Bull. Korean Math. Soc. **54** (2017), No. 1, pp. 225–242
https://doi.org/10.4134/BKMS.b160070
pISSN: 1015-8634 / eISSN: 2234-3016

STRUCTURE OF APÉRY-LIKE SERIES AND MONOTONICITY PROPERTIES FOR BINOMIAL SUMS

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Abstract. A family of Apéry-like series involving reciprocals of central binomial coefficients is studied and it is shown that they represent transcendental numbers. The structure of such series is further examined in terms of finite combinations of logarithms and arctangents with arguments and coefficients belonging to a suitable algebraic extension of rationals. Monotonicity of certain quotients of weighted binomial sums which arise in the study of competitive cheap talk models is established with the help of a continuous extension of the discrete model at hand. The monotonic behavior of such quotients turns out to have important applications in game theory.

1. Introduction

This paper is devoted to a study of sums and series involving binomial coefficients. In the first part, we consider a family of series concerning reciprocals of central binomial coefficients and establish structural properties of them in the context of Mahler's classification of transcendental numbers. We are then led to representations of such series as finite combinations of logarithms and arctangents with arguments and coefficients belonging to a specified algebraic extension of $\mathbb{Q}$. This motivates us to look at more general finite combinations of logarithms and arctangents with algebraic arguments and coefficients, and investigate their structure from a similar point of view. In the second part, we look at certain quotients of weighted binomial sums which arise naturally in the study of competitive cheap talk models belonging to game theory. The desired monotonicity of such quotients which has potential applications in game theory is obtained by introducing a continuous extension of the discrete model at hand. In particular, an analytic function interpolating these quotients at positive integers is shown to exist and then one is able to infer the stronger result that this function is monotonic on all of its domain. Fast converging series using reciprocals of binomial coefficients became a central topic of research

Received January 26, 2016; Revised June 21, 2016.
2010 *Mathematics Subject Classification.* 11B65, 11J82, 05A10.
Key words and phrases. Apéry-like series, transcendental number, central binomial coefficient, monotonicity, competitive cheap talk.

after Apéry's seminal work [3] on the irrationality of $\zeta(3)$, where

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

is the Riemann zeta function for $\Re(s)>1$. The series

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\binom{2n}{n}n^3}$$

was the starting point of Apéry in his proof. For other forms of these series arising from Ramanujan's notebooks such as

$$\zeta(3)=\frac{7\pi^3}{180}-2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)},$$

the reader is referred to the works of Berndt [5], [6], especially Chap. 14 of [7]. Another striking example

$$\frac{\zeta(2)}{3}=\sum_{n=1}^{\infty}\frac{1}{\binom{2n}{n}n^2}$$

represents the transcendental number $\frac{\pi^2}{18}$. The author [1] found new families of rapidly converging series converging to the special values of L-functions and to Catalan's constant defined as

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Apart from these sporadic examples, our first result is concerned with the study of an infinite family of Apéry-like series and their structure in terms of Mahler's approach to classifying real numbers. Mahler [10] originally partitioned real numbers into four subsets taking into account the accuracy with which nonzero polynomials with integer coefficients approximate zero when evaluated at a given real number. These four subsets are called A, S, T and U-numbers, where A-numbers correspond to the set of real algebraic numbers and S, T and U-numbers form a partition of the real transcendental numbers. To be precise, for a given positive integer n and real number $H \geq 1$, define

$$w_n(\xi,H):=\min\{|P(\xi)| \ : \ P(x)\in\mathbb{Z}[x], H(P)\leq H, \deg(P)\leq n, P(\xi)\neq 0\},$$

where $\deg(P)$ and $H(P)$ denote the degree and the height of the polynomial, namely the maximum of moduli of its coefficients. Based on this, Mahler's classification makes use of the quantities

$$w_n(\xi):=\limsup_{H\to\infty}\frac{-\log w_n(\xi,H)}{\log H} \quad\text{and}\quad w(\xi):=\limsup_{n\to\infty}\frac{w_n(\xi)}{n}.$$

A transcendental number is a U-number if $w_n(\xi)=\infty$ from some n onwards. Historically, the very first examples of transcendental numbers, namely the

Liouville numbers, were among the U-numbers and one can say that transcendental numbers that are not U-numbers are harder to deal with. For some recent results and literature on this topic, see [2]. We are now ready to give the statement.

Theorem 1. *Let $\{\epsilon_1, \ldots, \epsilon_k\}$ be all kth roots of i. Consider the field*

$$F = \mathbb{Q}(\sqrt{1-4\epsilon_1}, \sqrt{1-4\overline{\epsilon_1}}, \ldots, \sqrt{1-4\epsilon_k}, \sqrt{1-4\overline{\epsilon_k}}) \cap \mathbb{R}.$$

Then for $k \geq 1$,

$$M_k = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{4kn-2k}{2kn-k}(2n-1)(4kn-2k+1)}$$

is transcendental but not a U-number with $w_n(M_k) = O_k(n^{2k+2})$ for $n \geq 1$, where the implied constant depends only on k. Moreover, M_k can be written as an $F^{\times}$-linear combination of k logarithms and $2k$ arctangents of numbers that are also in $F^{\times}$, where the logarithmic terms are all nonvanishing and $F^{\times} = F - \{0\}$. Consequently,

$$\lambda = \pi\sqrt{\frac{1+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-1}{2}}\, \log\left(\frac{1+\sqrt{17}+\sqrt{2+2\sqrt{17}}}{4}\right)$$
$$-\sqrt{2+2\sqrt{17}}\, \arctan\left(\frac{1+\sqrt{17}+\sqrt{2+2\sqrt{17}}}{4}\right)$$

is transcendental but not a U-number with $w_n(\lambda) = O(n^4)$ for $n \geq 1$.

Motivated by Theorem 1, consider sums formed with algebraic combinations of logarithms and arctangents. The result below generalizes and improves Theorem 3 of [2].

Theorem 2. *Let α_j, β_j for $1 \leq j \leq r$ and μ_j, γ_j for $1 \leq j \leq m$ be positive algebraic numbers such that $\gamma_j > 1$ for all j and β_j's are distinct. Then*

$$K_{r,m} = \sum_{j=1}^{r} \alpha_j \arctan \beta_j + \sum_{j=1}^{m} \mu_j \log \gamma_j$$

is transcendental but is not a U-number with

$$w_n(K_{r,m}) = O_{\alpha_j,\beta_j,\mu_j,\gamma_j}(n^{2r+m+2})$$

if no $\beta_j > 1$,

$$w_n(K_{r,m}) = O_{\alpha_j,\beta_j,\mu_j,\gamma_j}(n^{2r+m+4})$$

if some β_j's are > 1 but no $\beta_j = 1$ and

$$w_n(K_{r,m}) = O_{\alpha_j,\beta_j,\mu_j,\gamma_j}(n^{2r+m+2})$$

if some β_j's are > 1 and exactly one $\beta_j = 1$. If $\alpha \neq 0$, $\beta \neq 0$, $\mu \neq 1$, $\mu > 0$ are real algebraic numbers, then

$$L_1 = \alpha \arctan \beta + \log \mu$$

is transcendental but not a U-number with $w_n(L_1) = O_{\alpha,\beta,\mu}(n^5)$ *if* $|\beta| \leq 1$ *and* $w_n(L_1) = O_{\alpha,\beta,\mu}(n^7)$ *if* $|\beta| > 1$*. Moreover, if* $\alpha \neq 0$ *and* $\beta \notin \mathbb{Q}$ *are real algebraic numbers, then*

$$L_2 = \beta\pi + \arctan\alpha$$

is transcendental but not a U-number with $w_n(L_2) = O_{\alpha,\beta}(n^6)$.

Defining $(a)_n = a(a+1)\cdots(a+n-1)$, the family of series

$$N_m = -2m^2 + \sum_{n=1}^{\infty} \frac{(n-1)!\left(\frac{1}{m}\right)_n}{\left(\frac{1}{m}\right)_{2n+1}}$$

was studied in [2] and it was shown that N_m can be written as an algebraic combination of π and m logarithms of algebraic numbers. Thus taking $r = 1$ in the case when no $\beta_j > 1$, one infers from Theorem 2 above, the better bound $w_n(N_m) = O_m(n^{m+4})$ for all $m \geq 1$ and $n \geq 1$. Similarly, for the numbers

$$\gamma = \frac{\pi}{2} + \sqrt{3}\ \log(2+\sqrt{3})$$

and

$$\eta = \frac{\pi}{\sqrt{3}}\cos\left(\frac{2\pi}{9}\right) - \cos\left(\frac{\pi}{9}\right)\log\sin\left(\frac{\pi}{18}\right) + \cos\left(\frac{2\pi}{9}\right)\log\cos\left(\frac{\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right)\log\cos\left(\frac{2\pi}{9}\right)$$

that appear in Corollary 1 of [2], the improvements $w_n(\gamma) = O(n^5)$ and $w_n(\eta) = O(n^7)$ follow from Theorem 2.

Our next problem arises in competitive cheap talk models of game theory. This is done in the setting of simultaneous communication with $n \geq 2$ agents. One can then focus on the symmetric communication equilibrium, where the agents use the same message set and ties are broken in a random and symmetric manner among the agents. As a result of the indifference equation for equilibrium, one is naturally led to analyze certain quotients of weighted binomial sums in the form

$$\frac{\sum_{j=1}^{n}\binom{n}{j}P(m_n)^j P(m < m_n)^{n-j}\frac{j}{1+j}}{\sum_{j=1}^{n}\binom{n}{j}P(m_{n+1})^j P(m < m_{n+1})^{n-j}\frac{1}{1+j}},$$

where $P(-)$ denotes the probability of a certain event. For details, let us refer to the paper of Li, Rantakari and Young [9] (see also [11]). The strict monotonicity of the above quotient, as the number of agents increases, becomes important since it is possible to show as a consequence of this that, in the case of symmetric equilibrium, the incremental step size of partitions decreases as n increases (see Proposition 10 in [9]). Here we show that such quotients are indeed strictly decreasing even with more general parameters not necessarily resulting from probabilities. It turns out that one can find a strictly decreasing analytic function interpolating these quotients at integers. Hence there is a universal behavior for the quotients not directly related to game theoretic considerations.

Theorem 3. *Let b, d, y be positive real numbers with $d < y$ and let $a = b/y$, $c = d/y$. Then there exists an analytic function $S : (1, \infty) \to \mathbb{R}$ depending only on a and c such that $S(x)$ is strictly decreasing for $x > 1$ and*

$$S(n+1) = \frac{\sum_{j=1}^{n} \binom{n}{j} d^j (y-d)^{n-j} \frac{j}{1+j}}{\sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \frac{1}{1+j}}$$

for all positive integers n.

As a further application of our approach to Theorem 3, it is possible to give the following variation on an elegant problem of Erdös (see p. 10 of [13]) which serves as a converse to L'Hospital's rule under suitable growth conditions on the derivatives.

Theorem 4. *Let f be a $\mathcal{C}^\infty$ function defined for $x > 1$ and assume that*

$$|f^{(n+1)}(x)| = O_n(|f^{(n)}(x)|)$$

holds for all $n \geq 1$ and large enough x, where the implied constant depends only on n. If

$$\lim_{x\to\infty} \frac{f(x)}{e^x} = 1,$$

then

$$\lim_{x\to\infty} \frac{f^{(n)}(x)}{e^x} = 1$$

for all $n \geq 0$.

2. A preliminary result

For the proof of Theorem 3, we will need the following.

Theorem 5. *Assume that $f = f(x)$ and $g = g(x)$ are functions that are continuous for $x \geq x_0$ and differentiable for $x > x_0$ with $f(x_0) = 0 = g(x_0)$. If g and g' are positive and $\frac{f'}{g'}$ is strictly decreasing for $x > x_0$, then $\frac{f}{g}$ is also strictly decreasing for $x > x_0$.*

Proof. Since f and g are differentiable for $x > x_0$ and g is positive for $x > x_0$, $\frac{f}{g}$ is also differentiable for $x > x_0$ and it suffices to show that

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} < 0$$

for $x > x_0$. Noting that f and g are continuous for $x \geq x_0$ and differentiable for $x > x_0$, we may apply Cauchy's mean value theorem (see Chap. 4 of [12]) on any interval of the form $[x_0, x]$ with $x_0 < x$ to see that

$$(f(x) - f(x_0))g'(\lambda) = (g(x) - g(x_0))f'(\lambda)$$

for some $x_0 < \lambda < x$. Using $f(x_0) = 0 = g(x_0)$, we have

$$\frac{f(x)}{g(x)} = \frac{f'(\lambda)}{g'(\lambda)}$$

and since $\frac{f'}{g'}$ is strictly decreasing for $x > x_0$, the claim follows from

$$\frac{f'(x)}{g'(x)} < \frac{f'(\lambda)}{g'(\lambda)}$$

as g and g' are positive for $x > x_0$. □

3. Proof of Theorem 4

First let us write $f(x) = (1 + g(x))e^x$ with $\lim_{x\to\infty} g(x) = 0$. We have

$$g'(x) = \frac{f'(x) - f(x)}{e^x}, \tag{3.1}$$

$$f'(x) = (1+g(x)+g'(x))e^x \quad \text{and} \quad f''(x) = (1+g(x)+2g'(x)+g''(x))e^x. \tag{3.2}$$

By hypothesis, $|f''(x)| < c|f'(x)|$ when x is large enough, say for $x > a > 1$, where $c > 0$ is a constant. Thus $f'(x)$ is never zero when $x > a$ and (3.2) gives

$$|1 + g(x) + 2g'(x) + g''(x)| < c|1 + g(x) + g'(x)| \tag{3.3}$$

for $x > a$. By Cauchy's mean value theorem

$$(f'(x) - f'(a))f'(\lambda) = (f(x) - f(a))f''(\lambda) \tag{3.4}$$

follows with some $a < \lambda < x$. Using (3.4) and the fact that $f'(\lambda) \neq 0$, one obtains

$$|f'(x) - f'(a)| = \frac{|f''(\lambda)|}{|f'(\lambda)|}|f(x) - f(a)| < c|f(x) - f(a)|. \tag{3.5}$$

Since $f(x)/e^x$ is bounded as x tends to infinity, it is easy to see from (3.5) that $f'(x)/e^x$ is also bounded as x tends to infinity. Therefore, by (3.1), $g'(x)$ is bounded and we deduce from (3.3) that $g''(x)$ is bounded as x tends to infinity. Since $\lim_{x\to\infty} g(x) = 0$, using a classical result of Landau (see Chap. 4 of [12]), one has $\lim_{x\to\infty} g'(x) = 0$. This shows as a result of (3.2) that

$$\lim_{x\to\infty} \frac{f'(x)}{e^x} = 1.$$

The general formula can now be shown by induction on the number of derivatives. We may assume that $g, g', \ldots, g^{(n)}$ are bounded as x tends to infinity and $g, g', \ldots, g^{(n-1)}$ all tend to zero as x tends to infinity. Rewriting the hypothesis

$$|f^{(n+1)}(x)| = O_n(|f^{(n)}(x)|),$$

one gets

$$\begin{aligned} &\left|1 + g(x) + \sum_{j=1}^{n} k_j g^{(j)}(x) + g^{(n+1)}(x)\right| \\ &< c_n \left|1 + g(x) + \sum_{j=1}^{n-1} m_j g^{(j)}(x) + g^{(n)}(x)\right| \end{aligned} \tag{3.6}$$

for large enough x, where $c_n > 0$ is a constant, k_j and m_j are positive integers. It follows at once from (3.6) that $g^{(n+1)}(x)$ is bounded as x tends to infinity. Since $g^{(n-1)}(x)$ tends to zero as x tends to infinity, applying again Landau's result, one infers that $g^{(n)}(x)$ tends to zero as x tends to infinity. This completes the induction and shows that

$$\lim_{x\to\infty} \frac{f^{(n)}(x)}{e^x} = 1$$

for all $n \geq 0$.

4. Proof of Theorem 3

First we have

$$\sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \frac{1}{1+j} = \sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \int_0^1 t^j \; dt = \int_0^1 \sum_{j=1}^{n} \binom{n}{j} (bt)^j y^{n-j} \; dt. \tag{4.1}$$

Noting that

$$\sum_{j=1}^{n} \binom{n}{j} (bt)^j y^{n-j} = (bt+y)^n - y^n \tag{4.2}$$

and combining (4.1) and (4.2)

$$\sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \frac{1}{1+j} = y^n \left(\int_0^1 \left(\frac{bt}{y} + 1 \right)^n \; dt - 1 \right) \tag{4.3}$$

follows. Similarly, one obtains

$$\begin{aligned} &\sum_{j=1}^{n} \binom{n}{j} d^j (y-d)^{n-j} \frac{j}{1+j} \\ &= \sum_{j=1}^{n} \binom{n}{j} d^j (y-d)^{n-j} \left(1 - \int_0^1 t^j \; dt \right) \\ &= y^n - (y-d)^n - \int_0^1 \sum_{j=1}^{n} \binom{n}{j} (td)^j (y-d)^{n-j} \; dt \\ &= y^n - \int_0^1 (y-td)^n \; dt \\ &= y^n \left(1 - \int_0^1 \left(1 - \frac{td}{y} \right)^n \; dt \right). \end{aligned} \tag{4.4}$$

By (4.3) and (4.4), we have

$$\frac{\sum_{j=1}^{n} \binom{n}{j} d^j (y-d)^{n-j} \frac{j}{1+j}}{\sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \frac{1}{1+j}} = \frac{1 - \int_0^1 (1-ct)^n \; dt}{\int_0^1 (1+at)^n \; dt - 1}, \tag{4.5}$$

where $a = b/y$ and $c = d/y$. From (4.5), we see that

$$\frac{\sum_{j=1}^{n} \binom{n}{j} d^j (y-d)^{n-j} \frac{j}{1+j}}{\sum_{j=1}^{n} \binom{n}{j} b^j y^{n-j} \frac{1}{1+j}} = \frac{a}{c}\left(\frac{(1-c)^{n+1} + c(n+1) - 1}{(1+a)^{n+1} - a(n+1) - 1}\right). \tag{4.6}$$

Motivated by (4.6), let us define the function $S : (1, \infty) \to \mathbb{R}$ by

$$S(x) := \frac{a}{c}\left(\frac{(1-c)^x + cx - 1}{(1+a)^x - ax - 1}\right) \tag{4.7}$$

for any $x > 1$. As a consequence of (4.6) and (4.7), $S(n+1)$ interpolates the quotient of weighted binomial sums in (4.6). Observe that $S(x)$ itself is a quotient of analytic functions so is analytic at all $x > 1$ as soon as $(1+a)^x - ax - 1$ is not zero. $(1+a)^x - ax - 1$ is indeed positive for all $x > 1$ as will be shown below. To this end, let $f(x) = (1-c)^x + cx - 1$ and $g(x) = (1+a)^x - ax - 1$ and note that $f(1) = 0 = g(1)$. We have $g'(x) = (1+a)^x \log(1+a) - a$ and $g'(1) = (1+a)\log(1+a) - a$ so that $g'(x) > g'(1)$ for $x > 1$. If $0 < a < 1$, then using $\log(1+a) > a - \frac{a^2}{2}$,

$$g'(1) = (1+a)\log(1+a) - a > 0 \tag{4.8}$$

follows for $0 < a < 1$. Moreover, $\frac{d}{da}((1+a)\log(1+a) - a) = \log(1+a) > 0$ for $a > 0$ and $(1+a)\log(1+a) - a$ is an increasing function of $a > 0$. Consequently, (4.8) holds for all $a > 0$ and this gives $g'(x) > 0$ for $x > 1$. Since $g(1) = 0$, by the mean value theorem, one verifies that g and g' are both positive for $x > 1$. Next consider the quotient

$$\frac{f'(x)}{g'(x)} = \frac{(1-c)^x \log(1-c) + c}{(1+a)^x \log(1+a) - a}. \tag{4.9}$$

To show that (4.9) is strictly decreasing for $x > 1$, it suffices to check that

$$f''(x)g'(x) - f'(x)g''(x) < 0 \tag{4.10}$$

for $x > 1$. As f', g' are continuous for $x \geq 1$ and differentiable for $x > 1$, by Cauchy's mean value theorem, one obtains

$$\frac{f'(x) - f'(1)}{g'(x) - g'(1)} = \frac{f''(\lambda)}{g''(\lambda)} \tag{4.11}$$

for some $1 < \lambda < x$. On the other hand,

$$\frac{f''(x)}{g''(x)} = \frac{(1-c)^x \log^2(1-c)}{(1+a)^x \log^2(1+a)} \tag{4.12}$$

is clearly a strictly decreasing function for $x \geq 1$. (4.11) and (4.12) give that

$$\frac{f'(x) - f'(1)}{g'(x) - g'(1)} > \frac{f''(x)}{g''(x)} \tag{4.13}$$

for $x > 1$. As $g'(x) - g'(1) > 0$ and $g''(x) > 0$, we may rewrite (4.13) in the form

$$f''(x)g'(x) - f'(x)g''(x) < f''(x)g'(1) - g''(x)f'(1). \tag{4.14}$$

As a result of (4.14), verifying (4.10) reduces to showing

$$\frac{f''(x)}{g''(x)} < \frac{f'(1)}{g'(1)} \tag{4.15}$$

for $x > 1$. But (4.12) is strictly decreasing for $x \geq 1$ so that (4.15) further reduces to showing

$$\frac{f''(1)}{g''(1)} < \frac{f'(1)}{g'(1)}. \tag{4.16}$$

It is easy to see that

$$f'(1) = (1-c)\log(1-c) + c > 0$$

for $0 < c < 1$. Thus (4.16) is equivalent to

$$\frac{(1-c)\log^2(1-c)}{(1-c)\log(1-c)+c} = \frac{f''(1)}{f'(1)} < \frac{g''(1)}{g'(1)} = \frac{(1+a)\log^2(1+a)}{(1+a)\log(1+a)-a}. \tag{4.17}$$

Finally, to obtain (4.17), consider the function

$$F(x) = \frac{x\log^2 x}{x\log x - x + 1}.$$

Note that $\lim_{x\to 1} F(x) = 2$. Therefore, F has a removable discontinuity at $x = 1$ and we may take $F(1) = 2$. Observe that (4.17) is further equivalent to $F(1-c) < F(1+a)$. To see this, we show that $F'(x) > 0$ for $0 < x < 1$ and for $x > 1$. First assume $0 < x < 1$ and consider

$$F'(x) = \frac{\log x((1+x)\log x + 2 - 2x)}{(x\log x - x + 1)^2}.$$

Since $\log x < 0$ for $0 < x < 1$, let us see that $u(x) = (1+x)\log x + 2 - 2x < 0$ for $0 < x < 1$. As $\lim_{x\to 0^+} u(x) = -\infty$, it is enough to check that $u'(x) = \log x + \frac{1+x}{x} - 2 > 0$ for $0 < x < 1$. But this is equivalent to $\log x + \frac{1}{x} > 1$. Letting $x = 1 - t$ with $0 < t < 1$,

$$\log(1-t) + \frac{1}{1-t} = -\left(t + \frac{t^2}{2} + \cdots\right) + 1 + t + t^2 + \cdots > 1$$

follows. Using $u(1) = 0$, we see that $u(x) < 0$ and $F'(x) > 0$ for $0 < x < 1$. Next assume $x > 1$. Then $\log x > 0$. In this case we show that $u(x) > 0$ for $x > 1$. It is enough to show that $u'(x) > 0$ for $x > 1$. Letting $x = 1/(1-y)$ with $0 < y < 1$, this is equivalent to

$$1 - y - \log(1-y) = 1 - y + y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots > 1.$$

This finishes the argument showing that $\frac{f'}{g'}$ is strictly decreasing for $x > 1$. Therefore, all of the hypotheses of Theorem 5 hold and one concludes that $\frac{f}{g}$ and $S(x)$ are strictly decreasing for $x > 1$. The proof of Theorem 3 is now complete.

5. Proof of Theorem 1

For $k \geq 1$, we have

$$\max_{0\leq x\leq 1} (x(1-x))^k = \frac{1}{4^k}.$$

Taking this into account, integrating by parts, using Taylor series of arctangent and interchanging the order of summation with integration which is easily seen to be permissible, one obtains

$$\begin{aligned}\int_0^1 x\frac{\frac{d}{dx}(x^k(1-x)^k)}{1+(x(1-x))^{2k}}\ dx &= -\int_0^1 \arctan(x^k(1-x)^k)\ dx \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\int_0^1 x^{k(2n-1)}(1-x)^{k(2n-1)}\ dx.\end{aligned} \tag{5.1}$$

Moreover, by the beta function identity

$$\int_0^1 x^{k(2n-1)}(1-x)^{k(2n-1)}\ dx = \frac{\Gamma(2kn-k+1)^2}{\Gamma(4kn-2k+2)} \tag{5.2}$$

follows, where

$$\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}\ dx$$

is the gamma function for $\Re(s) > 0$. Combining (5.1), (5.2) and evaluating the gamma function in terms of factorials, one derives the formula

$$\int_0^1 x\frac{\frac{d}{dx}(x^k(1-x)^k)}{1+(x(1-x))^{2k}}\ dx = \sum_{n=1}^{\infty}\frac{(-1)^n}{\binom{4kn-2k}{2kn-k}(2n-1)(4kn-2k+1)}. \tag{5.3}$$

Let $P(x) = (x(1-x))^k$ and note that

$$\frac{P'(x)}{1+P(x)^2} = \frac{i}{2}\left(\frac{P'(x)}{i+P(x)} - \frac{P'(x)}{-i+P(x)}\right). \tag{5.4}$$

(5.4) will be used for the evaluation of the left hand side of (5.3) in another way. Clearly, $1+P(x)^2$ has no real roots and the set of roots of $i+P(x)$ and $-i+P(x)$, each containing $2k$ elements, form a partition of the set of roots of $1+P(x)^2$. Roots of $-i+P(x)$ arise from pairs of roots of k quadratic equations given by $x^2-x+\epsilon_j = 0$ for $1 \leq j \leq k$, where ϵ_j is a kth root of i. Let ω_j, τ_j be the roots of $x^2-x+\epsilon_j$. Then $\omega_j\tau_j = \epsilon_j$ and $\omega_j+\tau_j = 1$. As $|\omega_j||\tau_j| = |\epsilon_j| = 1$, one can assume $|\omega_j| \geq 1$ for each j. It is easy to see that both of the roots of $x^2-x+\epsilon_j$ can not have modulus ≥ 1 and the choice of ω_j is unique in each case. If ω_j, τ_j, $1 \leq j \leq k$, range over all roots of $-i+P(x)$, then $\overline{\omega_j}, \overline{\tau_j}$, $1 \leq j \leq k$, range over all roots of $i+P(x)$. In the light of these observations, the left hand side of (5.3) becomes

$$\int_0^1 \frac{xP'(x)}{1+P(x)^2}\ dx = \frac{i}{2}\left(\int_0^1 \frac{xP'(x)}{i+P(x)}\ dx - \int_0^1 \frac{xP'(x)}{-i+P(x)}\ dx\right) \tag{5.5}$$

and furthermore, we have

$$\int_0^1 \frac{xP'(x)}{-i+P(x)}\,dx = \int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\omega_j}\right)\,dx + \int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\tau_j}\right)\,dx. \tag{5.6}$$

Expanding the terms $\frac{x}{x-\omega_j}$ in (5.6) as a geometric series and interchanging the order of summation with integration which is justified by the bounded convergence theorem, one infers

$$\int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\omega_j}\right)\,dx = k + \sum_{n=0}^{\infty} \frac{-\sum_{j=1}^k (\overline{\epsilon_j}\tau_j)^n}{n+1}. \tag{5.7}$$

Next using $\omega_j + \tau_j = 1$, we see that

$$\int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\tau_j}\right)\,dx = \sum_{j=1}^k \int_0^1 \frac{1-x}{\omega_j - x}\,dx. \tag{5.8}$$

As $|\omega_j| \geq 1$,

$$\frac{1-x}{\omega_j - x} = (1-x)\sum_{n=0}^{\infty} \frac{x^n}{\omega_j^{n+1}} = \sum_{n=0}^{\infty} (\overline{\epsilon_j}\tau_j)^{n+1} x^n - \sum_{n=0}^{\infty} (\overline{\epsilon_j}\tau_j)^{n+1} x^{n+1} \tag{5.9}$$

holds for $0 \leq x < 1$. Again termwise integration is justified, and assembling (5.8) and (5.9), one has

$$\begin{aligned} \int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\tau_j}\right)\,dx &= \sum_{j=1}^k \sum_{n=0}^{\infty} \frac{(\overline{\epsilon_j}\tau_j)^{n+1}}{n+1} - \sum_{j=1}^k \sum_{n=0}^{\infty} \frac{(\overline{\epsilon_j}\tau_j)^{n+1}}{n+2} \\ &= k + \sum_{n=0}^{\infty} \frac{\left(\sum_{j=1}^k (\overline{\epsilon_j}\tau_j)^{n+1} - (\overline{\epsilon_j}\tau_j)^n\right)}{n+1}. \end{aligned} \tag{5.10}$$

Gathering now (5.6), (5.7) and (5.10),

$$\int_0^1 \frac{xP'(x)}{-i+P(x)}\,dx = 2k + \sum_{n=0}^{\infty} \frac{\left(\sum_{j=1}^k (\overline{\epsilon_j}\tau_j)^{n+1} - 2(\overline{\epsilon_j}\tau_j)^n\right)}{n+1} \tag{5.11}$$

follows. Conjugating (5.11), we see that

$$\begin{aligned} \int_0^1 \frac{xP'(x)}{i+P(x)}\,dx &= \int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\overline{\omega_j}}\right)\,dx + \int_0^1 \left(\sum_{j=1}^k \frac{x}{x-\overline{\tau_j}}\right)\,dx \\ &= 2k + \sum_{n=0}^{\infty} \frac{\left(\sum_{j=1}^k (\epsilon_j\overline{\tau_j})^{n+1} - 2(\epsilon_j\overline{\tau_j})^n\right)}{n+1} \end{aligned} \tag{5.12}$$

holds. Feeding (5.11) and (5.12) into (5.5), we deduce that

$$\int_0^1 \frac{xP'(x)}{1+P(x)^2}\,dx = \frac{i}{2}\left(\sum_{n=0}^{\infty} \frac{\sum_{j=1}^{k}(\epsilon_j\overline{\tau_j})^{n+1} - 2(\epsilon_j\overline{\tau_j})^n - (\overline{\epsilon_j}\tau_j)^{n+1} + 2(\overline{\epsilon_j}\tau_j)^n}{n+1}\right). \tag{5.13}$$

For every $1 \le j \le k$, $\overline{\epsilon_j}\tau_j$ is algebraic and we may define the constant polynomials $P_j(x) := 2 - \overline{\epsilon_j}\tau_j \in \overline{\mathbb{Q}}[x]$, where $\overline{\mathbb{Q}}$ is the set of all algebraic numbers over $\mathbb{Q}$. Letting $\alpha_j = \overline{\epsilon_j}\tau_j$, we may write

$$\sum_{j=1}^{k}(\epsilon_j\overline{\tau_j})^{n+1} - 2(\epsilon_j\overline{\tau_j})^n - (\overline{\epsilon_j}\tau_j)^{n+1} + 2(\overline{\epsilon_j}\tau_j)^n = \sum_{j=1}^{k} P_j(n)\alpha_j^n - \overline{P_j(n)}\ \overline{\alpha_j}^n. \tag{5.14}$$

Consider the exponential polynomial

$$g(x) = \sum_{j=1}^{k} P_j(x)\alpha_j^x - \overline{P_j(x)}\ \overline{\alpha_j}^x$$

with $2k$ terms. Then as a result of (5.3), (5.13) and (5.14), we have

$$\begin{aligned} M_k &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{4kn-2k}{2kn-k}(2n-1)(4kn-2k+1)} \\ &= -\int_0^1 \frac{xP'(x)}{1+P(x)^2}\,dx = -\frac{i}{2}\sum_{n=0}^{\infty}\frac{g(n)}{n+1}. \end{aligned} \tag{5.15}$$

Using (5.15), we may deduce as in the proof of Theorem 1 of [2] that M_k is zero or transcendental. It is clear from (5.1) that $M_k \ne 0$ and M_k is therefore transcendental. To deal with the w_n measure of M_k, it is more convenient to use a related quantity first introduced by Koksma [8] as an alternative to Mahler's, namely that

$$w_n^*(\xi, H) := \min\{|\xi - \alpha| : \alpha \in \overline{\mathbb{Q}} \cap \mathbb{R},\ d_\alpha \le n,\ H(\alpha) \le H,\ \alpha \ne \xi\} \tag{5.16}$$

and

$$w_n^*(\xi) := \limsup_{H\to\infty} \frac{-\log(Hw_n^*(\xi, H))}{\log H}, \tag{5.17}$$

where d_α and $H(\alpha)$ are the degree and height of the minimal polynomial of α over $\mathbb{Z}$ which is the integer polynomial vanishing at α with smallest degree having no nontrivial common divisors of its coefficients and positive leading coefficient. Moreover, choosing an algebraic number β^* with the property $w_n^*(M_k, H) = |M_k - \beta^*|$, using (5.16), (5.17) and

$$\frac{-\log|M_k - \beta^*|}{h(\beta^*)} \le c_k d_{\beta^*}^{2k+3},$$

where $h(\beta^*)$ is the absolute logarithmic height of β^* and $c_k > 0$ is a constant depending only on M_k, one derives the estimate

$$w_n(M_k) = O_k(n^{2k+2}) \tag{5.18}$$

for all $n \geq 1$. It follows from (5.18) that M_k is not a U-number for all k. Finally, to examine the structure of M_k as combinations, assume that $\sqrt{1-4\epsilon_j} = \pm(a_j + ib_j)$ with $a_j \geq 0$. Then the roots of $x^2 - x + \epsilon_j$ are

$$\frac{1+a_j}{2} + i\frac{b_j}{2}, \quad \frac{1-a_j}{2} - i\frac{b_j}{2} \quad \text{with} \quad \left|\frac{1+a_j}{2} + i\frac{b_j}{2}\right| \geq \left|\frac{1-a_j}{2} - i\frac{b_j}{2}\right|$$

so that

$$\omega_j = \frac{1+a_j}{2} + i\frac{b_j}{2}$$

for $1 \leq j \leq k$. There are k pairs of quadratic equations $x^2 - x + \epsilon_j = 0$ and $x^2 - x + \overline{\epsilon_j} = 0$ and if

$$\omega_j = \frac{1+a_j}{2} + i\frac{b_j}{2} \quad \text{and} \quad \tau_j = \frac{1-a_j}{2} - i\frac{b_j}{2}$$

with $b_j \neq 0$ (since ϵ_j are not real), are the roots of the former, then

$$\overline{\omega_j} = \frac{1+a_j}{2} - i\frac{b_j}{2} \quad \text{and} \quad \overline{\tau_j} = \frac{1-a_j}{2} + i\frac{b_j}{2}$$

are the roots of the latter. For the contribution of each such quadruple

$$(\omega_j, \tau_j, \overline{\omega_j}, \overline{\tau_j})$$

of roots to the right hand side of (5.5), note that

$$\begin{aligned} &\int x \left(\frac{1}{x-\overline{\omega_j}} - \frac{1}{x-\omega_j} + \frac{1}{x-\overline{\tau_j}} - \frac{1}{x-\tau_j} \right) dx \\ = &\; -i\, \Im(\omega_j) \log\left(x^2 - 2\Re(\omega_j) + |\omega_j|^2\right) - 2i\, \Re(\omega_j) \arctan\left(\frac{x - \Re(\omega_j)}{\Im(\omega_j)}\right). \end{aligned} \tag{5.19}$$

As a result of (5.5) and (5.19), we see that

$$\begin{aligned} M_k = &\; -\int_0^1 \frac{xP'(x)}{1+P(x)^2}\, dx \\ = &\; -\sum_{j=1}^{k} \frac{b_j}{2} \log\left(\frac{(1-a_j)^2 + b_j^2}{(1+a_j)^2 + b_j^2}\right) \\ &+ a_j \left(\arctan\left(\frac{1-a_j}{b_j}\right) + \arctan\left(\frac{1+a_j}{b_j}\right)\right). \end{aligned} \tag{5.20}$$

By assumption, $\sqrt{1-4\epsilon_j} = \pm(a_j + ib_j)$, $\sqrt{1-4\overline{\epsilon_j}} = \pm(a_j - ib_j)$ are in F. Obviously, i is also in F. Consequently, a_j, b_j are both in F. Since $(a_j + ib_j)^2 = 1 - 4\epsilon_j$ and ϵ_j is not real, a_j, b_j indeed belong to $F^\times$. Observe that the logarithmic terms in (5.20) are all nonvanishing. To verify the nonvanishing also for the terms involving the arctangent, assume for a contradiction that $a_j = 1$ for some j. Then $\omega_j = 1 + i\frac{b_j}{2}$ and $\tau_j = -i\frac{b_j}{2}$. Thus we have

$$\frac{b_j^2}{4} - i\frac{b_j}{2} = \omega_j \tau_j = \epsilon_j.$$

Hence $t = -\frac{b_j}{2}$ satisfies $t(t+i) = \epsilon_j$. Taking modulus, we see that $t^2(t^2+1) = 1$ and this gives $|t| = \sqrt{\frac{\sqrt{5}-1}{2}}$. One can similarly show that $\overline{\omega_j} = 1 - i\frac{b_j}{2}$ and $\overline{\tau_j} = i\frac{b_j}{2}$ give

$$\frac{b_j^2}{4} + i\frac{b_j}{2} = \overline{\epsilon_j}$$

and this time $t = \frac{b_j}{2}$ satisfies $t(t+i) = \overline{\epsilon_j}$ with $|t| = \sqrt{\frac{\sqrt{5}-1}{2}}$. As b_j or $-b_j$ is positive, $\epsilon_j \neq i$ is not real and is of the form

$$\epsilon_j = e^{\frac{\pi i}{2k} + \frac{2\pi m i}{k}}$$

for some integer m, there exists an integer $1 \leq r < k$ satisfying

$$\cos\left(\frac{r\pi}{2k}\right) = \frac{\sqrt{5}-1}{2}, \quad \sin\left(\frac{r\pi}{2k}\right) = \sqrt{\frac{\sqrt{5}-1}{2}}. \tag{5.21}$$

It follows from (5.21) that

$$w_d = e^{\frac{r\pi i}{2k}} \in \mathbb{Q}\left(i, \sqrt{\frac{\sqrt{5}-1}{2}}\right),$$

where w_d is a primitive dth root of unity for some d dividing $4k$. Note that

$$\left[\mathbb{Q}\left(i, \sqrt{\frac{\sqrt{5}-1}{2}}\right) : \mathbb{Q}\right] = 8 \quad \text{and} \quad [\mathbb{Q}(w_d) : \mathbb{Q}] = \phi(d), \tag{5.22}$$

where $\phi(d)$ is Euler's totient function. Since $\mathbb{Q}(w_d)$ is a subfield, we see from (5.22) that $\phi(d)$ is a divisor of 8 and a complete list of possibilities for d is

$$d \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30\}. \tag{5.23}$$

Using (5.21), one calculates the argument of w_d to be $0.90455\cdots$ in radians and it is easy to check that no positive integer power of a primitive dth root of unity, where d is as in (5.23), can coincide with this number in terms of their arguments. This contradiction shows that $a_j \neq 1$ for all j. (5.20) gives that M_k can be written as an $F^\times$-linear combination of k logarithms and $2k$ arctangents of numbers that are also in $F^\times$, where the logarithmic terms are all nonvanishing. To complete the proof, let us take $k = 1$, working with the equations $x^2 - x + i = 0$ and $x^2 - x - i = 0$, one finds that

$$a_1 = \sqrt{\frac{1+\sqrt{17}}{2}}, \quad b_1 = -\sqrt{\frac{\sqrt{17}-1}{2}}.$$

Combining the two arctangent terms appearing in M_1 and simplifying, one calculates λ as in the statement of the theorem. Furthermore, λ is transcendental but not a U-number with $w_n(\lambda) = O(n^4)$ for $n \geq 1$.

6. Proof of Theorem 2

Let α_j and β_j be positive algebraic numbers for $1 \leq j \leq r$, where β_j are distinct. Then note that

$$\begin{aligned} S_r = \sum_{j=1}^{r} \alpha_j \arctan \beta_j &= \sum_{\substack{1\leq j\leq s\\ \beta_j>1}} \alpha_j \arctan \beta_j + \sum_{\substack{s+1\leq j\leq r\\ 0<\beta_j\leq 1}} \alpha_j \arctan \beta_j \\ &= \sum_{j=1}^{s} \alpha_j \left(\frac{\pi}{2} - \arctan \lambda_j\right) + \sum_{\substack{s+1\leq j\leq r\\ 0<\beta_j\leq 1}} \alpha_j \arctan \beta_j, \end{aligned} \tag{6.1}$$

where $\lambda_j = 1/\beta_j$. If $s = 0$ in (6.1), then by our convention $0 < \beta_j \leq 1$ for all $1 \leq j \leq r$ and in this case, we may write

$$\arctan \beta_j = -\frac{i}{2} \log(1 + i\beta_j) - \frac{i}{2} \log\left(\frac{1}{1 - i\beta_j}\right), \tag{6.2}$$

where the complex logarithm has its principal value. Moreover, we have

$$\log(1 + i\beta_j) = \sum_{n=0}^{\infty} \frac{(-1)^n (i\beta_j)^{n+1}}{n+1} \tag{6.3}$$

and

$$\begin{aligned} \log\left(\frac{1}{1 - i\beta_j}\right) &= \log\left(1 + \frac{i\beta_j - \beta_j^2}{1 + \beta_j^2}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n i\beta_j((1 + i\beta_j)/(1 + \beta_j^2))((i\beta_j - \beta_j^2)/(1 + \beta_j^2))^n}{n+1} \end{aligned} \tag{6.4}$$

as

$$\left|\frac{i\beta_j - \beta_j^2}{1 + \beta_j^2}\right| = \frac{\beta_j}{\sqrt{1 + \beta_j^2}} < 1.$$

Next let

$$T_m = \sum_{j=1}^{m} \mu_j \log \gamma_j. \tag{6.5}$$

For each algebraic number $\gamma_j > 1$, define a new algebraic number φ_j as follows. If $1 < \gamma_j \leq 2$, then take $\varphi_j = \gamma_j - 1$. If $\gamma_j > 2$, then take $\varphi_j = \frac{1}{\gamma_j} - 1$. Note that in both cases $-1 < \varphi_j \leq 1$ and $\pm\log(1+\varphi_j) = \log\gamma_j$ holds. Here the plus sign is chosen only when $1 < \gamma_j \leq 2$. It follows that

$$\log\gamma_j = \pm\log(1+\varphi_j) = \pm\sum_{n=0}^{\infty}\frac{(-\varphi_j)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{s_j(-\varphi_j)^{n+1}}{n+1} \tag{6.6}$$

for any $1 \leq j \leq m$, where $s_j \in \{-1, 1\}$ determines the sign. Assembling now (6.1)–(6.6), the representation

$$K_{r,m} = S_r + T_m = \sum_{n=0}^{\infty}\frac{\sum_{j=1}^{r}\frac{\alpha_j\beta_j}{2}(-i\beta_j)^n + \frac{\alpha_j\beta_j(1+i\beta_j)}{2(1+\beta_j^2)}((-i\beta_j+\beta_j^2)/(1+\beta_j^2))^n + \sum_{j=1}^{m} s_j\mu_j(-\varphi_j)(-\varphi_j)^n}{n+1} \tag{6.7}$$

follows. Define the constant polynomials

$$P_j(x) := \begin{cases} \frac{\alpha_j\beta_j}{2}, & 1 \leq j \leq r \\ \frac{\alpha_{j-r}\beta_{j-r}(1+i\beta_{j-r})}{2(1+\beta_{j-r}^2)}, & r+1 \leq j \leq 2r \\ -s_{j-2r}\mu_{j-2r}\varphi_{j-2r}, & 2r+1 \leq j \leq 2r+m, \end{cases} \tag{6.8}$$

where each case in (6.8) is clearly an algebraic number. Also define the algebraic numbers

$$\omega_j := \begin{cases} -i\beta_j, & 1 \leq j \leq r \\ \frac{-i\beta_{j-r}+\beta_{j-r}^2}{1+\beta_{j-r}^2} & r+1 \leq j \leq 2r \\ -\varphi_{j-2r}, & 2r+1 \leq j \leq 2r+m, \end{cases} \tag{6.9}$$

and the exponential polynomial

$$g(x) := \sum_{j=1}^{2r+m} P_j(x)\omega_j^x. \tag{6.10}$$

Gathering (6.7)–(6.10), we see that

$$0 < K_{r,m} = \sum_{n=0}^{\infty}\frac{g(n)}{n+1} \tag{6.11}$$

and $K_{r,m}$ is forced to be transcendental by (6.11). Note that $g(x)$ has $2r+m$ terms so that choosing an algebraic number κ in Koksma's formulation and using the inequality

$$\frac{-\log|K_{r,m}-\kappa|}{h(\kappa)} \leq c_{\alpha_j,\beta_j,\mu_j,\gamma_j} d_\kappa^{2r+m+3}$$

with a suitable constant $c_{\alpha_j,\beta_j,\mu_j,\gamma_j} > 0$ depending only on the algebraic numbers defining $K_{r,m}$, one arrives at the estimate

$$w_n(K_{r,m}) = O_{\alpha_j,\beta_j,\mu_j,\gamma_j}(n^{2r+m+2}) \tag{6.12}$$

when no $\beta_j > 1$. Thus $K_{r,m}$ is not a U-number. If $s \geq 1$ and none of $\beta_j = 1$, then we may similarly represent $K_{r,m}$ using an exponential polynomial with $2r + m + 2$ terms as

$$\frac{\pi}{4} = \arctan 1 = -\frac{i}{2}\log(1+i) - \frac{i}{2}\log\left(\frac{1}{1-i}\right).$$

In this case, $K_{r,m}$ is transcendental but not a U-number, satisfying

$$w_n(K_{r,m}) = O_{\alpha_j,\beta_j,\mu_j,\gamma_j}(n^{2r+m+4}).$$

If $s \geq 1$ and exactly one $\beta_j = 1$, then the two logarithmic terms arising from $\frac{\pi}{2}$ can be combined with the term involving $\arctan\beta_j = \arctan 1$ and one again obtains (6.12). Next consider

$$L_1 = \alpha\arctan\beta + \log\mu,$$

where $\alpha \neq 0$, $\beta \neq 0$, $\mu \neq 1$, $\mu > 0$ are real algebraic numbers. Let us see that $L_1 \neq 0$. For a contradiction, assume that $L_1 = 0$. Putting $\lambda_1 = \arctan\beta$ and $\lambda_2 = \log\mu$, we have $\alpha\lambda_1 + \lambda_2 = 0$. It follows that $\sin\lambda_1 = \beta\cos\lambda_1$ and $e^{\lambda_2} = \mu$. But $e^{\lambda_2} = e^{-\alpha\lambda_1}$ is one of the values of $(e^{i\lambda_1})^{i\alpha}$. Note that $i\alpha \notin \mathbb{Q}$. Since β is algebraic and $\sin^2\lambda_1 + \cos^2\lambda_1 = 1$, it is easy to see that $\sin\lambda_1$ and $\cos\lambda_1$ are both algebraic. Consequently, $e^{i\lambda_1}$ is algebraic. As μ is algebraic, one contradicts the Gelfond-Schneider theorem (see p. 119 of [4]) unless $e^{i\lambda_1} = 1$ and in that case $\beta = \tan\lambda_1 = 0$, another contradiction. Thus L_1 is transcendental. When $|\beta| \leq 1$, we may write

$$\arctan\beta = -\frac{i}{2}\log(1+i\beta) - \frac{i}{2}\log\left(\frac{1}{1-i\beta}\right)$$

and it follows as above that L_1 is not a U-number with $w_n(L_1) = O_{\alpha,\beta,\mu}(n^5)$. If $\beta > 1$, then using $\arctan\beta = \frac{\pi}{2} - \arctan\lambda$ with $\lambda = 1/\beta$ and writing both of $\frac{\pi}{2}$ and $\arctan\lambda$ as a combination of a pair of logarithms, we see that $w_n(L_1) = O_{\alpha,\beta,\mu}(n^7)$. If $\beta < -1$, then using $\arctan\beta = -\frac{\pi}{2} + \arctan(-1/\beta)$, one similarly obtains $w_n(L_1) = O_{\alpha,\beta,\mu}(n^7)$. Finally, if $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$ are real algebraic numbers, then

$$L_2 = \beta\pi + \arctan\alpha \neq 0$$

since otherwise $\beta\pi + \arctan\alpha = 0$ and $\lambda = \arctan\alpha$ give that $\tan\lambda = \alpha$, $\lambda = -\beta\pi$. Similarly as above, we see that $e^{i\lambda} = e^{-i\beta\pi}$ is algebraic. This is a contradiction unless β is rational contrary to our assumption. It follows that L_2 is transcendental. Writing π and $\arctan\alpha$ in terms of pairs of logarithms and distinguishing the cases $|\alpha| \leq 1$, $|\alpha| > 1$, we easily deduce that L_2 is not a U-number with $w_n(L_2) = O_{\alpha,\beta}(n^6)$. This completes the proof.

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