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SIMPLICIAL WEDGE COMPLEXES AND PROJECTIVE TORIC VARIETIES

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Abstract. Let K be a fan-like simplicial sphere of dimension $n-1$ such that its associated complete fan is strongly polytopal, and let v be a vertex of K. Let $K(v)$ be the simplicial wedge complex obtained by applying the simplicial wedge operation to K at v, and let v_0 and v_1 denote two newly created vertices of $K(v)$. In this paper, we show that there are infinitely many strongly polytopal fans Σ over such $K(v)$'s, different from the canonical extensions, whose projected fans $\mathrm{Proj}_{v_i}\Sigma$ $(i=0,1)$ are also strongly polytopal. As a consequence, it can be also shown that there are infinitely many projective toric varieties over such $K(v)$'s such that toric varieties over the underlying projected complexes $K_{\mathrm{Proj}_{v_i}\Sigma}$ $(i=0,1)$ are also projective.

1. Introduction and main results

There is a method of construction to obtain a new simplicial complex from a given one, called a *simplicial wedge operation*, which has recently attracted much attention in toric topology world (see, e.g., [1] and [2]). Among many other things, it is particularly interesting because, starting from a toric manifold with its associated simple convex polytope, one can construct an infinite family of new and meaningful toric manifolds, one for each sequence of positive integers.

In order to explain our results in more detail, we now want to briefly recall the construction of a simplicial wedge complex. To do so, let K be a simplicial complex of dimension $n-1$ on vertex set $\{v_1, v_2, \dots, v_m\}$, and let $J = (j_1, j_2, \dots, j_m)$ be a sequence of positive integers. A *minimal non-face* of K is a sequence of vertices of K which is not a simplex of K but any proper subset is a simplex of K. Let $K(J)$ be a simplicial complex on $j_1 + j_2 + \cdots + j_m$ vertices

$$v_{11}, \dots, v_{1j_1}, v_{21}, \dots, v_{2j_2}, \dots, v_{m1}, \dots, v_{mj_m}$$

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with the property that

$$\{v_{i_11},\ldots,v_{i_1j_{i_1}},v_{i_21},\ldots,v_{i_2j_{i_2}},\ldots,v_{i_k1},\ldots,v_{i_kj_{i_k}}\}$$

is a minimal non-face of $K(J)$ if and only if $\{v_{i_1},v_{i_2},\ldots,v_{i_k}\}$ is a minimal non-face of K.

In order to obtain an alternative description of the simplicial complex $K(J)$ that is our main interest, we next recall that the *link* of a simplex σ in K is the simplicial subcomplex of K given by

$$\mathrm{link}_K\sigma=\{\tau\in K\,|\,\sigma\cup\tau\in K,\ \sigma\cap\tau=\emptyset\},$$

while the *join* of two disjoint simplicial subcomplexes K_1 and K_2 is the simplicial complex given by

$$K_1*K_2=\{\sigma_1\cup\sigma_2\,|\,\sigma_i\in K_i,\ i=1,2\}.$$

Now, fix a vertex v_i in K. Let I denote a 1-simplex whose vertices are v_{i1} and v_{i2}, and let ∂I denote the boundary complex of I consisting of two vertices v_{i1} and v_{i2}. We then define a new simplicial complex $K(v_i)$, called a *simplicial wedge complex*, with $m+1$ vertices

$$v_1,v_2,\ldots,v_{i-1},v_{i1},v_{i2},v_{i+1},\ldots,v_m$$

by

$$K(v_i)=(I*\mathrm{link}_K\{v_i\})\cup(\partial I*K\backslash\{v_i\})\,.$$

It is easy to see that the new simplicial complex $K(v_i)$ is same as $K(J)$ with

$$J=(1,\ldots,\overset{i\text{-th coordinate}}{2},1,\ldots,1).$$

By applying this construction repeatedly starting from $J=(1,\ldots,1)$, one can also obtain $K(J)$ for any sequence $J=(j_1,\ldots,j_m)$ with positive integer entries (see [1], Section 2 for more details). Let K be dual to the boundary complex of a simple convex polytope P of dimension n with m facets, and let $d(J)=j_1+\cdots+j_m$ for $J=(j_1,\ldots,j_m)$. Then it can be shown as in [1], Theorem 2.4 that $K(J)$ is dual to the boundary of a simple convex polytope $P(J)$ of dimension $d(J)-m+n$ with $d(J)$ facets.

Let K be a simplicial complex of dimension $n-1$, as before. We say that K is a *simplicial sphere* of dimension $n-1$ if its geometric realization $|K|$ of K is homeomorphic to a sphere S^{n-1}. On the other hand, K is said to be *polytopal* if there is an embedding of the geometric realization $|K|$ into $\mathbb{R}^n$ which is given by the boundary of a simplicial polytope P^* of dimension n.

There is also a notion between a simplicial sphere and polytopality. That is, we say that a simplicial sphere K of dimension $n-1$ is *star-shaped* if there is an embedding of the geometric realization $|K|$ of K into $\mathbb{R}^n$ so that there exists a point p with the property that each ray emanating from p meets $|K|$ in one and only one point. In this case, p is called a *kernel point.* Clearly every polytopal sphere is also star-shaped, even though the converse is not true in general, as the Barnette sphere shows (see [6], p. 90).

A *rational fan* (or simply *fan*) Σ of dimension n is a collection of strongly convex rational cones in $\mathbb{R}^n$ such that each face of a cone and the intersection of a finite number of cones are again in the fan. Here a cone is *strongly convex* if it does not contain any non-trivial linear subspace, and is *rational* if every generator of a one-dimensional cone can be taken in the integer lattice $\mathbb{Z}^n$. A rational cone is called *non-singular* if its generators form a part of an integral basis of $\mathbb{Z}^n$, while it is called *simplicial* if its generators are simply linearly independent. We can associate a simplicial complex K_Σ to each simplicial fan Σ, called the *underlying simplicial complex*, in such a way that vertices of K_Σ are generators of one-dimensional cones of Σ and faces of K_Σ are the sets of generators of cones of Σ. Recall also that an ordinary fan is said to be *complete* if the union of all cones cover the whole space $\mathbb{R}^n$. We say that a simplicial sphere K is *fan-like* (or, equivalently *star-shaped*) if there is a complete fan whose underlying simplicial complex is same as K. Note that a simplicial sphere is fan-like if and only if so is its simplicial wedge.

A fan Σ is said to be *weakly polytopal* if its underlying simplicial complex K_Σ is polytopal with a simplicial polytope P^*, and is said to be *strongly polytopal* if, in addition, P^* satisfies the following two conditions:

- $0 \in P^*$.
- $\Sigma = \{\operatorname{pos}\sigma \mid \sigma \in \partial P^*\}$. Here $\operatorname{pos}\sigma$ is the set of all positive linear combinations of σ, and ∂P^* denotes the boundary complex of P^*.

Note that a rational fan Σ of dimension n is completely determined by the underlying simplicial complex K_Σ and a map $\lambda : V(K_\Sigma) \to \mathbb{Z}^n$, called the *characteristic map*, obtained by mapping each vertex of K_Σ to the primitive generator of the corresponding one-dimensional cone of Σ, and vice versa. Here $V(K_\Sigma)$ denotes the vertex set of K_Σ.

Let K be a simplicial complex of dimension $n-1$, equipped with a characteristic map $\lambda : V(K) \to \mathbb{Z}^n$ such that for each face σ of K the vectors $\lambda(i)$, $i \in \sigma$, form a part of an integral basis of $\mathbb{Z}^n$. Then we can obtain a new simplicial complex $\operatorname{link}_K\sigma$, equipped with a new characteristic map $\operatorname{Proj}_\sigma\lambda$ defined by

$$\operatorname{Proj}_\sigma(\lambda)(v) = [\lambda(v)], \quad v \in V(\operatorname{link}_K\sigma)$$

in the quotient space $\mathbb{Z}^n/\langle\lambda(w) \mid w \in \sigma\rangle$ isomorphic to $\mathbb{Z}^{n-|\sigma|}$. In a similar way, we can also define the notion of a projected fan $\operatorname{Proj}_\sigma\Sigma$ of a fan Σ with respect to a face σ of K_Σ (refer to [7], Section 2).

In the paper [5], Ewald introduced the notion of a canonical extension which is a particular way to obtain a simplicial wedge complex, and proved that Theorem 1.1 below always holds for canonical extensions ([5], Theorem 2). Here, a *canonical extension* of a simplicial complex K equipped with a characteristic map λ is a simplicial wedge complex $K(v)$ equipped with a characteristic map λ' such that $\operatorname{Proj}_{v_i}\lambda' = \lambda$ for all $i = 0, 1$ (see, e.g., Section 3 for a precise definition).

Our main aim of this paper is to significantly generalize the results of Ewald in [5] to more general simplicial wedge complexes. In addition, we shall provide a very simple and also efficient algorithm to construct certain particular simplicial wedge complexes, which will be another important point of this paper (see the proof of Theorem 3.2 for more details). In fact, we have the following.

Theorem 1.1. *Let K be a fan-like simplicial sphere of dimension $n-1$ such that its associated complete fan is strongly polytopal, and let v be a vertex of K. Let $K(v)$ be the simplicial wedge complex obtained by applying the simplicial wedge operation to K at v, and let v_0 and v_1 denote two newly created vertices of $K(v)$. Then there are infinitely many strongly polytopal fans Σ over such $K(v)$'s, different from the canonical extensions, whose projected fans $\mathrm{Proj}_{v_i}\Sigma$ $(i = 0, 1)$ are also strongly polytopal.*

As a consequence of Theorem 1.1 and its proof, we can easily construct many examples of a complete, non-singular, strongly polytopal fan Σ over the simplicial wedge complex $K(v)$ whose projected fans $\mathrm{Proj}_{v_i}\Sigma$ $(i = 0, 1)$ are also complete, non-singular, and strongly polytopal (see, e.g., Example 3.5). In sharp contrast, according to the paper [3], Section 7 there exists an example of a complete, singular, non-strongly polytopal fan Σ over the simplicial wedge complex $K(v)$ whose projections $\mathrm{Proj}_{v_i}\Sigma$ $(i = 0, 1)$ are complete, singular, and strongly polytopal. We also remark that Theorem 1.1 somehow answers a related question posed in the paper [3] (refer to Question 7.2).

It is well known that there is a one-to-one correspondence between the collection of toric varieties and the collection of rational fans, up to some equivalence. So, given a complete rational fan Σ there is always a compact toric variety M which corresponds to the underlying simplicial complex K_Σ. In this case, we shall say that M is a toric variety *over* K_Σ. Recall that M is projective if and only if its corresponding fan Σ is strongly polytopal ([4], p. 118).

Theorem 1.2. *Let K, v, $K(v)$, v_0, v_1, and Σ be the same as in Theorem 1.1. Then there are infinitely many projective toric varieties over such $K(v)$'s such that toric varieties over $K_{\mathrm{Proj}_{v_i}\Sigma}$ $(i = 0, 1)$ are also projective.*

This paper is organized as follows. In Section 2, we briefly review necessary facts which play an important role in the proof of Theorem 1.1. In particular, we recall the definition of a Gale transform and the Shephard's criterion which gives a convenient and useful way to determine whether or not a complete fan is strongly polytopal. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2. Gale transforms and Shephard's criterion

The aim of this section is to set up some basic notation and definitions, and to collect some important facts necessary for the proof of Theorem 1.1. To do so, we first begin with reviewing linear transforms and Gale transforms. Refer to [6], Chapter II-Section 4 for more details.

Let $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ be a finite sequence of vectors x_i in $\mathbb{R}^n$ which linearly spans $\mathbb{R}^n$. Then we consider the space of linear dependencies (or linear relations) of X which is given by the $(m-n)$-dimensional space

$$\{(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} \alpha_i x_i = 0\}.$$

By choosing a basis $\{\Theta^1, \dots, \Theta^{m-n}\}$ of the space of linear dependencies as above, it is convenient to write it as a matrix of size $(m-n) \times m$, as follows.

$$\begin{aligned}(\Theta^1, \dots, \Theta^{m-n})^T &= \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1m} \\ \vdots & \vdots & \vdots \\ \alpha_{(m-n)1} & \dots & \alpha_{(m-n)m} \end{pmatrix}_{(m-n)\times m} \\ &= (\bar{x}_1, \dots, \bar{x}_m) =: \bar{X}.\end{aligned}$$

The finite sequence $\bar{X}$ is called a *linear transform* (or *linear representation*) of X. Clearly, a linear transform is not unique and depends only on a choice of a basis. Note also that we have the following relationship between X and $\bar{X}$

$$X\bar{X}^T = 0. \tag{2.1}$$

It is also easy to see that $\bar{X}X^T = 0$ by taking the transpose of the equation (2.1). Thus, if $\bar{X}$ is a linear transform of X, then X is also a linear transform of $\bar{X}$.

Next, in order to define a Gale transform by using the notion of a linear transform, as before let $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ be a finite sequence of vectors x_i in $\mathbb{R}^n$ which linearly spans $\mathbb{R}^n$. Then we identify $\mathbb{R}^n$ as an affine space with a hyperplane H in a linear space $\mathbb{R}^{n+1}$ by the natural embedding

$$j : \mathbb{R}^n \to \mathbb{R}^{n+1}, \quad v \mapsto (v, 1).$$

Then $H = \{(v, 1) \in \mathbb{R}^{n+1} \mid v \in \mathbb{R}^n\}$ does not contain the origin of $\mathbb{R}^{n+1}$. Thus it follows from [6], Lemma 4.15 that a linear transform $\bar{\hat{X}} = (\bar{x}_1, \dots, \bar{x}_m) \in (\mathbb{R}^{m-n-1})^m$ of

$$j(X) = ((x_1, 1), \dots, (x_m, 1)) =: (\hat{x}_1, \dots, \hat{x}_m)$$

in $\mathbb{R}^{n+1}$ satisfies

$$\sum_{i=1}^{m} \bar{x}_i = 0,$$

and $\bar{X}$ is called a *Gale transform* (or an *affine transform*) of X.

Now, we are ready to characterize a complete fan that is strongly polytopal. To be more precise, we have the following criterion given by Shephard in the paper [8] (or [6], Theorem 4.8 and [5], Section 2) for a complete fan to be strongly polytopal.

Theorem 2.1. *Let $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ be a finite sequence of lattice points x_i in $\mathbb{Z}^n \subset \mathbb{R}^n$ that span the 1-dimensional cones of a complete fan Σ, and let $\bar{X}$ be a Gale transform of X. For each proper face $\sigma = \mathrm{pos}\{x_{j_1}, \ldots, x_{j_k}\}$ of Σ, let $C(\sigma)$ denote the convex hull generated by $\bar{X}\backslash\{\bar{x}_{j_1}, \ldots, \bar{x}_{j_k}\}$. That is,*

$$C(\sigma) = \mathrm{conv}(\bar{X}\backslash\{\bar{x}_{j_1}, \ldots, \bar{x}_{j_k}\}).$$

Then Σ is strongly polytopal if and only if we have

$$\bigcap_{\sigma\in\Sigma} \mathrm{relint}\, C(\sigma) \neq \emptyset.$$

Here, $\mathrm{relint}\, C(\sigma)$ means the *relative interior* of $C(\sigma)$. Recall also that, when σ is a proper face of Σ generated by $\{x_{j_1}, \ldots, x_{j_k}\}$, $\bar{X}\backslash\{\bar{x}_{j_1}, \ldots, \bar{x}_{j_k}\}$ is called a *coface* of σ in X.

In fact, in order to use the Shepherd's criterion for a complete fan to be strongly polytopal, we shall start with a finite sequence X whose column sum is equal to zero. Then we obtain a linear transform $\bar{X}$ of X, and use it to prove our main Theorems 1.1 and 1.2. Refer to Section 3 for more detail.

3. Proofs of Theorems 1.1 and 1.2

The aim of this section is to give proofs of Theorems 1.1 and 1.2. In this section, we also provide an example of a complete, non-singular, strongly polytopal fan Σ over the simplicial wedge complex whose projected fans are also complete, non-singular, and strongly polytopal.

To do so, let K be a fan-like simplicial sphere of dimension $n-1$ whose vertex set $V(K)$ is equal to $\{w_1, w_2, \ldots, w_m\}$. Then choose any vertex v, say w_1, from $V(K)$. Let $K(v)$ be the simplicial complex obtained by applying the simplicial wedge operation to K at v, and let v_0 and v_1 denote two newly created vertices of $K(v)$. Let $V(K(v))$ be the vertex set of $K(v)$ such that $V(K(v)) = \{v_0, v_1, v_2, \ldots, v_m\}$ is given by $v_i = w_i$ for each $i = 2, 3, \ldots, m$.

Let Σ be a complete fan associated with the simplicial complex $K(v)$. Then choose a point x_i in $\mathbb{R}^{n+1}$ from each 1-dimensional cone corresponding to a vertex v_i in $V(K(v))$ so that a finite sequence

$$X = (x_0, x_1, x_2, \ldots, x_m) \in (\mathbb{R}^{n+1})^{m+1}$$

positively spans $\mathbb{R}^{n+1}$. Thus we have the identity

$$x_0 + x_1 + \cdots + x_m = 0.$$

For later use, let us write the finite sequence X as

$$X = \begin{pmatrix} a_0 & 0 & c \\ 0 & b_0 & d \\ 0 & 0 & \\ \vdots & \vdots & G \\ 0 & 0 & \end{pmatrix}_{(n+1)\times(m+1)}, \tag{3.1}$$

where a_0 and b_0 are non-zero real numbers, c and d are row vectors of size $m-1$, and G is a real matrix of size $(n-1)\times(m-1)$. In particular, if $a_0 = b_0 = 1$ and $c = d$, then X (or Σ) will be called a *canonical extension* of a complete fan associated to the simplicial complex K.

Now, let

$$\bar{X} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m) \in (\mathbb{R}^{m-n})^{m+1}$$

be a linear transform of X. Then it follows from [6], Theorem 4.14 that $\operatorname{pos}\bar{X}$ is a strongly positive cone C in $\mathbb{R}^{m-n}$. Let H denote any hyperplane in $\mathbb{R}^{m-n}$ such that $H \cap C$ is a polytope $\hat{P}$ of dimension $m-n-1$. For each $\bar{x}_i$, let $\hat{x}_i$ be an intersection point in $H \cap \{r\bar{x}_i \mid r > 0\}$. Then the finite sequence

$$\hat{X} = (\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m) \in H$$

is called a *Shephard diagram* (or simply *diagram*) of X.

For the sake of notational convenience, from now on we set

$$\hat{X}_0 = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m) \text{ and } \hat{X}_1 = (\hat{x}_0, \hat{x}_2, \ldots, \hat{x}_m).$$

Recall that a subsequence Y of X is said to be a *coface* of Σ if $\operatorname{pos}(X\backslash Y)$ is a face of Σ. Note also that $\hat{X}$ has a face poset which consists of subsequences of X of the form $X\backslash Y$ for a subsequence Y of X such that

$$0 \in \operatorname{relin}\operatorname{conv}(\hat{X}|_{X\backslash Y}).$$

Thus, it follows from Theorem 2.1 that we have the following Shephard's criterion for a complete fan to be strongly polytopal (see also [8]).

Theorem 3.1. *A complete fan Σ is strongly polytopal if and only if*

$$S(\Sigma, \hat{X}) := \bigcap_{Y \text{ coface of } \Sigma} \operatorname{relint}\operatorname{conv}(\hat{X}|_Y) \neq \emptyset.$$

With these understood, our first main result of this section is:

Theorem 3.2. *For any $n \geq 2$, there are infinitely many complete fans Σ over such $K(v)$'s, different from the canonical extensions, such that*

$$S(\Sigma, \hat{X}) = S(\operatorname{Proj}_{v_0}\Sigma, \hat{X}_0) = S(\operatorname{Proj}_{v_1}\Sigma, \hat{X}_1).$$

Proof. To prove it, for a finite sequence X as in (3.1) let us write

$$G = (y_1, y_2, \ldots, y_{m-1}) \in (\mathbb{R}^{n-1})^{m-1}.$$

Then we have the identity

$$y_1 + y_2 + \cdots + y_{m-1} = 0.$$

Thus there is a Shephard diagram $\hat{G} = (\hat{y}_1, \ldots, \hat{y}_{m-1}) \in (\mathbb{R}^{m-n})^{m-1}$ of G.

Since $\hat{G}$ can be considered as a real matrix of size $(m-n)\times(m-1)$, it defines a linear map $L_{\hat{G}}$ from $\mathbb{R}^{m-1}$ to $\mathbb{R}^{m-n}$ in the natural way. Note that the dimension of the kernel of $L_{\hat{G}}$ is greater than or equal to $m-1-(m-n) = n-1 \geq 1$. Thus we can always choose two linearly independent vectors $c = (c_1, \ldots, c_{m-1})$

and $d = (d_1, \dots, d_{m-1})$ in $\mathbb{R}^{m-1}$, and two non-zero real numbers a_0 and b_0 such that

$$L_{\hat{G}}(b_0 c - a_0 d)^T = \hat{G}(b_0 c - a_0 d)^T = 0,\ a_0 = -\sum_{i=1}^{m-1} c_i,\ b_0 = -\sum_{i=1}^{m-1} d_i. \tag{3.2}$$

In fact, there is an easy way to take two vectors c and d, and non-zero real numbers a_0 and b_0 satisfying the above condition (3.2). To be more precise, note first that all row vectors G^i of G lie in the kernel of $\hat{G}$ by the definition of a linear transform. So choose any row vector, say G^1, of G, and then write

$$G^1 = \sum_{i=1}^{m-1} r_i e_i,$$

where $e_1, e_2, \dots, e_{m-1}$ denote the standard basis vectors of $\mathbb{R}^{m-1}$. Assume without loss of generality that the first component of G^1 is not zero, that is, $r_1 \neq 0$. Since $\sum_{i=1}^{m-1} y_i = 0$, we have $\sum_{i=1}^{m-1} r_i = 0$. So it is possible to rewrite G^1 as

$$G^1 = \sum_{i=1}^{m-1} r_i e_i = -(\sum_{i=2}^{m-1} r_i)e_1 + \sum_{i=2}^{m-1} r_i e_i.$$

Now, let

$$a_0 = \sum_{i=2}^{m-1} r_i \neq 0,\ b_0 = 1,\ d = -e_1,\ \text{and } c = -\sum_{i=2}^{m-1} r_i e_i.$$

Then we have

$$\begin{aligned} -G^1 &= -(\sum_{i=2}^{m-1} r_i)(-e_1) + (-\sum_{i=2}^{m-1} r_i e_i) = -a_0 d + b_0 c, \\ \hat{G}(-G_1)^T &= -\hat{G}G_1^T = 0, \end{aligned}$$

as required.

Next, for each $i = 1, 2, \dots, m-n$ let

$$\alpha_i = -\frac{c \cdot \hat{G}^i}{a_0},\ \beta_i = -\frac{d \cdot \hat{G}^i}{b_0}, \tag{3.3}$$

where $\cdot$ denotes the standard inner product and $\hat{G}^i$ denotes the i-th row of $\hat{G}$. It is easy to see from (3.2) and (3.3) that

$$\alpha_i = \beta_i, \quad i = 1, 2, \dots, m-n. \tag{3.4}$$

With these a_0, b_0, c, and d as in (3.2), let us define a new finite sequence X, as follows:

$$X = (x_0, x_1, \dots, x_m) = \begin{pmatrix} a_0 & 0 & c \\ 0 & b_0 & d \\ 0 & 0 & \\ \vdots & \vdots & G \\ 0 & 0 & \end{pmatrix}_{(n+1)\times(m+1)}.$$

Note that, by the way of construction, it is possible to take an integral finite sequence X satisfying the required conditions. Here an integral sequence means that all components of the sequence are integers. So we let Σ be a complete rational fan whose associated finite sequence is X.

Since by the choices of a_0 and b_0 the identity $\sum_{i=0}^{m} x_i = 0$ continues to hold, we can also find a Shephard diagram of Σ. Indeed, let $\hat{X}$ be

$$\begin{pmatrix} \alpha_1 & \beta_1 & \\ \vdots & \vdots & \hat{G} \\ \alpha_{m-n} & \beta_{m-n} & \end{pmatrix}_{(m-n)\times(m+1)}.$$

Then it follows from (3.3) that $X\hat{X}^T = 0$. Hence $\hat{X}$ is a Shephard diagram of Σ. Moreover, it is easy to see that in this case

$$\hat{X}_0 = ((\alpha_1, \dots, \alpha_{m-n})^T, \hat{G}) \text{ and } \hat{X}_1 = ((\beta_1, \dots, \beta_{m-n})^T, \hat{G})$$

are Shephard diagrams of $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$, respectively. Since by (3.4) $\alpha_i = \beta_i$ for all $i = 1, 2, \dots, m-n$, it is also important to notice that we have

$$\hat{X}_0 = \hat{X}_1. \tag{3.5}$$

By the construction of a simplicial wedge complex, two underlying simplicial complexes K_0 and K_1 of $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$, respectively, are combinatorially equivalent so that $\mathrm{link}_{K_0}(v_1)$ coincides with $\mathrm{link}_{K_1}(v_0)$. Moreover, it follows from (3.5) that two intersections $S(\mathrm{Proj}_{v_0}\Sigma, \hat{X}_0)$ and $S(\mathrm{Proj}_{v_1}\Sigma, \hat{X}_1)$ should be identical. Finally, note that every coface of the simplicial wedge complex $K(v)$ is a coface of K_0 or K_1. Hence, as in [3], Proposition 5.9, we have

$$S(\Sigma, \hat{X}) = S(\mathrm{Proj}_{v_0}\Sigma, \hat{X}_0) \cap S(\mathrm{Proj}_{v_1}\Sigma, \hat{X}_1) = S(\mathrm{Proj}_{v_i}\Sigma, \hat{X}_i)$$

for all $i = 0, 1$.

Starting from any matrix G whose sum of column vectors is equal to zero, it is now clear that we can produce infinitely many complete fans Σ over such $K(v)$'s satisfying the conclusion of the theorem. This completes the proof of Theorem 3.2. □

As a consequence of Theorem 3.2, we have the following theorem that is same as Theorem 1.1.

Theorem 3.3. *Let K be a fan-like simplicial sphere of dimension $n-1$ such that its associate complete fan is strongly polytopal, and let v be a vertex of K. Let $K(v)$ be the simplicial wedge complex obtained by applying the simplicial wedge operation to K at v, and let v_0 and v_1 denote two newly created vertices of $K(v)$. Then there are infinitely many strongly polytopal fans Σ over such $K(v)$'s, different from the canonical extensions, whose projected fans $\mathrm{Proj}_{v_i}\Sigma$ $(i = 0, 1)$ are all strongly polytopal.*

Proof. To prove the theorem, first take a finite sequence X satisfying the conclusion of Theorem 3.2. By the way of construction of a simplicial wedge complex, we can identify K with one of two simplices K_0 and K_1, say K_0. So we may assume that $S(\mathrm{Proj}_{v_0}\Sigma, \hat{X}_0)$ is not empty. This together with Theorems 3.1 and 3.2 implies that the corresponding fans Σ, $\mathrm{Proj}_{v_0}\Sigma$, and $\mathrm{Proj}_{v_1}\Sigma$ over $K(v)$, K_0, and K_1, respectively, should be strongly polytopal. This completes the proof of Theorem 3.3. □

The following corollary follows immediately.

Corollary 3.4. *Let K, v, $K(v)$, v_0, v_1, and Σ be the same as in Theorem 3.3. Then there are infinitely many projective toric varieties over such $K(v)$'s such that toric varieties over $K_{\mathrm{Proj}_{v_i}\Sigma}$ $(i = 0, 1)$ are also projective.*

Proof. To prove it, recall that there is a one-to-one correspondence between the collection of compact toric varieties and the collection of complete rational fans, up to some equivalence. So there are always compact toric varieties which correspond to the underlying simplicial complexes $K(v)$, K_0, and K_1, constructed in the proof of Theorem 3.3. Moreover, it follows from Theorem 3.3 that the corresponding fans Σ, $\mathrm{Proj}_{v_0}\Sigma$, and $\mathrm{Proj}_{v_1}\Sigma$ over $K(v)$, K_0, and K_1, respectively, are now strongly polytopal. Therefore their corresponding compact toric varieties should be all projective. This completes the proof of Corollary 3.4. □

Finally, we close this section with an example of how to apply the algorithm given in the proof of Theorem 3.2 in order to obtain a complete, non-singular, strongly polytopal fan whose projected fans are also complete, non-singular, and strongly polytopal.

Example 3.5. Let G be an integral matrix of size 2×3 given by

$$G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then take the first row $G^1 = (1, 0, -1)$ of G. By applying our algorithm given in the proof of Theorem 3.2 to G^1, it is easy to obtain

$$a_0 = -1,\ b_0 = 1,\ c = (0, 0, 1),\ d = (-1, 0, 0).$$

Thus our complete fan Σ is given by the following characteristic matrix λ given by

$$\lambda = \begin{pmatrix} a_0 & 0 & c \\ 0 & b_0 & d \\ 0 & 0 & \\ \vdots & \vdots & G \\ 0 & 0 & \end{pmatrix}_{(3+1)\times(4+1)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}_{4\times 5} =: X.$$

Note that every 4×4-minor of λ has determinant equal to ± 1. Thus the complete fan Σ is actually non-singular.

Let λ_0 and λ_1 be the 3×4-matrices obtained from λ given by

$$\lambda_0 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then λ_0 and λ_1 can be considered as characteristic maps associated with the projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$, respectively. Note also that every 3×3-minor of λ_i has determinant equal to ± 1 for each $i = 0, 1$. Thus the projected fans $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ are indeed non-singular (and also complete). Moreover, observe that $\mathrm{Proj}_{v_0}\Sigma$ and $\mathrm{Proj}_{v_1}\Sigma$ are strongly polytopal. Thus Σ is also strongly polytopal by Theorem 3.3. It can be seen directly by using a Shephard diagram $\hat{X}$ of X. More precisely, in this case $\hat{X}$ can be taken to be $(1, 1, 1, 1, 1) \in (\mathbb{R}^1)^5$, and $\mathrm{relint\,conv}\,\{1\} = \{1\}$. Thus clearly we have

$$S(\Sigma, \hat{X}) = \{1\} \neq \emptyset.$$

As a consequence, we can see that their associated toric varieties are actually toric manifolds and also projective by Corollary 3.4.

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