INJECTIVE LINEAR MAPS ON $\mathcal{T}_\infty(F)$ THAT PRESERVE THE ADDITIVITY OF RANK

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ABSTRACT. We consider $\mathcal{T}_\infty(F)$ – the space of upper triangular infinite matrices over a field F. We investigate injective linear maps on this space which preserve the additivity of rank, i.e., the maps ϕ such that $\mathrm{rank}(x + y) = \mathrm{rank}(x) + \mathrm{rank}(y)$ implies $\mathrm{rank}(\phi(x + y)) = \mathrm{rank}(\phi(x)) + \mathrm{rank}(\phi(y))$ for all $x, y \in \mathcal{T}_\infty(F)$.

1. Introduction

In recent years, or even decades, many authors have dealt with the linear preserver problem, i.e., with characterizing the linear operators on some spaces which preserve some properties, functions or sets invariant. One of the most intensively studied are problems concerning preserving the rank of matrices and issues related to it.

Take into consideration the maps ϕ on some space $\mathcal{S}$ satisfying

(1.1)
$$\mathrm{rank}(x + y) = \mathrm{rank}(x) + \mathrm{rank}(y) \ \Rightarrow \ \mathrm{rank}(\phi(x + y)) = \mathrm{rank}(\phi(x)) + \mathrm{rank}(\phi(y))$$

for all $x, y \in \mathcal{S}$.

If (1.1) holds, then we say that ϕ preserves the additivity of rank, or we say that ϕ preserves pairs x, y satisfying extreme rank properties (as in [4] and [16]). It was shown in [9] that the above condition is equivalent to preserving the substractivity of rank. It is also worth mentioning that the rank additivity has some connections with the range additivity (see [1]).

The properties of the additivity of rank were studied in [10]. It was a natural question to arise how maps that preserve the additivity of rank can be described. Some authors applied partial orderings to solve it - this was done for instance in [8], some made use of properties of ranks of finite matrices - like in [2] (for more information about subspaces of matrices with the same rank see [3, 15]). This problem was also generalized to describing all additive maps satisfying (1.1) and solved in many cases in works [6, 7, 13, 14].

In this article we would like to consider the raised above problem in $\mathcal{T}_\infty(F)$ – the space of upper triangular infinite matrices over a field F. Many linear preserver problems for $\mathcal{T}_n(F)$ – the space of upper triangular matrices of dimension n over F, were solved in [5, 11]. However, we need to note that quite often the arguments given for finite dimensional spaces do not apply to infinite dimensional ones. To establish that difference, let us note that in the space of all square matrices over a field F all nonzero maps fulfilling (1.1) are given either by the formula $\phi(x) = axb$ or $\phi(x) = ax^Tb$, where a and b are nonsingular matrices. In our case the result is somewhat different. We will prove:

Theorem 1.1. *Let F be a field. If ϕ is an injective linear map on $\mathcal{T}_\infty(F)$ satisfying condition* (1.1), *then either*

(1) *$\mathcal{T}_\infty(F)$ contains some matrices x such that $\mathrm{rank}(x) < \infty$ and $\mathrm{rank}(\phi(x)) = \infty$ or*

(2) *there exists a number $k \in \mathbb{N}$ for which we have*

(1.2) $$\phi(x) = a_1xb_1 + \cdots + a_kxb_k,$$

where a_1, b_1, ..., a_k, b_k are infinite matrices such that
- *$a_ixb_i \in \mathcal{T}_\infty(F)$ for all $1 \le i \le k$ and $x \in \mathcal{T}_\infty(F)$,*
- *the columns of all a_i form a linearly independent set,*
- *the rows of all b_i form a linearly independent set.*

2. Proofs of results

Let e_{nm} stand for the infinite matrix with 1 in the position (n, m) and 0 in every other position, and let e_∞ be a matrix equal to $\sum_{n \in \mathbb{N}} e_{nn}$. By $M_{\infty \times 1}^{fin}(F)$ we denote the space of all vectors of the form $(x_1, x_2, \ldots)^T$ with $x_n \in F$ for all $n \in \mathbb{N}$, whose support is finite (x^T is the transposition of x). We write $M_{1 \times \infty}(F)$ for the space of all vectors of the form $(x_1, x_2, \ldots)$ with $x_n \in F$. By e_n we denote the vector from $M_{\infty \times 1}^{fin}(F)$ such that $(e_n)_n = 1$ and $(e_n)_i = 0$ for $i \ne n$. The symbol f_n is used for e_n^T. Clearly, $M_{\infty \times 1}^{fin}(F)$, $M_{1 \times \infty}(F)$ are linear spaces. Moreover, $\{e_n : n \in \mathbb{N}\}$ is a basis of $M_{\infty \times 1}^{fin}(F)$.

Moreover, by $\langle a_i : i \in I \rangle$ we understand the subspace generated by the set $\{a_i : i \in I\}$.

We also note that 0 may denote the element of a field, as well as the zero vector of any of aforementioned spaces; its exact meaning will follow from the context.

If we write that some infinite matrix x is an infinite sum of matrices x_i such that for all pairs n, m the condition $(x_i)_{nm} \ne 0$ only for a finite number of indices i, we simply mean that

$$x_{nm} = \sum_{i,\ (x_i)_{nm} \ne 0} (x_i)_{nm},$$

i.e., such Σ is rather a notation than a symbol of an algebraic operation.

We begin with two simple observations.

Remark 2.1. Every nonzero $x \in \mathcal{T}_\infty(F)$ can be written as a sum $\sum_{i \in I} u_i v_i$, where

- $I \subseteq \mathbb{N}$,
- $u_i \in M_{\infty \times 1}^{fin}(F)$, $v_i \in M_{1 \times \infty}(F)$ for all $i \in I$,
- $\{u_i : i \in I\}$ and $\{v_i : i \in I\}$ are linearly independent sets,
- $u_i v_i \in \mathcal{T}_\infty(F)$.

In particular, if $\mathrm{rank}(x)$ is finite, then the set I is finite as well.

Lemma 2.1. *Let ϕ be an injective linear map satisfying condition (1.1). If $x \in \mathcal{T}_\infty(F)$ has rank n $(n \in \mathbb{N})$, then $\mathrm{rank}(\phi(x)) \geq n$.*

Proof. Let x have rank one. Since ϕ is linear and injective we have $\phi(x) \neq 0$, so $\mathrm{rank}(\phi(x)) \geq 1$.

Suppose now that we have proved our claim for all matrices x with rank at most n $(n \geq 1)$. Consider a matrix x having rank equal to $n+1$. It can be written as a sum of matrices y and z, whose ranks are equal to n and 1 respectively. Thus $\mathrm{rank}(x) = \mathrm{rank}(y + z) = \mathrm{rank}(y) + \mathrm{rank}(z)$. As ϕ fulfills (1.1), the latter fact and the inductive assumption yield

$$\mathrm{rank}(\phi(x)) = \mathrm{rank}(\phi(y + z)) = \mathrm{rank}(\phi(y)) + \mathrm{rank}(\phi(z)) \geq n + 1.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Now we cite a result that follows from proofs given in [12].

Corollary 2.1. *Let $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ be a linear map. Assume that there exist $u(n) \in M_{\infty \times 1}^{fin}(F)$, $v(m) \in M_{1 \times \infty}(F)$ such that $\{u(n) : n \in \mathbb{N}\}$, $\{v(m) : m \in \mathbb{N}\}$ are linearly independent sets and $\phi(e_{nm}) = u(n)v(m)$. Then $\phi(\sum_{n \leq m} x_{nm} e_{nm}) = \sum_{n \leq m} x_{nm} \phi(e_{nm})$.*

After the above preparations we give a proof of our main result.

Proof of Theorem 1.1. Consider an arbitrary pair of natural numbers n, m, $n \leq m$. If $\mathrm{rank}(\phi(e_{nm})) = \infty$, then the first point of the claim holds. Suppose then the opposite – that $\mathrm{rank}(\phi(e_{nm})) < \infty$ for all $n \leq m$. In this case we can write all these matrices in the following form.

$$\phi(e_{nm}) = \sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m),$$

where $u_i(n,m) \in M_{\infty \times 1}^{fin}(F)$, $v_i(n,m) \in M_{1 \times \infty}(F)$ for all i, $1 \leq i \leq I(n,m)$, and by Lemma 2.1 $I(n,m) \in \mathbb{N}$. Moreover,

$$\{u_i(n,m) : i \in I(n,m)\}, \quad \{v_i(n,m) : i \in I(n,m)\}$$

are both linearly independent sets.

Notice that since $\phi(e_{nm}) \in \mathcal{T}_\infty(F)$, we may assume that for any i the product $u_i(n,m)v_i(n,m)$ is in $\mathcal{T}_\infty(F)$.

Let now $n, m, k, l \in \mathbb{N}$ be such that $n \leq m$, $k \leq l$, $n \neq k$, $m \neq l$. Clearly $\operatorname{rank}(e_{nm} + e_{kl}) = \operatorname{rank}(e_{nm}) + \operatorname{rank}(e_{kl})$, so by (1.1)

$$\operatorname{rank}(\sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m) + \sum_{i=1}^{I(k,l)} u_i(k,l)v_i(k,l))$$

$$= \operatorname{rank}(\sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m)) + \operatorname{rank}(\sum_{i=1}^{I(k,l)} u_i(k,l)v_i(k,l)).$$

Therefore the set

$$\{u_i(n,m) : 1 \leq i \leq I(n,m)\} \cup \{u_i(k,l) : 1 \leq i \leq I(k,l)\}$$

is linearly independent, as well as

$$\{v_i(n,m) : 1 \leq i \leq I(n,m)\} \cup \{v_i(k,l) : 1 \leq i \leq I(k,l)\}.$$

Assume now that $m \leq l$ and consider the triple e_{nm}, e_{nl}, e_{kl}. From

$$\operatorname{rank}(e_{nm} + e_{nl} + e_{kl}) = \operatorname{rank}(e_{nm} + e_{nl}) + \operatorname{rank}(e_{kl}),$$
$$\operatorname{rank}(e_{nm} + e_{nl} + e_{kl}) = \operatorname{rank}(e_{nm}) + \operatorname{rank}(e_{nl} + e_{kl}),$$

it follows that

$$\operatorname{rank}(\sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m) + \sum_{i=1}^{I(n,l)} u_i(n,l)v_i(n,l))$$

$$+ \operatorname{rank}(\sum_{i=1}^{I(k,l)} u_i(k,l)v_i(k,l))$$

$$= \operatorname{rank}(\sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m) + \sum_{i=1}^{I(n,l)} u_i(n,l)v_i(n,l)$$

$$+ \sum_{i=1}^{I(k,l)} u_i(k,l)v_i(k,l))$$

$$= \operatorname{rank}(\sum_{i=1}^{I(n,m)} u_i(n,m)v_i(n,m)) + \operatorname{rank}(\sum_{i=1}^{I(n,l)} u_i(n,l)v_i(n,l)$$

$$+ \sum_{i=1}^{I(k,l)} u_i(k,l)v_i(k,l)).$$

As all ranks in the above equations are finite we must either have

(2.1)
$$\langle u_i(n,l) : 1 \leq i \leq I(n,l) \rangle \subseteq \langle u_i(n,m) : 1 \leq i \leq I(n,m) \rangle$$
$$\text{and}$$
$$\langle v_i(n,l) : 1 \leq i \leq I(n,l) \rangle \subseteq \langle v_i(k,l) : 1 \leq i \leq I(k,l) \rangle$$

or

$$\langle u_i(n,l) : 1 \leq i \leq I(n,l) \rangle \subseteq \langle u_i(k,l) : 1 \leq i \leq I(k,l) \rangle$$
(2.2)
$$\text{and}$$
$$\langle v_i(n,l) : 1 \leq i \leq I(n,l) \rangle \subseteq \langle v_i(n,m) : 1 \leq i \leq I(n,m) \rangle.$$

Suppose that for some n, m, k, l it holds (2.1) and for some it holds (2.2). More precisely, assume that for some n, m, k, l it holds (2.1), and for some n', m', k', l' it holds (2.2). (With no loss of generality we assume that $n \leq n'$.) Then

$$\langle u_i(n,l) : 1 \leq i \leq I(n,l) \rangle \subseteq \langle u_i(k,l) : 1 \leq i \leq I(k,l) \rangle$$
$$\text{and}$$
$$\langle u_i(n',l') : 1 \leq i \leq I(n',l') \rangle \subseteq \langle u_i(k',l') : 1 \leq i \leq I(k',l') \rangle.$$

Hence

$$\langle u_i(n,l') : 1 \leq i \leq I(n,l') \rangle$$
$$\subseteq \langle v_i(n,m) : 1 \leq i \leq I(n,m) \rangle \cap \langle v_i(k',l') : 1 \leq i \leq I(k',l') \rangle.$$

However we have already learnt that the sets

$$\langle v_i(n,m) : 1 \leq i \leq I(n,m) \rangle, \quad \langle v_i(k',l') : 1 \leq i \leq I(k',l') \rangle$$

are linearly independent as well as their union. This yields that their intersection consists only of the zero matrix, i.e., $\phi(e_{nl'}) = 0$ – this contradicts the injectivity.

Assume now that (2.2) holds. Consider $\langle v_i(1,m) : 1 \leq i \leq I(1,m) \rangle$. Let s_i be the smallest number such that $(v_i(1,m))_{s_i} \neq 0$. Notice that there exist numbers $p \in \mathbb{N}$ and i, $1 \leq i \leq I(1,m)$, such that $p > s_i$ and $(u_i(1,m))_p \neq 0$. This forces $(u_i(1,m)v_i(1,m))_{ps_i} \neq 0$ and consequently $u_i(1,m)v_i(1,m) \notin \mathcal{T}_\infty(F)$ – a contradiction. Hence it is (2.1) that must hold.

Notice that from (2.1) we get that $I(n,m) \leq I(n,l)$ and $I(n,l) \leq I(k,l)$ for any $n \leq m < l$, $k \leq l$, i.e., the numbers $I(n,m)$ are increasing when n is increasing and m is fixed, and analogously, in the case when m is increasing and n fixed. Repeating the arguments given above for e_{nl}, e_{np}, e_{kl} and for e_{nl}, e_{kl}, e_{kp}, where $p > l$, we obtain that $I(n,m)$ are decreasing when n is increasing and m is fixed, and the same in the case when m is increasing and n fixed. Hence $I(n',m) = I(n,m) = I(n,m')$ for all n, m, n', m' such that $n, n' \leq m$, $n \leq m, m'$. Denote this number by I. We have

$$\langle u_i(n,m) : 1 \leq i \leq I \rangle = \langle u_i(n,l) : 1 \leq i \leq I \rangle,$$
(2.3)
$$\langle v_i(n,m) : 1 \leq i \leq I \rangle = \langle v_i(k,m) : 1 \leq i \leq I \rangle.$$

For fixed n denote by $u_1(n)$, $u_2(n)$, ..., $u_I(n)$ the vectors $u_1(n,n)$, $u_2(n,n)$, ..., $u_I(n,n)$. Since (2.3) is fulfilled, there exist vectors $v_i'(n,m) \in M_{1\times\infty}(F)$ such that

$$\phi(e_{nm}) = \sum_{i=1}^{I} u_i(n,m)v_i(n,m) = \sum_{i=1}^{I} u_i(n)v_i'(n,m).$$

Notice that since $\phi(e_{nn})$, $\phi(e_{nn} + e_{nm}) \in \mathcal{T}_\infty(F)$ and $u_i(n)v_i'(n, n) \in \mathcal{T}_\infty(F)$, the latter substitution does not change the fact that $u_i(n)v_i'(n, m) \in \mathcal{T}_\infty(F)$.

Now we wish to prove that for any i, $1 \le i \le I$, $\langle v_i'(n, m) \rangle = \langle v_i'(n', m) \rangle$ for all $n, n' \le m$. Observe that since for $n > 1$ we have

$$
\begin{aligned}
& \text{rank}(e_{1m} + e_{1l} + e_{1s} + e_{nl} + e_{ns}) \\
= \ & \text{rank}(e_{1m} + e_{1l} + e_{1s}) + \text{rank}(e_{nl} + e_{ns}) \\
= \ & \text{rank}(e_{1m}) + \text{rank}(e_{1l} + e_{1s} + e_{nl} + e_{ns}),
\end{aligned}
$$

then

$$
\begin{aligned}
& \text{rank}(\phi(e_{1m} + e_{1l} + e_{1s} + e_{nl} + e_{ns})) \\
(2.4) \qquad = \ & \text{rank}(\sum_{i=1}^{I} u_i(1)(v_i'(1, m) + v_i'(1, l) + v_i'(1, s))) \\
& + \text{rank}(\sum_{i=1}^{I} u_i(n)(v_i'(n, l) + v_i'(n, s))) = I + I = 2I.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& \text{rank}(\phi(e_{1m} + e_{1l} + e_{1s} + e_{nl} + e_{ns})) \\
= \ & \text{rank}(\sum_{i=1}^{I} u_i(1)v_i'(1, m)) \\
& + \text{rank}(\sum_{i=1}^{I} u_i(1)v_i'(1, l) + \sum_{i=1}^{I} u_i(1)v_i'(1, s) + \sum_{i=1}^{I} u_i(n)v_i'(n, l) + \\
& + \sum_{i=1}^{I} u_i(n)v_i'(n, s)) \\
= \ & I + \text{rank}(\sum_{i=1}^{I} u_i(1)v_i'(1, l) + \sum_{i=1}^{I} u_i(1)v_i'(1, s) + \sum_{i=1}^{I} u_i(n)v_i'(n, l) + \\
& + \sum_{i=1}^{I} u_i(n)v_i'(n, s)) \\
= \ & I + \text{rank}(\sum_{i=1}^{I} u_i(1)(v_i'(1, l) + v_i'(1, s)) + \sum_{i=1}^{I} u_i(n)(v_i'(n, l) + v_i'(n, s))).
\end{aligned}
$$

By (2.4) the rank of

$$
\sum_{i=1}^{I} u_i(1)(v_i'(1, l) + v_i'(1, s)) + \sum_{i=1}^{I} u_i(n)(v_i'(n, l) + v_i'(n, s))
$$

must be equal to I. This is possible only if for every i there exists exactly one j such that

$$v_i'(n,l) + v_i'(n,s) = \alpha(v_j'(1,l) + v_j'(1,s))$$

for some $\alpha \in F^*$. Since

$$\langle v_i'(n,l) : 1 \le i \le I \rangle \ni v_i'(n,l) - \alpha v_j'(n_1,l)$$
$$= -v_i'(n,s) + \alpha v_j'(n_1,s) \in \langle v_i'(n,s) : 1 \le i \le I \rangle$$

and $\langle v_i'(n,l) : 1 \le i \le I \rangle \cap \langle v_i'(n,s) : 1 \le i \le I \rangle = \{0\}$ for $l \ne s$, we must have $v_i'(n,l) = \alpha v_j'(1,l)$ and $v_i'(n,s) = \alpha v_j'(1,s)$. Now we change the enumeration of $u_1(n)$, ..., $u_I(n)$ for the one satisfying the following condition. If $v_i'(n,s) = \alpha v_j'(1,s)$, then we replace $u_i(n)$ with $u_j(n)$. Then we have $v_i'(n,l) = \alpha v_i'(1,l)$. As α is the same for all i, it depends only on n. Hence we may denote by $u_i'(n)$ the vectors $\alpha u_i(n)$. Moreover, denote by $v_i(m)$ the vectors $v_i'(n,m)$.

Summing up, we can write that for each n, m there exist $u_1(n)$, $u_2(n)$, ..., $u_I(n) \in M_{\infty \times 1}^{fin}(F)$, $v_1(m)$, $v_2(m)$, ..., $v_I(m) \in M_{1 \times \infty}(F)$ such that $\phi(e_{nm}) = \sum_{i=1}^I u_i'(n) v_i(m)$. Consider now the maps $\phi_i : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ ($1 \le i \le I$) defined by the conditions $\phi_i(e_{nm}) = u_i'(n) v_i(m)$. Clearly, $\phi = \phi_1 + \cdots + \phi_I$. By Corollary 2.1 we have $\phi_i(x) = \sum_{n \le m} x_{nm} u_i'(n) v_i(m)$. Define the matrices a_i, b_i by the following

$$(a_i)_{nm} = (u_i(m))_n, \qquad (b_i)_{nm} = (v_i(n))_m.$$

Then we have $\phi_i(x) = a_i x b_i$. Consequently $\phi(x) = \sum_{i=1}^I a_i x b_i$. $\qquad \square$

Let us now present some examples.

Example 2.1. Let ϕ be such that

$$\phi\left(\begin{pmatrix} x_{11} & x_{12} & \cdots \\ & x_{22} & \\ & & \ddots \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{11} & x_{12} & x_{12} & & \\ & x_{11} & 0 & x_{12} & & \\ & & x_{22} & x_{22} & & \\ & & & x_{22} & & \\ & & & & \ddots \end{pmatrix}.$$

This map preserves the additivity of rank. According to the notation from our proof we have

$$u_1(n) = e_{2n-1}, \quad v_1(m) = f_{2m-1} + f_{2m}, \qquad u_2(n) = e_{2n}, \quad v_2(m) = f_{2m}.$$

Hence

$$a_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & & \ddots \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \ddots \end{pmatrix},$$

$$a_2 = \begin{pmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 0 & & \\ & & & \ddots & \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ & & & & & & \ddots \end{pmatrix}.$$

Example 2.2. Let ϕ be given by

$$u_1(2n-1) = \sum_{i=-3n+3}^{-2n+2} e_{n^2+i}, \qquad u_1(2n) = \sum_{i=-2n+3}^{-n} e_{n^2+i}$$
$$v_1(m) = \sum_{i=-3m+4}^{-m+2} f_{\frac{m^2+i}{2}}, \qquad \text{for } n, m \in \mathbb{N},$$

i.e.,

$$\phi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} & \\ & & x_{33} & \\ & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} & x_{12} & x_{13} & x_{13} & x_{13} & \\ & x_{22} & x_{22} & x_{23} & x_{23} & x_{23} & \\ & & & x_{33} & x_{33} & x_{33} & \\ & & & x_{33} & x_{33} & x_{33} & \\ & & & & & & \ddots \end{pmatrix}.$$

Then ϕ preserves the additivity of rank. Notice that although blocks which are images of e_{nm} are getting bigger, all $\phi(e_{nm})$ have rank one.

The next example shows that not all maps ϕ of form (1.2) fulfill condition (1.1).

Example 2.3. We put

$$\phi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} & \\ & & x_{33} & \\ & & & \ddots \end{pmatrix}\right)$$
$$= \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ & x_{11}+x_{22} & x_{12}+x_{23} & x_{13}+x_{24} & \\ & & x_{22}+x_{33} & x_{23}+x_{34} & \\ & & & x_{33}+x_{44} & \\ & & & & \ddots \end{pmatrix}.$$

We can write that $\phi(x) = a_1 x b_1 + a_2 x b_2$, where

$$a_1 = b_1 = e_\infty, \quad a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

One can observe that this ϕ does not preserve the additivity of rank. In particular $\mathrm{rank}(e_{11} + e_{22}) = \mathrm{rank}(e_{11}) + \mathrm{rank}(e_{22})$, but

$$\mathrm{rank}(\phi(e_{11} + e_{22})) = 3 \neq 4 = \mathrm{rank}(\phi(e_{11})) + \mathrm{rank}(\phi(e_{22})).$$

The reason for this is that the columns of a_1, a_2 and rows of b_1, b_2 do not form linearly independent sets.

The last two examples concern the maps ϕ such that $\phi(\mathcal{T}_\infty(F))$ contains some matrices $\phi(x)$ that have infinite rank although the rank of x is finite.

Example 2.4. Consider ϕ given by

$$\phi(e_{1n}) = \sum_{i=1}^{\infty} e_{2i-1,2i+2n-3} \quad \text{for } n \in \mathbb{N}, \qquad \phi(e_{nm}) = e_{2n,2m} \quad \text{for } m \geq n \geq 2,$$

i.e.,

$$\phi\left(\left(\begin{matrix} x_{11} & x_{12} & x_{13} \\ & x_{22} & x_{23} \\ & & x_{33} \\ & & & \ddots \end{matrix}\right)\right) = \left(\begin{matrix} x_{11} & 0 & x_{12} & 0 & x_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & x_{11} & 0 & x_{12} & 0 & x_{13} \\ & & & x_{22} & 0 & x_{23} & 0 \\ & & & & x_{11} & 0 & x_{12} \\ & & & & & x_{33} & 0 \\ & & & & & & \ddots \end{matrix}\right).$$

In this case if $x_{1n} \neq 0$ for any $n \in \mathbb{N}$, then $\mathrm{rank}(\phi(x)) = \infty$. However, if $\mathrm{rank}(x) < \infty$ and $x_{1n} = 0$ for all n, we have $\mathrm{rank}(\phi(x)) < \infty$.

Example 2.5. Let

$$\phi(e_{nm}) = \sum_{i=1}^{\infty} e_{2^{m-1}(2i-1),2^{m-1}(2i-1)+n-1}.$$

We have

$$\phi\left(\left(\begin{matrix} x_{11} & x_{12} & x_{13} & \cdots \\ & x_{22} & x_{23} \\ & & x_{33} \\ & & & \ddots \end{matrix}\right)\right)$$

$$= \left(\begin{matrix} x_{11} & 0 & 0 \\ & x_{12} & x_{22} & 0 \\ & x_{11} & 0 & 0 \\ & & & x_{13} & x_{23} & x_{33} \\ & & & x_{11} & 0 & 0 \\ & & & & x_{12} & x_{22} \\ & & & & & \ddots \end{matrix}\right).$$

In this case ranks of all matrices in $\phi(\mathcal{T}_\infty(F))$, except $\phi(0) = 0$, are infinite.

We wish to focus on one more issue. Consider the maps ϕ on $\mathcal{T}_\infty(F)$ satisfying the condition
(2.5)
$$\text{rank}(x+y) = \text{rank}(x) + \text{rank}(y) \iff \text{rank}(\phi(x+y)) = \text{rank}(\phi(x)) + \text{rank}(\phi(y))$$

for all $x, y \in \mathcal{T}_\infty(F)$. On some assumptions such maps are described by the following.

Theorem 2.1. *Let ϕ be an injective linear map on $\mathcal{T}_\infty(F)$ satisfying condition $\text{rank}(\phi(x)) = \infty$ if only if $\text{rank}(x) = \infty$. Such ϕ satisfies (1.1) if and only if it satisfies (2.5).*

Proof. Obviously, it suffices to prove that if ϕ fulfills (1.1), then from $\text{rank}(\phi(x+y)) = \text{rank}(\phi(x)) + \text{rank}(\phi(y))$, it follows that $\text{rank}(x+y) = \text{rank}(x) + \text{rank}(y)$.

Consider then $\phi_1, \ldots, \phi_I$ which were defined as in the proof of Theorem 1.1.

Let x be a matrix of rank one. From the construction of a_i, b_i, we conclude that $\text{rank}(\phi_i(x)) = 1$ and $\text{rank}(\sum_{i=1}^I \phi_i(x)) = |I|$.

Assume now that $\text{rank}(x) = r + 1$. Then $x = y + z$ for some y, z such that $\text{rank}(y) = r$, $\text{rank}(z) = 1$. By the inductive assumption

$$\text{rank}(\phi(x)) = \text{rank}(\phi(y) + \phi(z)) = \text{rank}(\phi(y)) + \text{rank}(\phi(z)) = rI + I = (r+1)I.$$

Consequently, if $\text{rank}(\phi(x)) = rI$, then $\text{rank}(x) = r$.

Let us now get back to condition (2.5). Suppose that $\text{rank}(\phi(x+y)) = \text{rank}(\phi(x)) + \text{rank}(\phi(y))$.

If $\text{rank}(\phi(x))$ or $\text{rank}(\phi(y))$ is infinite, then $\text{rank}(\phi(x) + \phi(y))$ is infinite as well. This implies the fact that either $\text{rank}(x)$ or $\text{rank}(y)$ is infinite, and so is $\text{rank}(x + y)$, which yields the equality $\text{rank}(x + y) = \text{rank}(x) + \text{rank}(y)$.

If ranks of $\phi(x)$, $\phi(y)$ are both finite, then there exist $r_1, r_2 \in \mathbb{N}$ such that $\text{rank}(\phi(x)) = r_1 I$, $\text{rank}(\phi(y)) = r_2 I$. Consequently $\text{rank}(\phi(x) + \phi(y)) = (r_1 + r_2)I$. From these facts and the paragraph above we obtain that $\text{rank}(x) = r_1$, $\text{rank}(y) = r_2$ and $\text{rank}(x + y) = r_1 + r_2$, which completes the proof. $\square$

At the end of the paper we wish to raise two questions that are connected to our considerations.

Problem 2.1. How to characterize all bijective maps on $\mathcal{T}_\infty(F)$ satisfying (1.1)?

Problem 2.2. How to characterize all maps on $\mathcal{T}_\infty(F)$ satisfying (1.1) that are not injective?

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