

CIRCLES ALONG A RIEMANNIAN MAP AND CLAIRAUT RIEMANNIAN MAPS

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ABSTRACT. We first extend Yano-Nomizu's theorem, which characterizes extrinsic spheres in a Riemannian manifold, for Riemannian maps. Then we introduce Clairaut Riemannian maps, give an example and obtain necessary and sufficient conditions for a Riemannian map to be Clairaut type.

1. Introduction

A smooth curve α on a Riemannian manifold M parametrized by its arclength is called a circle if it satisfies

$$(1.1) \quad \nabla_{\dot{\alpha}} \nabla_{\dot{\alpha}} \dot{\alpha} = -\kappa^2 \dot{\alpha}$$

with some nonnegative constant κ , where $\nabla_{\dot{\alpha}}$ denotes the covariant differentiation along α with respect to the Riemannian connection ∇ on M . This condition is equivalent to the condition that there exist a nonnegative constant κ and a field of unit vectors Y along this curve which satisfies the following differential equations:

$$(1.2) \quad \nabla_{\dot{\alpha}} \dot{\alpha} = \kappa Y,$$

$$(1.3) \quad \nabla_{\dot{\alpha}} Y = -\kappa \dot{\alpha}.$$

Here κ is called curvature of α . For a given point $p \in M$, an orthonormal pair of tangent vectors $u, v \in T_p M$ and a positive constant κ , by the existence and uniqueness theorem on solutions for ordinary differential equations we have locally a unique circle $\alpha = \alpha(s)$ with initial condition that $\alpha(0) = p$; $\dot{\alpha}(0) = u$ and $\nabla_{\dot{\alpha}} \dot{\alpha} = \kappa v$. In [19] Nomizu-Yano showed that α is a circle if and only if the following is satisfied

$$(1.4) \quad \nabla_{\dot{\alpha}}^2 \dot{\alpha} + g(\nabla_{\dot{\alpha}} \dot{\alpha}, \nabla_{\dot{\alpha}} \dot{\alpha}) \dot{\alpha} = 0,$$

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where g is the metric and $\nabla_{\alpha}^2 \dot{\alpha}$ is $\nabla_{\alpha} \nabla_{\alpha} \dot{\alpha}$. We also recall that a submanifold M^n of a Riemannian manifold \bar{M}^m is called an (extrinsic) sphere if it is umbilical and has parallel mean curvature vector. Nomizu-Yano also proved that if every circle of radius r in M^n is a circle in \bar{M}^m for some $r > 0$, then M^n is a sphere. Conversely, if M is a sphere in \bar{M} , then every circle in M is also a circle or a geodesic in \bar{M} . Many authors have studied circles on Riemannian manifolds and they showed that it is possible to obtain certain properties of a submanifold by observing the extrinsic shape of circles on this submanifold, see: [1], [3], [8], [14], [15], [17], [18], [21], [22].

In elementary differential geometry, if θ is the angle between the velocity vector of a geodesic and a meridian, and r is the distance to the axis of a surface of revolution, Clairaut's relation states that $r \sin \theta$ is constant. In the submersion theory, this notion was defined by Bishop. According to his definition, a submersion $F : M \rightarrow N$ is called a Clairaut submersion if there is a function $r : M \rightarrow \mathbb{R}^+$ such that for every geodesic, making angles θ with the horizontal subspaces, $r \sin \theta$ is constant. Clairaut submersions have been studied in Lorentzian spaces and timelike, spacelike and null geodesics of Lorentzian Clairaut submersion with one-dimensional fibers have been investigated in details. It is shown that such submersions have their applications in static space times [4]. In [13] the author also showed that the notion of Clairaut submersion is an useful tool for obtaining decomposition theorems on Riemannian manifolds. Moreover, Clairaut submersions have been further generalized in [5] and [9].

On the other hand, in 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [12] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F : (M, g_M) \rightarrow (N, g_N)$ be a smooth map between Riemannian manifolds such that $0 < \text{rank} F \leq \min\{m, n\}$, where $\dim M = m$ and $\dim N = n$. Then we denote the kernel space of F_* by $\mathcal{V}_p = \ker F_{*p}$ at $p \in M$ and consider the orthogonal complementary space $\mathcal{H}_p = (\ker F_{*p})^{\perp}$ to $\ker F_{*p}$. Then $T_p M$ of M at p has the following decomposition

$$T_p M = \ker F_{*p} \oplus (\ker F_{*p})^{\perp} = \mathcal{V}_p \oplus \mathcal{H}_p.$$

We denote the range of F_* by $\text{range} F_{*p}$ at $p \in M$ and consider the orthogonal complementary space $(\text{range} F_{*p})^{\perp}$ to $\text{range} F_{*p}$ in the tangent space $T_{F(p)} N$ of N at $p \in M$. Since $\text{rank} F \leq \min\{m, n\}$, we have $(\text{range} F_*)^{\perp} \neq \{0\}$. Thus the tangent space $T_{F(p)} N$ of N at $F(p) \in N$ has the following decomposition

$$T_{F(p)} N = (\text{range} F_{*p}) \oplus (\text{range} F_{*p})^{\perp}.$$

Now, a smooth map $F : (M_1^m, g_M) \rightarrow (M_2^n, g_N)$ is called Riemannian map at $p_1 \in M$ if the horizontal restriction $F_{*p_1}^h : (\ker F_{*p_1})^{\perp} \rightarrow (\text{range} F_{*p_1})$ is a linear isometry between the inner product spaces

$$((\ker F_{*p_1})^{\perp}, g_M(p_1)) |_{(\ker F_{*p_1})^{\perp}}$$

and

$$(\text{range}F_{*p_1}, g_N(p_2) |_{(\text{range}F_{*p_1})}),$$

$p_2 = F(p_1)$. Therefore Fischer stated in [12] that a Riemannian map is a map which is as isometric as it can be. In another words, F_* satisfies the equation

$$(1.5) \quad g_N(F_*X, F_*Y) = g_M(X, Y)$$

for X, Y vector fields tangent to \mathcal{H} . It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\text{range}F_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion [12] and this fact implies that the rank of the linear map $F_{*p} : T_pM \rightarrow T_{F(p)}N$ is constant for p in each connected component of M , [2] and [12]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics, [12].

The first aim of this paper is to extend Nomizu-Yano's result to the Riemannian maps. The second aim of this paper is to introduce notion of Clairaut Riemannian maps, give an example and obtain characterizations. Thus we are going to show that one can investigate the geometry of a Riemannian map itself, and domain manifold and target manifold of a Riemannian map by using circles and Clairaut maps. Because Riemannian maps include isometric immersions and Riemannian submersions as subclasses, the topics studied in this paper have potential for further research.

2. Preliminaries

In this section, we recall fundamental formulas for Riemannian maps similar to the Gauss-Weingarten formulas of isometric immersions. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \rightarrow N$ is a smooth map between them. Then the differential F_* of F can be viewed as a section of the bundle $\text{Hom}(TM, F^{-1}TN) \rightarrow M$, where $F^{-1}TN$ is the pullback bundle whose fibres at $p \in M$ is $(F^{-1}TN)_p = T_{F(p)}N, p \in M$. The bundle $\text{Hom}(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connections ∇^F . Then the second fundamental form of F is given by

$$(2.1) \quad (\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X^M Y)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. First note that in [23] we showed that the second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((\ker F_*)^\perp)$, of a Riemannian map has no components in $\text{range}F_*$. More precisely, we have

$$(2.2) \quad (\nabla F_*)(X, Y) \in \Gamma((\text{range}F_*)^\perp), \forall X, Y \in \Gamma((\ker F_*)^\perp).$$

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (N, g_N) and of its pullback along F . Then according to [20], for any vector field X on M and any section V of $(\text{range}F_*)^\perp$, where $(\text{range}F_*)^\perp$ is the subbundle of $F^{-1}(TN)$ with fiber $(F_*(T_pM))^\perp$ -orthogonal complement of

$F_*(T_pM)$ for g_N over p , we have $\nabla_X^{F^\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. In [20], the author also showed that ∇^{F^\perp} is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F^\perp} g_N = 0$. We now define \mathcal{S}_V as

$$(2.3) \quad \nabla_X^2 V = -\mathcal{S}_V F_* X + \nabla_X^{F^\perp} V,$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla_X^2 V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and $F_* X$ and $\mathcal{S}_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$(2.4) \quad g_N(\mathcal{S}_V F_* X, F_* Y) = g_N(V, (\nabla F_*)(X, Y))$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that \mathcal{S}_V is a symmetric linear transformation of $\text{range } F_*$.

Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N) . Then we define \mathcal{A} and \mathcal{T} as

$$(2.5) \quad \mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \quad \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields E, F on M , where ∇ is the Levi-Civita connection of g_M , \mathcal{V} and \mathcal{H} denote the projections to vertical subbundle and horizontal subbundle, respectively. In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM)$, \mathcal{T}_E and \mathcal{A}_E are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that \mathcal{T} is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and \mathcal{A} is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field \mathcal{T} satisfies

$$(2.6) \quad \mathcal{T}_U W = \mathcal{T}_W U, \quad \forall U, W \in \Gamma(\ker F_*).$$

On the other hand, from (2.5) we have

$$(2.7) \quad \nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W,$$

$$(2.8) \quad \nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V,$$

$$(2.9) \quad \nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

Umbilical maps have been defined in [20] and [25]. In fact, in [20] the author gave many definitions of umbilical maps with respect to the Riemannian metrics of the source manifolds and target manifolds. But we note that the definition given in [25] is same with the definition of g -umbilicity map given in [20]. We recall that a map $F : (M^m, g_M) \rightarrow (N^n, g_N)$ between Riemannian manifolds is called umbilical ([25]) if

$$(2.10) \quad \nabla F_* = \frac{1}{m} g_M \otimes \tau,$$

where τ denotes the tension field of the map F . In [24], we showed that every umbilical Riemannian map is totally geodesic, it means that this notion does not work for Riemannian maps. Therefore we presented the following definition for Riemannian maps.

Definition 2.1 ([24]). Let F be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N) . Then we say that F is an umbilical Riemannian map at $p_1 \in M$ if

$$(2.11) \quad \mathcal{S}_V F_* p_1(X_{p_1}) = \lambda F_* p_1(X_{p_1})$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$, where λ is a differentiable function on M . If F is umbilical for every $p_1 \in M$, then we say that F is an umbilical Riemannian map.

We also showed that a Riemannian map F is an umbilical Riemannian map if and only if

$$(2.12) \quad (\nabla F_*)(X, Y) = g_M(X, Y)H_2$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, where H_2 is the mean curvature vector field of the distribution $\text{range } F_*$.

3. A characterization of Riemannian maps in terms of circles

Let $F : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian map and $\alpha : I \rightarrow M$ a curve parameterized by its arclength. Then we say that α is a horizontal curve if $\dot{\alpha}(t) \in \Gamma(\ker F_{*\alpha(t)})^\perp$ for any $t \in I$. In this section, we are going to prove the following theorem.

Theorem 3.1. *Let F be a Riemannian map from a connected Riemannian manifold (M, g_M) , $\dim M \geq 2$, to a Riemannian manifold (N, g_N) . For some $\kappa > 0$, let α be a horizontal circle of radius κ on M , then F is umbilical and the mean curvature vector field H_2 is parallel if and only if for every horizontal circle α on M the curve $F \circ \alpha$ is a circle on N .*

Proof. Suppose that $p \in M$ and $\alpha(s)$, $|s| < \varepsilon$, is a horizontal circle on M . Then $F \circ \alpha : I \rightarrow N$ is also a curve and for each given vector field X_s along α , we can define a vector field $F_* X$ along $F \circ \alpha$ by

$$(3.1) \quad (F_* X)(s) = F_{*\alpha(s)} X(s),$$

here s is the arc length parameter and the vector field X_s is always the unit tangent vector field along α . Now suppose that $F \circ \alpha$ is a circle on N . Then from (1.4) we have

$$(3.2) \quad (\nabla_{X_s}^F)^2 F_*(X_s) + g_N(\nabla_{X_s}^F F_*(X_s), \nabla_{X_s}^F F_*(X_s)) F_*(X_s) = 0.$$

On the other hand, using (2.1), (2.2) and (2.3) we derive

$$(3.3) \quad \begin{aligned} (\nabla_{X_s}^F)^2 F_*(X_s) = & -\mathcal{S}_{(\nabla F_*)(X_s, X_s)} F_*(X_s) + \nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) \\ & + (\nabla F_*)(X_s, \nabla_{X_s}^1 X_s) + F_*(\nabla_{X_s}^1 X_s), \end{aligned}$$

where ∇^1 denotes the Levi-Civita connection on M . Substituting (2.1) and (3.3) in (3.2) and using (1.2) and (1.3) we obtain

$$(\nabla F_*)(X_s, \nabla_{X_s}^1 X_s) + g_N((\nabla F_*)(X_s, X_s), (\nabla F_*)(X_s, X_s)) F_*(X_s)$$

$$(3.4) \quad -\mathcal{S}_{(\nabla F_*)(X_s, X_s)} F_*(X_s) + \nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) = 0$$

due to α is a horizontal circle. By looking at $(\text{range} F_*)$ and $(\text{range} F_*)^\perp$ components of (3.4) we have

$$(3.5) \quad (\nabla F_*)(X_s, \nabla_{X_s}^1 X_s) + \nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) = 0$$

and

$$(3.6) \quad -\mathcal{S}_{(\nabla F_*)(X_s, X_s)} F_*(X_s) + g_N((\nabla F_*)(X_s, X_s), (\nabla F_*)(X_s, X_s)) F_*(X_s) = 0.$$

We now define

$$(\tilde{\nabla}_X (\nabla F_*))(Y, Z) = \nabla_X^\perp (\nabla F_*)(Y, Z) - (\nabla F_*)(\nabla_X^1 Y, Z) - (\nabla F_*)(Y, \nabla_X^1 Z)$$

for $X, Y, Z \in \Gamma(TM)$. Then we can write

$$(3.7) \quad (\tilde{\nabla}_{X_s} (\nabla F_*))(X_s, X_s) = \nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) - 2(\nabla F_*)(\nabla_{X_s}^1 X_s, X_s).$$

Using (3.7) in (3.5) we arrive at

$$(3.8) \quad 3(\nabla F_*)(\nabla_{X_s}^1 X_s, X_s) = -(\tilde{\nabla}_{X_s} (\nabla F_*))(X_s, X_s).$$

Thus we get

$$(3.9) \quad (\nabla F_*)(X, Y) = -\frac{1}{3\kappa} (\tilde{\nabla}_{X_s} (\nabla F_*))(X_s, X_s), \forall X, Y \in \Gamma((\ker F_*)^\perp).$$

This equation shows that given a unit vector $X \in T_p M$, $(\nabla F_*)(X, Y)$ does not depend on a unit vector $Y \in T_p M$ provided Y is orthogonal to X . Changing Y into $-Y$, we have

$$(3.10) \quad (\nabla F_*)(X, Y) = 0.$$

On the other hand, $\frac{1}{\sqrt{2}}(X + Y)$, $\frac{1}{\sqrt{2}}(X - Y)$ are orthogonal, hence we derive

$$(3.11) \quad (\nabla F_*)(\frac{1}{\sqrt{2}}(X + Y), \frac{1}{\sqrt{2}}(X - Y)) = 0.$$

Since ∇F_* is linear, we obtain

$$(3.12) \quad (\nabla F_*)(X, X) = (\nabla F_*)(Y, Y).$$

Now let $\{X_1, \dots, X_n\}$ be an orthonormal basis in $\Gamma((\ker F_*)^\perp)$, then we have

$$(\nabla F_*)(X_1, X_1) = (\nabla F_*)(X_2, X_2) = \dots$$

Thus we have

$$H_2 = (\nabla F_*)(X_1, X_1).$$

Moreover, choosing $X = \sum a_i X_i$, $Y = \sum b_j X_j$, we get

$$(\nabla F_*)(X, Y) = \sum_{i,j} a_i b_j (\nabla F_*)(X_i, X_j) = g_M(X, Y) H_2,$$

which shows that F is umbilical. On the other hand, from (3.5) we have

$$\nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) = 0$$

due to $(\nabla F_*)(X_s, \nabla_{X_s}^1 X_s) = 0$. This implies that H_2 is parallel. Conversely suppose that F is an umbilical Riemannian map and H_2 is parallel. From (2.1) we have

$$g_N(\nabla_{X_s}^F F_*(X_s), \nabla_{X_s}^F F_*(X_s)) = g_N((\nabla F_*)(X_s, X_s), (\nabla F_*)(X_s, X_s)) + g_N(F_*(\nabla_{X_s}^1 X_s), F_*(\nabla_{X_s}^1 X_s)).$$

Then Riemannian map F and (2.12) imply that

$$(3.13) \quad g_N(\nabla_{X_s}^F F_*(X_s), \nabla_{X_s}^F F_*(X_s)) = \|H_2\|^2 + g_M(\nabla_{X_s}^1 X_s, \nabla_{X_s}^1 X_s).$$

On the other hand, since F is an umbilical Riemannian map and H_2 is parallel, we have

$$\nabla_{X_s}^\perp (\nabla F_*)(X_s, X_s) = 0 \quad \text{and} \quad (\nabla F_*)(X_s, \nabla_{X_s}^1 X_s) = 0.$$

Then using (3.3) and (2.12) we get

$$(3.14) \quad (\nabla_{X_s}^F)^2 F_*(X_s) = -\|H_2\|^2 F_*(X_s) + F_*(\nabla_{X_s}^1 X_s).$$

Thus from (3.13) and (3.14) we obtain

$$\begin{aligned} & (\nabla_{X_s}^F)^2 F_*(X_s) + g_N(\nabla_{X_s}^F F_*(X_s), \nabla_{X_s}^F F_*(X_s)) F_*(X_s) \\ &= F_*(\nabla_{X_s}^1 X_s) + g_M(\nabla_{X_s}^1 X_s, \nabla_{X_s}^1 X_s) F_*(X_s). \end{aligned}$$

Since α is a circle on M , using (1.1) and (1.2) we arrive at

$$(\nabla_{X_s}^F)^2 F_*(X_s) + g_N(\nabla_{X_s}^F F_*(X_s), \nabla_{X_s}^F F_*(X_s)) F_*(X_s) = 0$$

which shows that $\gamma = F \circ \alpha$ is a circle on N . □

We recall that a diffeomorphism $F : (M, g_M) \rightarrow (N, g_N)$ between two Riemannian manifolds (M, g_M) and (N, g_N) is called concircular if it maps circles in M to circles in N . This notion was defined by Yano [27] and independently by Fialkow in the more general concept of conformal geodesics [11]. The original definition by Yano required a priori that F is conformal. But Vogel [26] obtained the following result.

Theorem 3.2 ((Vogel) [26], [16, p. 112]). *Every concircular diffeomorphism is necessarily conformal.*

Remark 1. Above theorem shows that a diffeomorphism F between Riemannian manifolds (M, g_M) and (N, g_N) which preserves circles is conformal. We note that the Riemannian map F in Theorem 3.1 can not be a diffeomorphism. Indeed, if F is a diffeomorphism which is a local diffeomorphism, then the inverse function theorem implies that $F_{*p} : T_p M \rightarrow T_{F(p)} N$, $p \in M$ is a linear isomorphism. It means that F_{*p} is a bijection which is not the case for a Riemannian map.

4. Clairaut Riemannian maps

As we mentioned in introduction, the notion of Clairaut Riemannian submersion was defined by Bishop. According to his definition, a submersion $F : M \rightarrow N$ is a Clairaut submersion if there is a function $r : M \rightarrow \mathbb{R}^+$ such that for every geodesic, making angles θ with the horizontal subspaces, $r \sin \theta$ is constant. He found the following characterization.

Theorem 4.1 ([6, 10]). *Let $F : (M, g_M) \rightarrow (B, g)$ be a Riemannian submersion with connected fibers. Then F is a Clairaut submersion with $r = e^f$ if and only if each fiber is totally umbilical and has mean curvature vector field $H = -\text{grad}f$.*

We now present the notion of Clairaut Riemannian maps as follows:

Definition 4.1. A Riemannian map $F : M \rightarrow N$ between Riemannian manifolds (M, g_M) and (N, g_N) is called a Clairaut Riemannian map if there is a function $r : M \rightarrow \mathbb{R}^+$ such that for every geodesic, making angles θ with the horizontal subspaces, $r \sin \theta$ is constant.

As we have seen above, the definition involves the notion of geodesic. Therefore we are going to find necessary and sufficient conditions for a curve on M to be a geodesic. From (3.1) and (2.7)-(2.9) we obtain the following conditions.

Corollary 4.1. *Let $F : M \rightarrow N$ be a Riemannian map. If $c : I \rightarrow M$ is a regular curve and U and X denote the vertical and the horizontal components of its tangent vector field, then c is a geodesic on M if and only if*

$$\hat{\nabla}_U U + \mathcal{T}_U X + \mathcal{V}\nabla_X U = 0$$

and

$$\nabla_X^F F_*(X) = -F_*(\mathcal{T}_U U + 2\mathcal{A}_X U) + (\nabla F_*)(X, X) = 0.$$

From Corollary 4.1, we have the following result.

Corollary 4.2. *Let $F : M \rightarrow N$ be a Riemannian map and $c : I \rightarrow M$ a geodesic with $U(t) = \mathcal{V}\dot{c}(t)$ and $X(t) = \mathcal{H}\dot{c}(t)$. Then the curve $\beta = F \circ c$ is a geodesic on N if and only if*

$$\mathcal{T}_U X + 2\mathcal{A}_X U = 0, (\nabla F_*)(X, X) = 0.$$

Proof. Since

$$(\nabla F_*)(X, X) \in \Gamma((\text{range} F_{*p})^\perp) \text{ and } F_*(\mathcal{T}_U U + 2\mathcal{A}_X U) \in \Gamma(\text{range} F_{*p}),$$

the assertion follows from Corollary 4.1. \square

We also have the following result.

Corollary 4.3. *The projection on N of a horizontal geodesic on M is a geodesic if and only if*

$$(\nabla F_*)(X, X) = 0, \quad X \in \Gamma((\ker F_{*\alpha(t)})^\perp).$$

We note that the assertion of Corollary 4.3 is valid for a Riemannian submersion without any condition.

Moreover we have the following result.

Theorem 4.2. *Let $F : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian map with connected fibers. Then F is a Clairaut Riemannian map with $r = e^f$ if and only if each fiber is totally umbilical and has mean curvature vector field $H = -\text{grad}f$.*

Proof. Let $c : I \rightarrow M$ be a geodesic on M with $U(t) = \mathcal{V}\dot{c}(t)$ and $X(t) = \mathcal{H}\dot{c}(t)$ and let $\omega(t)$ denote the angle in $[0, \pi]$ between $\dot{c}(t)$ and $X(t)$. Putting $a = \|\dot{c}(t)\|^2$, one can obtain

$$(4.1) \quad g_{c(t)}(X(t), X(t)) = a \cos^2 \omega(t), \quad g_{c(t)}(U(t), U(t)) = a \sin^2 \omega(t).$$

Thus, by considering the first relation of (4.1) and taking the derivative of it with respect to t , we get

$$(4.2) \quad \frac{d}{dt}g_{c(t)}(X(t), X(t)) = -2a \cos \omega(t) \sin \omega(t) \frac{d\omega(t)}{dt}.$$

On the other hand, since F is a Riemannian map, using (2.1) we have, along $c(t)$,

$$\frac{d}{dt}g_{c(t)}(X, X) = 2g_N(-(\nabla F_*)(\dot{c}, X) + \nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Since the second fundamental form of F is linear, from (2.2) we derive

$$\frac{d}{dt}g_{c(t)}(X, X) = 2g_N(-(\nabla F_*)(U, X) + \nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Then (2.1) and Riemannian map F imply

$$\frac{d}{dt}g_{c(t)}(X, X) = 2g_M(\nabla_X U, X) + 2g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Using (2.8) we obtain

$$\frac{d}{dt}g_{c(t)}(X, X) = 2g_M(\mathcal{A}_X U, X) + 2g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Thus skew-symmetric \mathcal{A} implies that

$$\frac{d}{dt}g_{c(t)}(X, X) = -2g_M(U, \mathcal{A}_X X) + 2g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Hence we obtain

$$(4.3) \quad \frac{d}{dt}g_{c(t)}(X, X) = 2g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)).$$

Then from (4.2) and (4.3) we have

$$(4.4) \quad g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)) = -a \cos \omega(t) \sin \omega(t) \frac{d\omega(t)}{dt}.$$

By direct computations, F is a Clairaut Riemannian map with $r = e^f$ if and only if $\frac{d}{dt}(e^{f \circ c} \sin \omega) = 0$. Multiplying this with the nonzero factor $a \sin \omega(t)$, we get

$$(4.5) \quad -a \sin \omega \cos \omega \frac{d\omega}{dt} = \frac{df}{dt} a \sin^2 \omega.$$

Thus from (4.1), (4.4) and (4.5) we find

$$(4.6) \quad g_N(\nabla_{\dot{c}}^F F_*(X), F_*(X)) = \frac{df}{dt} g_M(U, U).$$

Since $c(t)$ is geodesic on M , from the second equation of Corollary 4.1. we have

$$g_N(-F_*(\mathcal{T}_U U + 2\mathcal{A}_X U) + (\nabla F_*)(X, X), F_*(X)) = \frac{df}{dt} g_M(U, U).$$

Then Riemannian map F and (2.2) imply

$$-g_M(\mathcal{T}_U X + 2\mathcal{A}_X U, X) = \frac{df}{dt} g_M(U, U).$$

Hence we obtain

$$g_M(\mathcal{T}_U U, X) = \frac{df}{dt} g_M(U, U).$$

The rest of this proof is same with the calculations given in [10, p. 30]. \square

We note that the above condition does not imply that the Riemannian map itself is totally umbilical contrary to the Riemannian submersions. We now give an example of Clairaut Riemannian maps.

Example 1. Let (B, g_B) and (F, g_F) be two Riemannian manifolds, $f : B \rightarrow (0, \infty)$ and $\pi_1 : B \times F \rightarrow B$, $\pi_2 : B \times F \rightarrow F$ the projection maps given by $\pi_1(p_1, p_2) = p_1$ and $\pi_2(p_1, p_2) = p_2$ for every $(p_1, p_2) \in B \times F$. The warped product ([7]) $M = B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$g(X, Y) = g_B(\pi_{1*} X, \pi_{1*} Y) + (f \circ \pi_1)^2 g_F(\pi_{2*} X, \pi_{2*} Y)$$

for every X and Y of M , where $*$ denotes the tangent map. The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the warped product manifold M is said to be trivial. It is known that the first projection $\pi_1 : B \times F \rightarrow B$ is a Riemannian submersion whose vertical and horizontal spaces at any point $p = (p_1, p_2)$ are respectively identified with $T_{p_2} F$, $T_{p_1} B$. Moreover the fibers of π_1 is totally umbilical with mean curvature vector field $H = -\frac{1}{2f} \text{grad} f$. We now consider the isometric immersion $\pi : B \rightarrow B \times_f F$, then the composite map $\pi \circ \pi_1$ is a Riemannian map. Moreover the projection π_1 and the map $\pi \circ \pi_1$ have the same vertical distribution. Hence $\pi \circ \pi_1$ is a Clairaut Riemannian map with $r = \sqrt{f}$.

We also have another characterization.

Corollary 4.4. *Let $F : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian map with connected fibers. Let $c : I \rightarrow M$ be a geodesic on M with $U(t) = \mathcal{V}\dot{c}(t)$ and $X(t) = \mathcal{H}\dot{c}(t)$. Then F is a Clairaut Riemannian map with $r = e^f$ if and only if*

$$g_N(\nabla_{\dot{c}(t)}^F F_*(X), F_*(X)) = \frac{df}{dt} g_M(U, U).$$

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