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BOUNDED AND UNBOUNDED OPERATORS SIMILAR TO THEIR ADJOINTS

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Abstract. In this paper, we establish results about operators similar to their adjoints. This is carried out in the setting of bounded and also unbounded operators on a Hilbert space. Among the results, we prove that an unbounded closed operator similar to its adjoint, via a cramped unitary operator, is self-adjoint. The proof of this result works also as a new proof of the celebrated result by Berberian on the same problem in the bounded case. Other results on similarity of hyponormal unbounded operators and their self-adjointness are also given, generalizing well known results by Sheth and Williams.

1. Introduction

First, we notice that while we will be recalling most of the essential background we will assume the reader is familiar with any other result or notion which will appear in the present paper. Some of the standard textbooks on bounded and unbounded operator theory are [4], [7], [8], [12], [17], [18], [19] and [23].

The notions of bounded self-adjoint, normal, hyponormal and unitary operators are defined in their usual fashion. So are the notions of unbounded closed, symmetric, self-adjoint and normal operators. The spectrum and the approximate spectrum of an operator are denoted respectively by σ and σ_a. We shall not recall their definitions here.

The numerical range of a bounded operator T on a $\mathbb{C}$-Hilbert space H, denoted by $W(T)$, is defined as

$$W(T) = \{\langle Tx, x\rangle : \ x \in H, \ \|x\| = 1\}$$

(if T is unbounded, then replace H by $D(T)$).

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If S and T are two unbounded operators with domains $D(S)$ and $D(T)$ respectively, then $S \subset T$ means that T is an extension of S, that is,

$$D(S) \subset D(T) \text{ and } \forall x \in D(S) : \; Sx = Tx.$$

We also assume throughout this paper that all operators are linear and defined on a separable complex Hilbert space H, and that unbounded operators have a dense domain (so that the uniqueness of the adjoint is guaranteed). They are said to be densely defined.

We define the product ST of two unbounded operators with domains $D(S)$ and $D(T)$ respectively by:

$$(ST)x = S(Tx), \; \forall x \in D(ST) = \{x \in D(T) : \; Tx \in D(S)\}.$$

Recall that an unbounded operator S, defined on a Hilbert space H, is said to be invertible if there exists an *everywhere defined* (i.e., on the whole of H) bounded operator T, which then will be designated by S^{-1}, such that

$$TS \subset ST = I,$$

where I is the usual identity operator.

Recall also that if S, T and ST are all densely defined, then we have $T^*S^* \subset (ST)^*$. There are cases where equality holds in the previous inclusion, namely if S is bounded. The next result gives another case where the equality does hold.

Lemma 1 ([23]). *If S and T are densely defined and T is invertible with inverse T^{-1} in $B(H)$, then $(ST)^* = T^*S^*$.*

The next lemma is also known.

Lemma 2. *Let T be a densely defined and closed operator in a Hilbert space H, with domain $D(T) \subset H$. Assume that for some $k > 0$,*

$$\|Tx\| \geq k\|x\| \text{ for all } x \in D(T).$$

Then $\operatorname{ran}(T)$ *is closed.*

Finally, let us recall some other important results for us. But first, we have a definition:

Definition. A *unitary* operator U is said to be *cramped* if its spectrum is completely contained on some open semi-circle (of the unit circle), that is

$$\sigma(U) \subseteq \{e^{it} : \; \alpha < t < \alpha + \pi\}.$$

Theorem 1 ([2]). *If U is a cramped unitary element of $\mathcal{A}$ (where $\mathcal{A}$ is any C^*-algebra), and T is an element of $\mathcal{A}$ such that $UTU^* = T^*$, then T is self-adjoint.*

Remark. The previous theorem will be generalized to unbounded operators with a proof which works in the bounded case too. See Theorem 11 and the remark below it.

Theorem 2 ([3]). *Let T be a bounded operator for which $0 \notin \overline{W(T)}$. Then T is invertible and the unitary operator $T(T^*T)^{-\frac{1}{2}}$ is cramped.*

Theorem 3 ([20] or [24]). *Let T be a bounded hyponormal operator. If S is any bounded operator for which $0 \notin \overline{W(S)}$, then*

$$ST = T^*S \Longrightarrow T = T^*.$$

Proposition 1 ([10]). *Let T be an unbounded, closed and hyponormal operator in some Hilbert space H. Then $W(T) \subset conv\, \sigma(T)$, where $conv\, \sigma(T)$ denotes the convex hull of $\sigma(T)$.*

Proposition 2 ([16]). *Let T be a densely defined, closed and symmetric operator in a Hilbert space. If T is quasi-similar to its adjoint T^*, then T is self-adjoint (for the definition of quasi-invertibility, the reader may look at [16]).*

The notion of similarity of operators is important from matrices on finite-dimensional vector spaces to unbounded operators on a Hilbert space. Many authors have worked on this type of problems for bounded operators. We refer the interested reader to [1], [2], [3], [5], [6], [11], [20], [21] and [24].

There have been some works on similar unbounded operators but only a few compared to those in the bounded case. This is due probably to the complexity of the domains involved. Some of those papers are [14], [15], [16] and [22].

In the present article, we establish some new results on similarity in the setting of bounded and unbounded operators on a Hilbert space. We have two main sections, one dedicated to bounded operators and the other is devoted to unbounded operators.

2. Main results: The bounded case

The main result of this section is the following. It permits us to drop the hypothesis of hyponormality in Theorem 3 at the cost of imposing a commutativity-like assumption.

Theorem 4. *Let S, T be two bounded operators satisfying: $S^{-1}T^*S = T$, $S^*ST = TS^*S$ and $0 \notin \overline{W(S)}$. Then T is self-adjoint.*

Proof. Since $0 \notin \overline{W(S)}$, S is invertible. So, let $S = UP$ be its polar decomposition. Remember that $P = (S^*S)^{\frac{1}{2}}$ is positive and $U = S(S^*S)^{-\frac{1}{2}}$ is unitary. By Theorem 2, U is even cramped.

Since $S^*ST = TS^*S$, we have

$$P^2T = TP^2 \text{ or } PT = TP.$$

Hence we may write

$$\begin{aligned} & S^{-1}T^*S = T \\ \Longleftrightarrow & P^{-1}U^*T^*UP = T \\ \Longleftrightarrow & U^*T^*U = PTP^{-1} \end{aligned}$$

$$\begin{aligned}&\Longleftrightarrow U^*T^*U = TPP^{-1}\\&\Longleftrightarrow U^*T^*U = T\\&\Longleftrightarrow T^* = UTU^*.\end{aligned}$$

As U is cramped, Theorem 1 applies and yields the self-adjointness of T, establishing the result. □

Before giving another result on similarity, it appears to be convenient to recall the following result here:

Theorem 5 (Singh-Mangla, [21])**.** *If T is an invertible normaloid operator such that $T^* = S^{-1}T^{-1}S$, where $0 \notin \overline{W(S)}$, then T is unitary.*

Shortly afterwards, DePrima [5] found out that Theorem 5 was actually false by giving a counterexample! Then DePrima [5] gave some extra conditions for the conclusion of Theorem 5 to hold. One of the results there is the following:

Theorem 6 (DePrima, [5])**.** *Let T be an invertible normaloid or convexoid operator such that T^{-1} too is normaloid or convexoid. If $T^* = S^{-1}T^{-1}S$, where $0 \notin \overline{W(S)}$, then T is unitary.*

Our result in this spirit is the following:

Theorem 7. *Let S and T be two bounded operators such that $TS^*S = S^*ST$ and $S^{-1}T^*S = T^{-1}$, where $0 \notin \overline{W(S)}$ and where also T is invertible. Then T is unitary.*

Proof. Let $S = UP$ where U is unitary and P is positive (where $P = (S^*S)^{\frac{1}{2}}$). We then have

$$TS^*S = S^*ST \Longrightarrow S^*ST^{-1} = T^{-1}S^*S,$$

hence

$$P^2T^{-1} = T^{-1}P^2 \text{ so that } PT^{-1} = T^{-1}P.$$

Therefore,

$$\begin{aligned}&S^{-1}T^*S = T^{-1}\\&\Longleftrightarrow P^{-1}U^*T^*UP = T^{-1}\\&\Longleftrightarrow U^*T^*U = PT^{-1}P^{-1}\\&\Longleftrightarrow U^*T^*U = T^{-1}PP^{-1}\\&\Longleftrightarrow U^*T^*U = T^{-1}\\&\Longleftrightarrow T^* = UT^{-1}U^*.\end{aligned}$$

Since $0 \notin \overline{W(S)}$, U is cramped so that Theorem 2 of [21] applies and gives us $T^* = T^{-1}$, completing the proof. □

3. Main results: The unbounded case

The first result of the this section is Theorem 8 (it generalizes Theorem 3 to an unbounded operator setting). For readers convenience, we recall the following definition:

Definition. An unbounded densely defined operator A is called hyponormal if:

(1) $D(A) \subset D(A^*)$,
(2) $\|A^*x\| \leq \|Ax\|$ for all $x \in D(A)$.

Theorem 8. *Let S be a bounded operator on a $\mathbb{C}$-Hilbert space H such that $0 \notin \overline{W(S)}$. Let T be an unbounded and closed hyponormal operator with a dense domain $D(T) \subset H$. If $ST^* \subset TS$, then T is self-adjoint.*

Remark. We did add an extra condition (namely closedness) on T as regards to Theorem 3. This is fine for closed operators are considered as the natural substitutes of the bounded ones. Besides, if T is not closed, then it cannot be self-adjoint.

Now, we prove Theorem 8.

Proof. The proof is divided into three claims:

(1) **Claim 1:** $\sigma_a(T^*) = \sigma(T^*)$. By definition, $\sigma_a(T^*) \subset \sigma(T^*)$. To show the reverse inclusion, let $\lambda \notin \sigma_a(T^*)$. Then there exists a positive number k such that

$$\|T^*x - \lambda x\| \geq k\|x\| \text{ for all } x \in D(T^*).$$

Hence $T^* - \lambda I$ is clearly injective. Besides, and by Lemma 2, ran$(T^* - \lambda I)$ is closed as $T^* - \lambda I$ is closed for T^* is so. Now, since T is hyponormal, so is $T - \overline{\lambda}I$ which means that

$$\|Tx - \overline{\lambda}x\| \geq \|T^*x - \lambda x\| \geq k\|x\| \text{ for all } x \in D(T) \subset D(T^*).$$

Whence $T - \overline{\lambda}I$ is also injective so that

$$\operatorname{ran}(T^* - \lambda I)^\perp = \ker(T - \overline{\lambda}I) = \{0\} \text{ or } \overline{\operatorname{ran}(T^* - \lambda I)} = H.$$

Thus $T^* - \lambda I$ is onto since we already observed that its range was closed. Therefore, $\lambda \notin \sigma(T^*)$.

(2) **Claim 2:** $\sigma(T) \subset \mathbb{R}$. Let $\lambda \in \sigma(T^*) = \sigma_a(T^*)$. Then for some $x_n \in D(T^*)$ such that $\|x_n\| = 1$ we have $\|T^*x_n - \lambda x_n\| \to 0$ as $n \to \infty$. Since $ST^* \subset TS$ and $x_n \in D(T^*)$, we have $ST^*x_n = TSx_n$ so that we may write the following

$$\begin{aligned} 0 \leq |(\overline{\lambda} - \lambda)\langle Sx_n, x_n\rangle| &= |\langle (ST^*S^{-1} - \lambda + \overline{\lambda} - T)Sx_n, x_n\rangle| \\ &\leq |\langle (S(T^* - \lambda I)x_n, x_n\rangle| + |\langle (\overline{\lambda} - T)Sx_n, x_n\rangle| \\ &\leq \|S\|\ \|T^*x_n - \lambda x_n\| + |\langle Sx_n, (\lambda - T^*)x_n\rangle| \\ &\leq \|S\|\ \|T^*x_n - \lambda x_n\| + \|S\|\ \|T^*x_n - \lambda x_n\| \end{aligned}$$

$$= 2\|S\|\,\|T^*x_n - \lambda x_n\| \to 0.$$

(where in the second inequality we used the fact that $x_n \in D(T^*)$ and $Sx_n \in D(T)$ both coming from $ST^* \subset TS$). However, the condition $0 \notin \overline{W(S)}$ forces us to have $\lambda = \overline{\lambda}$. Accordingly, $\sigma(T^*) \subset \mathbb{R}$ or just $\sigma(T) \subset \mathbb{R}$ (remember that $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$).

(3) **Claim 3:** $T = T^*$. Since $\sigma(T) \subset \mathbb{R}$, Proposition 1 implies that $W(T) \subset \mathbb{R}$, which clearly implies that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in D(T)$, which, in its turn, means that T is symmetric. Hence T is quasi-similar to T^* via S and I, so that Proposition 2 applies and gives the self-adjointness of T. This completes the proof. □

The condition $ST^* \subset TS$ in the foregoing theorem is not purely conventional, i.e., we may not obtain the desired result by merely assuming instead that $ST \subset T^*S$, even with a slightly stronger condition (i.e., symmetricity in lieu of hyponormality). This is seen in the following proposition.

Proposition 3. *There exist a bounded operator S such that $0 \notin \overline{W(S)}$, and an unbounded and closed hyponormal T such that $ST \subset T^*S$ whereas $T \neq T^*$.*

Proof. Consider any unbounded, densely defined, closed and symmetric operator T which is *not self-adjoint*. Let $S = I$, i.e., the identity operator on the Hilbert space. Then S is bounded and $0 \notin \overline{W(S)}$. Also, T is closed and hyponormal. Finally, it is plain that

$$T = ST \subset T^* = T^*S.$$ □

Remark. We have not insisted on the explicitness of the example T in the previous proof. This was not too important. Besides, there are many of them. For instance, the interested reader may just consult Exercise 4 on page 316 of [4].

We can still obtain the self-adjointness of T from $ST \subset T^*S$ by imposing an extra condition on T. We have:

Theorem 9. *Let S be a bounded operator such that $0 \notin \overline{W(S)}$. Let T be an unbounded hyponormal and invertible operator. If $ST \subset T^*S$, then T is self-adjoint.*

Proof. Since T is invertible with an everywhere defined bounded inverse, we have

$$ST \subset T^*S \Longrightarrow ST^{-1} = (T^{-1})^*S.$$

Since T is hyponormal, the bounded T^{-1} too is hyponormal (see [9]). Hence by [20], T^{-1} is self-adjoint. Hence

$$T^{-1} = (T^{-1})^* \Longrightarrow T(T^{-1})^* = I \Longrightarrow T^* \subset T.$$

Now, since T is hyponormal, $D(T) \subset D(T^*)$ so that finally we have $T = T^*$, that is, T is self-adjoint. □

The condition of invertibility in the foregoing theorem may not simply be dispensed with. This is illustrated by the following example:

Example 10. Let A be an unbounded operator defined on a Hilbert space H, with domain $D(A) \subsetneqq H$. Set $T = A - A$, then T is not closed and hence it is surely not self-adjoint. Finally, let $S = I$ the identity operator on H. Now we verify that the remaining conditions (except for invertibility) of the theorem are fulfilled.

(1) T is hyponormal for $T^* = 0$ with $D(T^*) = H \supset D(A) = D(T)$ so that
$$\|T^*x\| = \|Tx\| = 0 \text{ for all } x \in D(T).$$

(2) Since $S = I$, obviously $0 \notin \overline{W(S)}$. Moreover,
$$T = 0_{D(A)} \subset T^* = 0_H \text{ so that } ST \subset T^*S.$$

Last but not least, we have a very nice and important result which generalizes Theorem 1 to unbounded operators.

Theorem 11. *Let U be a cramped unitary operator. Let T be a closed and densely defined operator such that $UT = T^*U$. Then T is self-adjoint.*

Proof. First we prove that $U^2T = TU^2$. Since U is bounded and invertible, we have
$$(UT)^* = T^*U^* \text{ and } (T^*U)^* = U^*T^{**} = U^*\overline{T} = U^*T$$
(by Lemma 1). Hence $T^*U^* = U^*T$. We may then write
$$\begin{aligned} U^2TU^{*2} &= U(UTU^*)U^* \\ &= UT^*U^* \\ &= UU^*T \\ &= T, \end{aligned}$$
giving $U^2T = TU^2$ or $TU^{*2} = U^{*2}T$ or $U^2T^* = T^*U^2$.

Next, we prove that $TU = UT^*$. We have
$$\begin{aligned} TU &= U^*T^*UU \\ &= U^*T^*U^2 \\ &= U^*UUT^* \\ &= UT^*. \end{aligned}$$
Hence also $U^*T^* = TU^*$.

The penultimate step in the proof is to prove that T is normal. To this end, set $S = \frac{1}{2}(U + U^*)$. Following [24], $S > 0$.

Then we show that $STT^* \subset T^*TS$. We have
$$UTT^* = T^*UT^* = T^*TU$$
and
$$U^*TT^* = T^*U^*T^* = T^*TU^*.$$

Hence

$$\begin{aligned} STT^* &= \frac{1}{2}(U+U^*)TT^* \\ &= \frac{1}{2}UTT^* + \frac{1}{2}U^*TT^* \text{ (as } U \text{ is bounded)} \\ &= \frac{1}{2}T^*TU + \frac{1}{2}T^*TU^* \\ &\subset T^*TS. \end{aligned}$$

So according to Corollary 5.1 in [22], $TT^* = T^*T$, and remembering that T is taken to be closed, we immediately conclude that T is normal. Accordingly, and by Corollary 3 in [14],

$$UT = T^*U \Longrightarrow T = T^*$$

as $0 \notin \overline{W(U)}$, establishing the result. □

Remark. Evidently, a hypothesis like $UT \subset T^*U$ would not yield the desired result. For example, take T to be any symmetric and closed unbounded operator T which is not self-adjoint. Let $U = I$ be the identity operator on the given Hilbert space. Then clearly $UT \subset T^*U$ while $T \neq T^*$.

The assumption U being cramped is indispensable even in the bounded case. This was already observed by Beck-Putnam [1] and Mc-Carthy [13].

Remark. Going back to the previous proof, we observe that this proof may well be applied to bounded operators. Hence we have just given a new proof of Theorem 1 which bypasses the Cayley transform.

Thanks to Theorem 11, we may prove an unbounded version of Theorem 4. It reads:

Corollary 1. *Let S be a bounded operator and T be an unbounded closed operator satisfying: $S^{-1}T^*S = T$, $S^*ST = TS^*S$ and $0 \notin \overline{W(S)}$. Then T is self-adjoint.*

Proof. The same proof as that of Theorem 4, mutatis mutandis. □

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