. Taking p_2 in evaluating the variety, we confirm that $\mathcal{V}$ may contain at most 7 points, explicitly,

$$\mathcal{V} = \left\{(0,0),(0,1),(0,k),(\pm\sqrt{A_1},0),(\pm\sqrt{A_1+A_2},1)\right\}. \tag{12}$$

To satisfy the variety condition, $\mathcal{V}$ must consist of exactly 7 distinct points; the problem indeed becomes extremal. We also should make sure

$$A_1 > 0 \qquad \text{and} \qquad A_1 + A_2 > 0. \tag{13}$$

It is possible to impose on moments of $M(3)$ the column relations $p_1(X,Y) = \mathbf{0}$, $p_2(X,Y) = \mathbf{0}$, and $p_3(X,Y) = \mathbf{0}$ without any conflict. Using arguments in the above, we have the result:

Theorem 4.1. *Suppose $M(3)$ is positive semidefinite with an invertible $M(2)$,* rank $M(3) = 7$ *with a column relation $Y^3 = -kY + (1+k)Y^2$, and $\mathcal{V}(M(3))$ is symmetric about the y-axis. Then $v = 7$ if and only if $M(3)$ admits a unique 7-atomic representing measure.*

Proof. ($\Longrightarrow$) If $v = 7$, then there are A_1 and A_2 satisfying inequalities in (13); this problem turns out to be extremal. Recall that the solution of an extremal $M(n)$ is positivity and consistency [5, Theorem 1.3]; we are about to check consistency of $M(3)$. We investigate the structure of the ideal-like set $J_6 := \{p \in \mathcal{P}_6 : p|_{\mathcal{V}} \equiv 0\}$. Consider the ideal $J := \{p \in \mathcal{P} : p|_{\mathcal{V}} \equiv 0\}$ and it is fortunate that we can easily calculate the Gröebner basis of J that is just $\{p_1, p_2, p_3\}$, where p_1, p_2, and p_3 as in (10). Using the division algorithm for a multivariable polynomial, a polynomial $p \in J_6 \subset J$ can be written as

$$p = fp_1(x,y) + gp_2(x,y) + hp_3(x,y) \qquad \text{for some } f,g,h \in \mathcal{P}_3.$$

Observe that if $M(n)$ has a column relation $q(X,Y) = \mathbf{0}$, then $\Lambda(x^i y^j q(x,y)) = 0$ for $0 \le i+j \le n$; this fact and the linearity of the Riesz functional would confirm:

$$\begin{aligned} \Lambda(p) = 0 \quad &\Longleftrightarrow \quad \Lambda(x^i y^j p_k) = 0 \qquad \text{for } 0 \le i+j \le n,\ 0 \le k \le 3; \\ &\Longleftrightarrow \quad p_k(X,Y) = \mathbf{0} \qquad \text{for } 0 \le k \le 3. \end{aligned}$$

That is, if the 3 column relations present in $M(3)$, then consistency of $M(3)$ is immediately established.

($\Longleftarrow$) Since $M(n)$ has a unique 7-atomic measure, the variety condition implies $v \ge 7$. On the other hand, earlier analysis before (12) shows $v \le 7$ if rank $M(3) = 7$; consequently, we know $v = 7$. $\square$

This result is very relevant to the main result of [6] in which the authors pay more attention to a specific cubic column relation associated to a symmetric variety with 7 points.

4.3. The case of rank 8.

Suppose $M(3)$ has the column relation $Y^3 = (1+k)Y^2 - kY$ for some $k \in \mathbb{R}$ and has rank 8. As mentioned earlier to satisfy the variety condition, the other column relation must be one of the two, $p_2(X,Y) = \mathbf{0}$ or $p_3(X,Y) = \mathbf{0}$. We can compute the variety of each case and observe first that if X^3 and Y^3 are dependent, then we know

$$\mathcal{V} = \left\{(0,0),(0,1),(0,k),(\pm\sqrt{A_1},0),(\pm\sqrt{A_1+A_2},1),(\pm\sqrt{A_1+A_2k},k)\right\}. \tag{14}$$

To have at least 8 distinct points, it is necessary to assume $A_1 > 0$, $A_1+A_2 > 0$, and $A_1 + A_2k > 0$. Note that $\mathcal{V}$ must contain exactly 9 distinct points rather than 8 points. This case of $M(n)$ is indeed recursively determined; in the view of Theorem 2.8 and Corollary 2.9, we know $M(3)$ admits a unique (RG) extension $M(4)$ and all we need to do is checking positivity of $M(4)$. The discussion so far proves:

Theorem 4.2. *Suppose $M(3)$ is positive semidefinite with an invertible $M(2)$, $\mathcal{V}(M(3))$ is symmetric on $y(y-1)(y-k) = 0$ about the y-axis, and $\{1, X, Y, X^2, XY, Y^2, X^2Y, XY^2\}$ is the basis for $\mathcal{C}_{M(3)}$. Then the unique* (RG) *extension $M(4)$ is positive definite and $v = 9$ if and only if $M(3)$ has a minimal 8-atomic representing measure.*

The following example illustrates how the above result determines whether a moment sequence admits a representing measure or not:

Example 4.3. Consider

$$M(3) = \begin{pmatrix} 7 & 0 & 18 & 42 & 0 & 78 & 0 & 168 & 0 & 378 \\ 0 & 42 & 0 & 0 & 168 & 0 & 546 & 0 & 808 & 0 \\ 18 & 0 & 78 & 168 & 0 & 378 & 0 & 808 & 0 & 1878 \\ 42 & 0 & 168 & 546 & 0 & 808 & 0 & 2592 & 0 & 4008 \\ 0 & 168 & 0 & 0 & 808 & 0 & 2592 & 0 & 4008 & 0 \\ 78 & 0 & 378 & 808 & 0 & 1878 & 0 & 4008 & 0 & 9378 \\ 0 & 546 & 0 & 0 & 2592 & 0 & 8322 & 0 & 12832 & 0 \\ 168 & 0 & 808 & 2592 & 0 & 4008 & 0 & 12832 & 0 & 20008 \\ 0 & 808 & 0 & 0 & 4008 & 0 & 12832 & 0 & 20008 & 0 \\ 378 & 0 & 1878 & 4008 & 0 & 9378 & 0 & 20008 & 0 & 46878 \end{pmatrix}.$$

This moment matrix is positive and has two column relations $X^3 = X + 3XY$ and $Y^3 = -5Y + 6Y^2$ ($k = 2$). The algebraic variety has 9 distinct points, and so satisfies the variety condition. Propagating the two cubic column relations forward, we build the (RG) extension $M(4)$ with the quartic column relations defined as

$$\begin{aligned} X^4 &:= X^2 + 3X^2Y, & X^3Y &:= XY + 3XY^2, \\ XY^3 &:= -5XY + 6XY^2, & Y^4 &:= -5Y^2 + 6Y^3. \end{aligned}$$

All the 7th- and 8th-order new moments are determined. However, $M(4)$ is not positive, and hence $M(3)$ dose not have a representing measure. If $M(4)$ were positive, then $M(3)$ should have admitted an 8-atomic measure.

The remaining case is when XY^2 and Y^3 are linearly dependent, and we have:

Theorem 4.4. *Suppose $M(3)$ is positive semidefinite with an invertible $M(2)$, $\mathcal{V}(M(3))$ is symmetric on $y(y-1)(y-k)=0$ about the y-axis, and $\{\mathit{1}, X, Y, X^2, XY, Y^2, X^3, X^2Y\}$ is the basis for $\mathcal{C}_{M(3)}$. Then $M(3)$ admits a minimal 8-atomic representing measure.*

Proof. If XY^2 and Y^3 are linearly dependent, as in the case of rank 7, we know p_3 can be selected among cubics in (11). An important observation is that we can eliminate the possibilities of $p_3(x,y)=xy(y-1)$ and $p_3(x,y)=xy(y-k)$. For, the intersection of $y(y-1)(y-k)=0$ and one of such $p_3(x,y)=0$ is the set of all points on $y=0$ and either $y=1$ or $y=k$. Thus, the algebraic variety of $M(3)$ would be either $y(y-1)=0$ or $y(y-k)=0$, which must bring a conic column relation in $\mathcal{C}_{M(3)}$ for the existence of a measure in the view of Proposition 2.5. This is absurd since we assumed $M(2)>0$. We thus may focus on the option $p_3(x,y)=x(y-1)(y-k)$. The algebraic variety is $\mathcal{V}=\{(0,0)\}\cup\{(x,1):x\in\mathbb{R}\}\cup\{(x,k):x\in\mathbb{R}\}$ whose cardinality is infinite. This is a case of rank 8 with an infinite variety; we may adopt a main result in Section 5 of [7] but we may use a simpler method. We start with a crucial observation that $(0,0)$ must be in the support of a measure. For, if not, $Y^2=(1+k)Y-k\mathit{1}$ must be a column relation in $M(3)$, which is not possible again by Proposition 2.5. Consequently, $M(3)$ have a representing measure if and only if M is decomposed as

$$M(3)=\widetilde{M}+uM[\delta_{(0,0)}],$$

where $\widetilde{M}$ must have a conic column relation $Y^2=(1+k)Y-k\mathit{1}$ and $u>0$.

In order to show $\widetilde{M}\geq 0$, let $I_1\equiv I_1(m)$ be the $m\times m$ matrix with 1 in the $(1,1)$-entry and 0 in all other entries (the moment matrix $M[\delta_{(0,0)}]$ is exactly of this type with $m=10$). One easily sees that for any $\alpha\in\mathbb{R}$

$$\det(A-\alpha I_1)=\det A-\alpha\det A_{\{2,3,\dots,m\}}.$$

This determinant formula gives

$$\det\widetilde{M}_{\{1,2,3,4,5,6,7,8\}}=\det M_{\{1,2,3,4,5,6,7,8\}}-u\det M_{\{2,3,4,5,6,7,8\}}. \tag{15}$$

If we set $u:=\det M_{\{1,2,3,4,5,6,7,8\}}/\det M_{\{2,3,4,5,6,7,8\}}$, then $\det\widetilde{M}_{\{1,2,3,4,5,6,7,8\}}=0$, and $u>0$ due to $M\geq 0$. The first essential observation is $7\leq\operatorname{rank}\widetilde{M}\leq 9$. Since M and $\widetilde{M}$ are different only at $(1,1)$-entry and the first column in M is not involved in the two column relations, $\widetilde{M}$ has the same two column relations. Thus rank $\widetilde{M}$ is 7 or 8. But by taking u as above we can bring another column relation $Y^2=-k\mathit{1}+(1+k)Y$ on $\widetilde{M}$ so that now rank $\widetilde{M}=7$.

If the eigenvalues of M and $\widetilde{M}$ are arranged in ascending order with the notation $\lambda_k(\cdot)$ that stands for the k-th greatest eigenvalue of the given matrix, then we know from Theorem 2.3 that $0<\lambda_3(M)\leq\lambda_4(\widetilde{M})$. Since rank $\widetilde{M}=7$, it follows $\widetilde{M}$ has the eigenvalue zero with multiplicity 3. Consequently, $\lambda_k(\widetilde{M}) = 0$ for $k = 1, 2, 3$ and $\lambda_k(\widetilde{M}) > 0$ for $k = 4, \ldots, 10$, that is, $\widetilde{M}$ is positive semidefinite. Applying Theorem 2.10, we know $\widetilde{M}$ admits a representing measure; so does M. Since a minimal measure of $\widetilde{M}$ is 7-atomic, we conclude that M has a minimal 8-atomic representing measure. $\square$

Example 4.5. Consider

$$M(3) = \begin{pmatrix} 8 & 0 & 10 & 28 & 0 & 16 & 0 & 36 & 0 & 28 \\ 0 & 28 & 0 & 0 & 36 & 0 & 196 & 0 & 52 & 0 \\ 10 & 0 & 16 & 36 & 0 & 28 & 0 & 52 & 0 & 52 \\ 28 & 0 & 36 & 196 & 0 & 52 & 0 & 228 & 0 & 84 \\ 0 & 36 & 0 & 0 & 52 & 0 & 228 & 0 & 84 & 0 \\ 16 & 0 & 28 & 52 & 0 & 52 & 0 & 84 & 0 & 100 \\ 0 & 196 & 0 & 0 & 228 & 0 & 1588 & 0 & 292 & 0 \\ 36 & 0 & 52 & 228 & 0 & 84 & 0 & 292 & 0 & 148 \\ 0 & 52 & 0 & 0 & 84 & 0 & 292 & 0 & 148 & 0 \\ 28 & 0 & 52 & 84 & 0 & 100 & 0 & 148 & 0 & 196 \end{pmatrix}.$$

$M(3)$ has the two column relations

$$XY^2 = -2X + 3XY \qquad \text{and} \qquad Y^3 = -2Y + 3Y^2$$

and the algebraic variety consists of $(0, 0)$ and all the points on $(y-1)(y-2) = 0$. It easy to find $u = 1$ in (15) and we write

$$M(3) \equiv \widetilde{M} + 1 \cdot I_1 = \begin{pmatrix} 7 & 0 & 10 & 28 & 0 & 16 & 0 & 36 & 0 & 28 \\ 0 & 28 & 0 & 0 & 36 & 0 & 196 & 0 & 52 & 0 \\ 10 & 0 & 16 & 36 & 0 & 28 & 0 & 52 & 0 & 52 \\ 28 & 0 & 36 & 196 & 0 & 52 & 0 & 228 & 0 & 84 \\ 0 & 36 & 0 & 0 & 52 & 0 & 228 & 0 & 84 & 0 \\ 16 & 0 & 28 & 52 & 0 & 52 & 0 & 84 & 0 & 100 \\ 0 & 196 & 0 & 0 & 228 & 0 & 1588 & 0 & 292 & 0 \\ 36 & 0 & 52 & 228 & 0 & 84 & 0 & 292 & 0 & 148 \\ 0 & 52 & 0 & 0 & 84 & 0 & 292 & 0 & 148 & 0 \\ 28 & 0 & 52 & 84 & 0 & 100 & 0 & 148 & 0 & 196 \end{pmatrix} + I_1.$$

$\widetilde{M}$ has the additional conic relations $Y^2 = -2\mathit{1} + 3Y$ and satisfies all the required conditions in Theorem 2.10. Thus, $\widetilde{M}$ has a 7-atomic measure and we conclude that $M(3)$ has a minimal 8-atomic measure.

Finally, for the case of rank 9, we have a single column relation. We can solve this problem with the results in Section 3. We should mention that when the moment sequence has a representing measure, the support of the measure may or may not be symmetric.

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References

[1] R. Curto and L. Fialkow, *Solution of the truncated complex moment problem with flat data*, Memoirs Amer. Math. Soc. no. 568, Amer. Math. Soc., Providence, 1996.

[2] ———, *Flat extensions of positive moment matrices: relations in analytic or conjugate terms*, Nonselfadjoint operator algebras, operator theory, and related topics, 59–82, Oper. Theory Adv. Appl., 104, Birkhäuser, Basel, 1998.

[3] ———, *Solution of the singular quartic moment problem*, J. Operator Theory **48** (2002), no. 2, 315–354.

[4] ———, *Recursively determined representing measures for bivariate truncated moment sequences*, J. Operator Theory **70** (2013), no. 2, 401–436.

[5] R. Curto, L. Fialkow, and H. M. Möller, *The extremal truncated moment problem*, Integral Equations Operator Theory **60** (2008), no. 2, 177–200.

[6] R. Curto and S. Yoo, *Cubic column relations in truncated moment problems*, J. Funct. Anal. **266** (2014), no. 3, 1611–1626.

[7] ———, *Non-extremal sextic moment problems*, J. Funct. Anal. **269** (2015), no. 3, 758–780.

[8] ———, *Concrete solution to the nonsingular quartic binary moment problem*, Proc. Amer. Math. Soc. **144** (2016), no. 1, 249–258.

[9] L. Fialkow, *Solution of the truncated moment problem with variety $y = x^3$*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 3133–3165.

[10] ———, *The truncated moment problem on parallel lines*, The varied landscape of operator theory, 109–126, Theta Ser. Adv. Math., Theta, Bucharest, 2014.

[11] ———, *Limits of positive flat bivariate moment matrices*, Trans. Amer. Math. Soc. **367** (2015), no. 4, 2665–2702.

[12] L. Fialkow and J. Nie, *Positivity of Riesz functionals and solutions of quadratic and quartic moment problems*, J. Funct. Anal. **258** (2010), 328–356.

[13] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[14] D. P. Kimsey, *Matrix-Valued Moment Problems*, ProQuest LLC, Ann Arbor, Ph.D. Thesis, Drexel University, 2011.

[15] ———, *The cubic complex moment problem*, Integral Equations Operator Theory **80** (2014), no. 3, 353–378.

[16] D. P. Kimsey and H. J. Woerdeman, *The truncated matrix-valued K-moment problem on $\mathbb{R}^d$, $\mathbb{C}^d$, and $\mathbb{T}^d$*, Trans. Am. Math. Soc. **365** (2013), no. 10, 5393–5430.

[17] J. L. Smul'jan, *An operator Hellinger integral*, Mat. Sb. **91** (1959), no. 1, 381–430.

[18] J. Stochel, *Solving the truncated moment problem solves the full moment problem*, Glasg. Math. J. **43** (2001), no. 3, 335–341.

[19] S. Yoo, *Extremal sextic truncated moment problems*, Thesis (Ph.D.) The University of Iowa, 2010.

[20] Wolfram Research, Inc., *Mathematica*, Version 10.3, Champaign, IL, 2015.

Seonguk Yoo
Department of Mathematics
Sungkyunkwan University
Suwon 16419, Korea
E-mail address: seyoo@skku.edu