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COMINIMAXNESS WITH RESPECT TO IDEALS OF DIMENSION ONE

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Abstract. Let R denote a commutative Noetherian (not necessarily local) ring and let I be an ideal of R of dimension one. The main purpose of this note is to show that the category $\mathscr{M}(R,I)_{com}$ of I-cominimax R-modules forms an Abelian subcategory of the category of all R-modules. This assertion is a generalization of the main result of Melkersson in [15]. As an immediate consequence of this result we get some conditions for cominimaxness of local cohomology modules for ideals of dimension one. Finally, it is shown that the category $\mathcal{C}^1_B(R)$ of all R-modules of dimension at most one with finite Bass numbers forms an Abelian subcategory of the category of all R-modules.

1. Introduction

Let R denote a commutative Noetherian ring, and let I be an ideal of R. In [9], Hartshorne defined an R-module L to be *I-cofinite*, if $\mathrm{Supp}(L) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, L)$ is finitely generated module for all i. He posed the following question:

Is the category $\mathscr{M}(R,I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cofinite modules, are $\ker f$ *and* $\mathrm{coker} f$ *I-cofinite?*

Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. On the other hand, in [8], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [11] generalized the Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. More recently, Melkersson in [15] have removed the local assumption on R.

The main purpose of this note is to generalize this result to other categories of modules. In this direction we present similar results for two new categories of modules. More precisely, we shall show that:

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Theorem 1.1. *Let R be a Noetherian ring and I be an ideal of R with $\dim R/I = 1$. Let $\mathscr{M}(R,I)_{com}$ denote the category of I-cominimax modules. Then $\mathscr{M}(R,I)_{com}$ forms an Abelian subcategory of the category of all R-modules.*

Our methods of the proof for Theorem 1.1, is based on an adaptation of the technique used in [5]. One of our tools for proving Theorem 1.1 is the following, which is a generalization of a similar result in [5].

Proposition 1.2. *Let I denote an ideal of a Noetherian ring R and let M be an R-module such that $\dim M \leq 1$ and $\operatorname{Supp}(M) \subseteq V(I)$. Then M is I-cominimax if and only if the R-modules $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are minimax.*

Recall that, we say an R-module M is minimax if there is a finitely generated submodule N of M, such that M/N is Artinian. The interesting class of minimax modules was introduced by H. Zöshinger in [17] and he has in [17] and [18] given many equivalent conditions for a module to be minimax. Also, the R-module M is said to be an I-cominimax if support of M is contained in $V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \geq 0$. The concept of the I-cominimax modules were introduced in [2] as a generalization of important notion of I-cofinite modules.

Recall that, for each R-module M, all integers $j \geq 0$ and all prime ideals $\mathfrak{p}$ of R, the j-th Bass number of M with respect to $\mathfrak{p}$ is defined as $\mu^j(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^j_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$, where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Our other important result in this paper is the following:

Theorem 1.3. *Let R be a Noetherian ring and let $\mathcal{C}^1_B(R)$ denote the category of all R-modules of dimension at most one with finite Bass numbers. Then $\mathcal{C}^1_B(R)$ forms an Abelian subcategory of the category of all R-modules.*

Our main tools for proving Theorem 1.3 is the following lemma.

Lemma 1.4. *Let R be a Noetherian ring and M be an R-module with $\dim M = d < \infty$. Then the followings are equivalent:*

(i) *For each $\mathfrak{p} \in \operatorname{Spec}(R)$ and any $0 \leq i \leq d$, the Bass numbers $\mu^i(\mathfrak{p}, M)$ are finite.*

(ii) *For each $\mathfrak{p} \in \operatorname{Spec}(R)$ and any $i \geq 0$, the Bass numbers $\mu^i(\mathfrak{p}, M)$ are finite.*

Finally, we get the following conditions for cominimaxness of local cohomology modules for ideals of dimension one.

Theorem 1.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R with $\dim R/I = 1$. Then for any R-module M, the following conditions are equivalent:*

(a) $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all* $i < \operatorname{cd}(I, M) + 2$.

(b) $H^i_I(M)$ *is I-cominimax for all i.*

(c) $\mathrm{Ext}^i_R(R/I, M)$ *is minimax for all* i.

(d) $\mathrm{Ext}^i_R(N, M)$ *is minimax for all* $i < \mathrm{cd}(I, M) + 2$ *and for any finitely generated* R*-module* N *with* $\mathrm{Supp}\, N \subseteq V(I)$.

(e) $\mathrm{Ext}^i_R(N, M)$ *is minimax for all* $i < \mathrm{cd}(I, M) + 2$ *and for some finitely generated* R*-module* N *with* $\mathrm{Supp}\, N = V(I)$.

(f) $\mathrm{Ext}^i_R(N, M)$ *is minimax for all* i *and for any finitely generated* R*-module* N *with* $\mathrm{Supp}\, N \subseteq V(I)$.

(g) $\mathrm{Ext}^i_R(N, M)$ *is minimax for all* i *and for some finitely generated* R*-module* N *with* $\mathrm{Supp}\, N = V(I)$.

(h) $\mu^i(\mathfrak{p}, M)$ *is finite for all* $\mathfrak{p} \in V(I)$ *and for all* $i < \mathrm{cd}(I, M) + 2$.

(i) $\mu^i(\mathfrak{p}, M)$ *is finite for all* $\mathfrak{p} \in V(I)$ *and for all* i.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. For an Artinian R-module A we denote by $\mathrm{Att}_R\, A$ the set of attached prime ideals of A. For each R-module L, we denote by $\mathrm{Assh}_R L$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R\, L : \dim R/\mathfrak{p} = \dim L\}$. We shall use $\mathrm{Max}\, R$ to denote the set of all maximal ideals of R. Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [7] and [12].

2. Two Abelian categories of modules

The following lemma is crucial for the proof of the next theorem.

Lemma 2.1. *Let* R *be a Noetherian ring and* M *be an* R*-module with* $\dim M = d < \infty$. *Then the followings are equivalent*:

(i) *For each* $\mathfrak{p} \in \mathrm{Spec}(R)$ *and any* $0 \leq i \leq d$, *the Bass numbers* $\mu^i(\mathfrak{p}, M)$ *are finite.*

(ii) *For each* $\mathfrak{p} \in \mathrm{Spec}(R)$ *and any* $i \geq 0$, *the Bass numbers* $\mu^i(\mathfrak{p}, M)$ *are finite.*

Proof. (ii)⇒(i) Is clear.

(i)⇒(ii) Using localization we may assume $(R, \mathfrak{m}, k)$ is a local Noetherian ring and $\mathfrak{p} = \mathfrak{m}$ is the unique maximal ideal of R. As by [7, Corollary 10.2.8] the R-module $E_R(k)$, the injective hull of k, is Artinian it follows from the hypothesis and from the definition of local cohomology modules that for any $0 \leq i \leq d$ the R-module $H^i_{\mathfrak{m}}(M)$ is Artinian. But in view of [7, Theorem 6.1.2] we have $H^i_{\mathfrak{m}}(M) = 0$ for all integers $i > d$. Therefore for all integers $i \geq 0$ the R-modules $H^i_{\mathfrak{m}}(M)$ are Artinian and hence are $\mathfrak{m}$-cofinite. So the assertion follows from the [14, Proposition 3.9]. □

Now we are ready to state and prove the first main result of this section.

Theorem 2.2. *Let R be a Noetherian ring and let $\mathcal{C}_B^1(R)$ denote the category of all R-modules of dimension at most one with finite Bass numbers. Then $\mathcal{C}_B^1(R)$ forms an Abelian subcategory of the category of all R-modules.*

Proof. Let $M, N \in \mathcal{C}_B^1(R)$ and $f : M \to N$ be an R-homomorphism. It is enough to prove that the R-modules $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{C}_B^1(R)$.
Now, the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

implies that for each $\mathfrak{p} \in \operatorname{Spec}(R)$ and any $0 \leq i \leq 1$ the Bass numbers $\mu^i(\mathfrak{p}, \ker(f))$ are finite. So by Lemma 2.1 it follows that $\ker(f)$ is in $\mathcal{C}_B^1(R)$. Now the reminder part of the proof follows from the exact sequences

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

and

$$0 \longrightarrow \operatorname{im}(f) \longrightarrow N \longrightarrow \operatorname{coker}(f) \longrightarrow 0,$$

as required. □

The following lemma will be useful in the proof of Proposition 2.4.

Lemma 2.3. *Let R be a Noetherian ring and I be an ideal of R. Then for every integer $t \geq 0$ and any R-module T the following conditions are equivalent*:
(i) $\operatorname{Ext}_R^n(R/I, T)$ *is minimax for all* $0 \leq n \leq t$.
(ii) *For any finitely generated R-module N with support in $V(I)$,* $\operatorname{Ext}_R^n(N, T)$ *is minimax for all* $0 \leq n \leq t$.

Proof. The assertion follows from [1, Lemma 2.2] using [3, Lemma 2.1]. □

Now we are ready to state and prove the second main result of this section.

Proposition 2.4. *Let I be an ideal of Noetherian ring R and M be a non-zero R-module such that* $\dim \operatorname{Supp}(M) \leq 1$ *and* $\operatorname{Supp}(M) \subseteq V(I)$. *Then the following statements are equivalent*:
(i) *The R-module M is I-cominimax.*
(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}_R^1(R/I, M)$ *are minimax.*

Proof. (i)⇒(ii) Is clear.
(ii)⇒(i) If $\dim \operatorname{Supp}(M) = 0$, then the R-module $\operatorname{Hom}_R(R/I, M)$ is minimnax of dimension 0 and hence is Artinian. Therefore, it follows from [7, Theorem 7.1.2] that M is Artinian. So the assertion is clear in this case. So we may assume $\dim \operatorname{Supp}(M) = 1$. Now, we use induction on $t = \operatorname{ara}_M(I) := \operatorname{ara}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M))$. If $t = 0$, then it follows from the definition that $I^n \subseteq \operatorname{Ann}_R(M)$ and so $M = (0 :_M I^n)$ for some natural number n. Therefore, the assertion holds by Lemma 2.3 in this case. So assume that $t > 0$ and the result has been proved for $0, 1, \ldots, t-1$. Let

$$T := \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \dim R/\mathfrak{p} = 1\}.$$

Now, it is easy to see that $T \subseteq \operatorname{Assh}_R(M)$. But, since M is I-torsion, it is easy to see that $\operatorname{Ass}_R M = \operatorname{Ass}_R \operatorname{Hom}_R(R/I, M)$ is finite. Therefore T is a finite set. Moreover, using the definition of the minimax modules and [12, Exercice 7.7] it is easy to see that, for each $\mathfrak{p} \in T$, the $R_\mathfrak{p}$-module $\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p})$ is finitely generated and $M_\mathfrak{p}$ is an $IR_\mathfrak{p}$-torsion $R_\mathfrak{p}$-module, with $\operatorname{Supp}(M_\mathfrak{p}) \subseteq V(\mathfrak{p}R_\mathfrak{p})$. Thus, the $R_\mathfrak{p}$-module $\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/IR_\mathfrak{p}, M_\mathfrak{p})$ is Artinian. Consequently, according to Melkersson's results [13, Theorem 1.3] and [14, Proposition 4.3], $M_\mathfrak{p}$ is an Artinian and $IR_\mathfrak{p}$-cofinite $R_\mathfrak{p}$-module. Let

$$T := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

By [4, Lemma 2.5], we have

$$V(IR_{\mathfrak{p}_j}) \cap \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq V(\mathfrak{p}_j R_{\mathfrak{p}_j})$$

for all $j = 1, 2, \ldots, n$. Next, let

$$U := \bigcup_{j=1}^{n} \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})\}.$$

Then it is easy to see that $U \cap V(I) \subseteq T$.

On the other hand, by definition of $t = \operatorname{ara}_M(I) \geq 1$, there exist elements $y_1, \ldots, y_t \in I$, such that

$$\operatorname{Rad}(I + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((y_1, \ldots, y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Now, as

$$I \not\subseteq \bigcup_{\mathfrak{q} \in U \setminus V(I)} \mathfrak{q},$$

it follows that

$$(y_1, \ldots, y_t) + \operatorname{Ann}_R(M) \not\subseteq \bigcup_{\mathfrak{q} \in U \setminus V(I)} \mathfrak{q}.$$

But, since for each $\mathfrak{q} \in U$ we have

$$\mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})$$

for some integer $1 \leq j \leq n$, it follows that

$$\operatorname{Ann}_R(M)R_{\mathfrak{p}_j} \subseteq \operatorname{Ann}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq \mathfrak{q}R_{\mathfrak{p}_j},$$

which implies $\operatorname{Ann}_R(M) \subseteq \mathfrak{q}$. Therefore from the fact that

$$\operatorname{Ann}_R(M) \subseteq \bigcap_{\mathfrak{q} \in U \setminus V(I)} \mathfrak{q},$$

it follows that

$$(y_1, \ldots, y_t) \not\subseteq \bigcup_{\mathfrak{q} \in U \setminus V(I)} \mathfrak{q}.$$

Therefore, by [12, Exercise 16.8] there is $a \in (y_2, \ldots, y_t)$ such that

$$y_1 + a \notin \bigcup_{\mathfrak{q} \in U \setminus V(I)} \mathfrak{q}.$$

Let $x := y_1 + a$. Then $x \in I$ and

$$\mathrm{Rad}(I + \mathrm{Ann}_R(M)/\mathrm{Ann}_R(M)) = \mathrm{Rad}((x, y_2, \ldots, y_t) + \mathrm{Ann}_R(M)/\mathrm{Ann}_R(M)).$$

Next, let $N := (0 :_M x)$. Now, it is easy to see that

$$\mathrm{ara}_N(I) = \mathrm{ara}(I + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N)) \leq t - 1.$$

(Note that $x \in \mathrm{Ann}_R(N)$ and hence

$$\mathrm{Rad}(I + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N)) = \mathrm{Rad}((y_2, \ldots, y_t) + \mathrm{Ann}_R(N)/\mathrm{Ann}_R(N)).)$$

Now, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/I, N) \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, xM)$$
$$\longrightarrow \mathrm{Ext}^1_R(R/I, N) \longrightarrow \mathrm{Ext}^1_R(R/I, M),$$

which implies the R-modules $\mathrm{Hom}_R(R/I, N)$ and $\mathrm{Ext}^1_R(R/I, N)$ are minimax. Consequently, by the inductive hypothesis, the R-module N is I-cominimax. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0$$

induces an exact sequence

$$\mathrm{Ext}^1_R(R/I, M) \longrightarrow \mathrm{Ext}^1_R(R/I, xM) \longrightarrow \mathrm{Ext}^2_R(R/I, N),$$

which implies that the R-module $\mathrm{Ext}^1_R(R/I, xM)$ is minimax.

Now, from the exact sequence

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

we get an exact sequence

$$\mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, M/xM) \longrightarrow \mathrm{Ext}^1_R(R/I, xM)$$

which implies that the R-module $\mathrm{Hom}_R(R/I, M/xM)$ is minimax.

Now, from [4, Lemma 2.4], it is easy to see that $(M/xM)_{\mathfrak{p}_j}$ is of finite length for all $j = 1, \ldots, n$. Therefore there exists a finitely generated submodule L_j of M/xM such that $(M/xM)_{\mathfrak{p}_j} = (L_j)_{\mathfrak{p}_j}$. Let $L := L_1 + \cdots + L_n$. Then L is a finitely generated submodule of M/xM such that

$$\mathrm{Supp}_R(M/xM)/L \subseteq \mathrm{Supp}(M) \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Max}\, R.$$

Now, from the sequence

$$0 \longrightarrow L \longrightarrow M/xM \longrightarrow (M/xM)/L \longrightarrow 0,$$

we get the following exact sequence:

$$\mathrm{Hom}_R(R/I, M/xM) \longrightarrow \mathrm{Hom}_R(R/I, (M/xM)/L) \longrightarrow \mathrm{Ext}^1_R(R/I, L),$$

which implies that the R-module $\mathrm{Hom}_R(R/I, (M/xM)/L)$ is minimax. We show that M/xM is a minimax R-module. To do this, since $\mathrm{Supp}(M/xM)/L \subseteq$

Max R and $(M/xM)/L$ is I-torsion, so that, according to Melkersson [13, Theorem 1.3] $(M/xM)/L$ is an Artinian R-module. That is M/xM is a minimax R-module. So M/xM is I-cominimax. Now, since the R-modules $N = (0 :_M x)$ and M/xM are I-cominimax, it follows from [14, Corollary 3.4] and [3, Lemma 2.1] that M is I-cominimax. This completes the inductive step, as required. □

The following result is the third main result of this section.

Theorem 2.5. *Let I be an ideal of Noetherian ring R and let $\mathcal{C}^1_{com}(I)$ denote the set of all I-cominimax R-modules M, with* $\dim \operatorname{Supp}(M) \leq 1$. *Then $\mathcal{C}^1_{com}(I)$ forms an Abelian subcategory of the category of all R-modules.*

Proof. Let $M, N \in \mathcal{C}^1_{com}(I)$ and $f : M \to N$ be an R-homomorphism. It is enough to prove that the R-modules $\ker(f)$ and $\operatorname{coker}(f)$ are I-cominimax.

Now, the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, \ker(f)) \longrightarrow \operatorname{Hom}_R(R/I, M) \longrightarrow \operatorname{Hom}_R(R/I, \operatorname{im}(f))$$
$$\longrightarrow \operatorname{Ext}^1_R(R/I, \ker(f)) \longrightarrow \operatorname{Ext}^1_R(R/I, M),$$

that implies the R-modules $\operatorname{Hom}_R(R/I, \ker(f))$ and $\operatorname{Ext}^1_R(R/I, \ker(f))$ are minimax. Therefore it follows from Proposition 2.4 that $\ker(f)$ is I-cominimax. Now the reminder part of the proof follows from the exact sequences

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \operatorname{im}(f) \longrightarrow 0,$$

and

$$0 \longrightarrow \operatorname{im}(f) \longrightarrow N \longrightarrow \operatorname{coker}(f) \longrightarrow 0,$$

as required. □

The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Let $\mathscr{M}(R, I)_{com}$ denote the category of I-cominimax modules over R. Then $\mathscr{M}(R, I)_{com}$ forms an Abelian subcategory of the category of all R-modules.*

Proof. Since for each $M \in \mathscr{M}(R, I)_{com}$, by the definition we have $\operatorname{Supp}(M) \subseteq V(I)$ it follows that $\dim(M) \leq 1$. Therefore, the assertion follows from Theorem 2.5. □

The following results are some applications of Theorem 2.6.

Corollary 2.7. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Let $\mathscr{M}(R, I)_{com}$ denote the category of I-cominimax modules over R. Let*

$$X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots,$$

be a complex such that $X^i \in \mathscr{M}(R,I)_{com}$ for all $i \in \mathbb{Z}$. Then the i^{th} homology module $H^i(X^\bullet)$ is in $\mathscr{M}(R,I)_{com}$.

Proof. The assertion follows immediately from Theorem 2.6. □

Corollary 2.8. *Let I be an ideal of a commutative Noetherian ring R of dimension one. Let $M \in \mathscr{M}(R,I)_{com}$. Then for each finitely generated R-module N, the R-modules $\operatorname{Ext}^j_R(N,M)$ and $\operatorname{Tor}^R_j(N,M)$ are in $\mathscr{M}(R,I)_{com}$, for all integers $j \geq 0$.*

Proof. Since N is finitely generated, there is a free resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

for N, such that the free R-modules F_i are finitely generated, for all $i \geq 0$. Therefore the assertion easily follows from the definition of the R-modules $\operatorname{Ext}^j_R(N,A)$ and $\operatorname{Tor}^R_j(N,M)$, $j = 0,1,2,\dots$ and Corollary 2.7. □

3. Local cohomology modules and cominimaxness

In this section we apply the main results of previous section to local cohomology modules. But, first we need the following well known results.

Lemma 3.1. *Let R be a Noetherian ring and I be an ideal of R. Let s be a non-negative integer and M be an R-module. Then for any Serre subcategory of the category of all R-modules as $\mathscr{S}$ the following statements hold:*

(i) *If $\operatorname{Ext}^j_R(R/I, H^i_I(M)) \in \mathscr{S}$ for all $i \leq s$ and all $j \geq 0$ and $\operatorname{Ext}^s_R(R/I,M) \in \mathscr{S}$, then $\operatorname{Hom}_R(R/I, H^s_I(M)) \in \mathscr{S}$.*

(ii) *If $\operatorname{Ext}^j_R(R/I, H^i_I(M)) \in \mathscr{S}$ for all $i < s$ and all $j \geq 0$ and $\operatorname{Ext}^{s+1}_R(R/I,M) \in \mathscr{S}$, then $\operatorname{Ext}^1_R(R/I, H^s_I(M)) \in \mathscr{S}$.*

Proof. See [1, Lemma 2.3]. □

Lemma 3.2. *Let $(R,\mathfrak{m})$ be a local (Noetherian) ring and I be an ideal of R such that $\dim R/I = 1$. Suppose that M is an R-module and $n \geq 0$ an integer. Then the following conditions are equivalent:*

(i) *$\mu^i(\mathfrak{p},M)$ is finite for all $\mathfrak{p} \in V(I)$ and for all $i \leq n$.*

(ii) *$\operatorname{Ext}^i_R(R/I,M)$ is minimax for all $i \leq n$.*

Proof. See [6, Theorem 2.3]. □

Recall that, for an R-module N, the *cohomological dimension of N with respect to* an ideal $\mathfrak{a}$ of R, denoted by $\operatorname{cd}(\mathfrak{a},N)$, is defined as

$$\operatorname{cd}(\mathfrak{a},N) := \sup\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(N) \neq 0\}.$$

Now we are ready to state and prove the main result of this section.

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let* I *be an ideal of* R *with* $\dim R/I = 1$. *Then for any* R*-module* M, *the following conditions are equivalent*:

(a) $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all* $i < \operatorname{cd}(I, M) + 2$.

(b) $H^i_I(M)$ *is* I*-cominimax for all* i.

(c) $\operatorname{Ext}^i_R(R/I, M)$ *is minimax for all* i.

(d) $\operatorname{Ext}^i_R(N, M)$ *is minimax for all* $i < \operatorname{cd}(I, M) + 2$ *and for any finitely generated* R*-module* N *with* $\operatorname{Supp} N \subseteq V(I)$.

(e) $\operatorname{Ext}^i_R(N, M)$ *is minimax for all* $i < \operatorname{cd}(I, M) + 2$ *and for some finitely generated* R*-module* N *with* $\operatorname{Supp} N = V(I)$.

(f) $\operatorname{Ext}^i_R(N, M)$ *is minimax for all* i *and for any finitely generated* R*-module* N *with* $\operatorname{Supp} N \subseteq V(I)$.

(g) $\operatorname{Ext}^i_R(N, M)$ *is minimax for all* i *and for some finitely generated* R*-module* N *with* $\operatorname{Supp} N = V(I)$.

(h) $\mu^i(\mathfrak{p}, M)$ *is finite for all* $\mathfrak{p} \in V(I)$ *and for all* $i < \operatorname{cd}(I, M) + 2$.

(i) $\mu^i(\mathfrak{p}, M)$ *is finite for all* $\mathfrak{p} \in V(I)$ *and for all* i.

Proof. (a) $\Rightarrow$ (b) Since in view of [3, Lemma 2.2], the category of minimax modules forms a Serre subcategory of the category of all R-modules, the assertion follows using an induction argument on i and applying Lemma 3.1 and Proposition 2.4.

(b) $\Rightarrow$ (c) follows from [14, Proposition 3.9].

(c) $\Rightarrow$ (a) is clear.

(c) $\Rightarrow$ (f) follows from Lemma 2.3.

(f) $\Rightarrow$ (g) and (g) $\Rightarrow$ (e) are trivial.

(e) $\Rightarrow$ (a) We argue by induction on i. Since $\operatorname{Supp} N = V(I) = \operatorname{Supp} R/I$, it follows from Gruson's theorem [16, Theorem 4.1], that there is a chain

$$0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = R/I,$$

such that the factors N_j/N_{j-1} are homomorphic images of a direct sum of finitely many copies of N. Using this it is easy to see that the R-module $\operatorname{Hom}_R(R/I, M)$ is minimax. Now let $0 < i < \operatorname{cd}(I, M) + 2$ and suppose that the result has been proved all smaller values of i. Since by inductive hypothesis the R-modules $\operatorname{Ext}^j_R(R/I, M)$ are minimax for $j = 0, 1, \ldots, i-1$ it follows from Lemma 2.3 that for any finitely generated R-module K with $\operatorname{Supp} K \subseteq V(I)$ the R-modules $\operatorname{Ext}^j_R(K, M)$ are minimax for $j = 0, 1, \ldots, i-1$. Now using this it follows that $\operatorname{Ext}^i_R(N_j, M)$ is minimax for all $0 < j \leq k$ and consequently the R-module $\operatorname{Ext}^i_R(R/I, M)$ is minimax. This completes the inductive step.

(a) $\Rightarrow$ (d) follows from Lemma 2.3.

(d) $\Rightarrow$ (e) is clear.

(i) $\Leftrightarrow$ (c) and (h) $\Leftrightarrow$ (a) follow from Lemma 3.2. $\square$

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