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LOCAL CONVERGENCE OF FUNCTIONAL ITERATIONS FOR SOLVING A QUADRATIC MATRIX EQUATION

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Abstract. We consider fixed-point iterations constructed by simple transforming from a quadratic matrix equation to equivalent fixed-point equations and assume that the iterations are well-defined at some solutions. In that case, we suggest real valued functions. These functions provide radii at the solution, which guarantee the local convergence and the uniqueness of the solutions. Moreover, these radii obtained by simple calculations of some constants. We get the constants by arbitrary matrix norm for coefficient matrices and solution. In numerical experiments, the examples show that the functions give suitable boundaries which guarantee the local convergence and the uniqueness of the solutions for the given equations.

1. Introduction

In this paper, we consider the *quadratic matrix equation* (QME)

$$Q(X) := AX^2 - BX + C = 0, \tag{1.1}$$

where $A, B, C \in \mathbb{C}^{m\times m}$. If $S \in \mathbb{C}^{m\times m}$ satisfies $Q(S) = 0$ in (1.1), then we call that S is a solvent [6].

From the QME we construct the fixed-point iterative methods as follows

$$\begin{cases} \text{Given } X_0 \in \mathbb{C}^{m\times m} \\ X_{k+1} = \mathcal{F}_i(X_k), \end{cases} \quad k = 0, 1, 2, \ldots \tag{1.2}$$

where $\mathcal{F}_i(X_k)$ can be defined at each steps.

$$\mathcal{F}_1(X) = B^{-1}(AX^2 + C), \tag{1.3}$$

$$\mathcal{F}_2(X) = (B - AX)^{-1}C, \tag{1.4}$$

$$\mathcal{F}_3(X) = A^{-1}\left(B - CX^{-1}\right). \tag{1.5}$$

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To find an approximated solution, we usually use fixed-point iterative methods. Sometimes, fixed-point methods don't converge to the solution even though a given initial matrix is sufficiently close to the solution. So, finding sufficient conditions which guarantee the convergence of given functional iterative methods is needed.

The existence of a solution of QME (1.1) and the convergence analysis of the iteration (1.2) on Banach space were studied in [1, 7, 15]. Higham and Kim [11] studied the convergence of the iteration with (1.3) and (1.4) to the special solution. From [11] we know that in over-damped quadratic eigenvalue problems and the quasi-birth-death problem (QBD) the Bernoulli iteration method converges to the solvent efficiently.

Bai and Gao [2] modified the iteration with (1.4) by techniques of the Gauss-Seidal iteration. Furthermore, they showed the local linear convergence of iterations sequentially with norm of matrix calculation under suitable conditions.

For iterations (1.3) and (1.4), the convergence to the elementwise minimal nonnegative solution of QME (1.1) arising in quasi-birth-death processes (QBDs) [4, 8, 14] is referred in [9] when the starting matrix is stochastic. When the leading coefficient matrix is an identity matrix and the starting matrix is the zero matrix, the iterations (1.3) and (1.4) were proposed by Guo [10] and Bai et al. [3], respectively.

The usual problem to apply the contraction mapping theorem in practice is to find the proper domain which is mapped into it [17]. The strategy to prove the local convergence of the iteration (1.2) is following tricks.

(1) Set a domain on which a fixed point function is defined.
(2) Find a set which satisfies one of following two theorems.
    (a) (Local Convergence Theorem) Find a proper open ball centered at a solution. Iterative sequence derived from the fixed point function converges to the solution for any initial point in the ball.
    (b) (Contraction Mapping Theorem) Choose a closed domain, which is a subset of the ball, in which the solution is unique.
(3) Investigate the rate of the convergence.

The difference of (a) and (b) is the closeness of sets.

Previously, we need complicated matrix calculation to find a subset in the above strategy (2) [2]. Even if we find the subset, the way to find the subset is hard to apply to another equation.

In this paper, we suggest two simple real valued functions. These functions are constructed by constants made from norms of coefficient matrices and solvent. We can easily calculate the radius which guarantees the uniqueness and the local convergence of a fixed point iteration when we know the solvent. The results on the finite dimensional real vector space referred in Ortega and Rheinboldt [16] are often adjusted to get along with our analysis. Moreover, theorems in this paper is easy to apply to another matrix equations.

2. Related definitions and theorems

Definition 2.1 ([5, Def. A.13]). Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ be given. The *Kronecker product* $A \otimes B$ of A and B is the $m \times n$ block matrix given by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}, \quad \text{where } A = [a_{ij}].$$

The properties of the Kronecker product and the following well-known lemmas are used in our works.

Lemma 2.2 (Neumann Lemma, [16, 2.3.1]). *For $A, B \in \mathbb{C}^{m\times m}$, if A is nonsingular and $\rho(A^{-1}B) < 1$, then $A - B$ is also nonsingular and represented by*

$$(A - B)^{-1} = A^{-1} + A^{-1}B\,(A - B)^{-1}. \tag{2.1}$$

$\|A^{-1}B\| < 1$ instead of $\rho\left(A^{-1}B\right) < 1$ also leads to the same conclusion in Lemma 2.2.

Remark 2.3. The equality (2.1) is represented by

$$\begin{aligned}(A - B)^{-1} &= \left(\sum_{k=0}^{\infty}(A^{-1}B)^k\right)A^{-1} \\ &= A^{-1} + \left(\sum_{k=1}^{\infty}(A^{-1}B)^k\right)A^{-1}. \end{aligned} \tag{2.2}$$

One-step stationary iterations have the form

$$X_{k+1} = F(X_k), \quad k = 0, 1, 2, \ldots, \tag{2.3}$$

where $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$. This include Newton's method and some of minimization methods.

The determination of fixed-point and estimation of the rate of convergence will depend upon finding sufficient conditions of which the following simple theorem is satisfied.

Theorem 2.4 (Local Convergence Theorem, [16, Thm. 10. 1. 2]). *Let $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, and suppose that there exist a ball $\mathcal{B} := \mathcal{B}(S, \delta) \subset \mathbf{D}$ and a constant $\alpha < 1$ such that*

$$\|F(X) - S\| \le \alpha\|X - S\| \quad \textit{for all } X \in \mathcal{B}.$$

Then, for any $X_0 \in \mathcal{B}$, the iterations from (2.3) remain in $\mathcal{B}$ and converges to the fixed point S of $F(X)$ and

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - S\|} \le \alpha.$$

In Theorem 2.4 since X_0 is arbitrary, F is invariant on $\mathcal{B}$ (i.e., F maps into itself). Note that we can not assure that S is a fixed-point of F unless F is continuous at S.

To guarantee the uniqueness of the solvent we need the contraction mapping theorem. So we introduce the following definition.

Definition 2.5 ([16, Def. 5.1.2]). A matrix function $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ is *contractive* on a set $\mathbf{D}_0 \subset \mathbf{D}$ if there is an $\alpha < 1$ such that $\|F(X) - F(Y)\| \leq \alpha\|X - Y\|$ for all $X, Y \in \mathbf{D}_0$.

The existence of a fixed point and the convergence of an iteration (2.3) are given by the following basic result.

Theorem 2.6 (Contraction Mapping Theorem, [12, Thm 5.1.2]). *Let $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, and suppose that F maps a closed set $\mathbf{D}_0 \subset \mathbf{D}$ into itself and contractive. Then F has the unique fixed point in $\mathbf{D}_0$.*

From Theorem 2.4 and Theorem 2.6, we have the following theorem which is used through the analysis.

Theorem 2.7. *Let $S \in \mathbb{R}^{m\times m}$ be a fixed point of $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ and suppose that there exist constants δ, δ' and balls $\mathcal{B}_1 := \mathcal{B}(S, \delta) \subset \mathbf{D}$, $\bar{\mathcal{B}}_2 := \bar{\mathcal{B}}(S, \delta') \subset \mathcal{B}_1$ such that*

$$\begin{aligned}
\|F(X) - F(Y)\| &\leq \Gamma(X, Y) \cdot \|X - Y\|, \qquad \forall X, Y \in \mathbf{D},\\
\Gamma(X, S) &\leq \bar{\mu}(\|X - S\|) < \bar{\mu}(\delta) \leq 1, \quad \forall X \in \mathcal{B}_1,\\
\Gamma(Y, Z) &\leq \bar{\nu}(\|Y - S\|, \|Z - S\|) \leq \bar{\nu}(\delta', \delta') < 1, \quad \forall Y, Z \in \bar{\mathcal{B}}_2,
\end{aligned}$$

and

$$\bar{\mu}(0) = \lim_{x\to 0} \bar{\mu}(x) = \mu < 1,$$

where $\Gamma(X, Y)$, $\bar{\mu}$ and $\bar{\nu}$ are real nonnegative valued functions on $\mathbf{D} \times \mathbf{D}$, $[0, \delta]$ and $[0, \delta'] \times [0, \delta']$ respectively, and $\bar{\mu}$ is increasing on $[0, \delta]$. Then, for any $X_0 \in \mathcal{B}_1$, the sequence $\{X_k\}$ generated by (2.3) converges to S. Moreover, S is the unique fixed point of F in $\bar{\mathcal{B}}_2$ and

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - S\|} \leq \alpha. \tag{2.4}$$

Proof. From $\bar{\mathcal{B}}_2 \subset \mathcal{B}_1$, we have $\delta' + \epsilon < \delta$ for a sufficiently small $\epsilon > 0$.

Let $\mathcal{B}_3 := \mathcal{B}(S, \delta' + \epsilon)$. Then since $\bar{\mu}$ is an increasing function we have

$$\Gamma(X, S) \leq \bar{\mu}(\|X - S\|) < \bar{\mu}(\delta' + \epsilon) < \bar{\mu}(\delta) \leq 1, \quad \forall X \in \mathcal{B}_3.$$

Therefore from the local convergence theorem we get

$$\|F(X) - S\| < \delta' + \epsilon, \quad \forall X \in \mathcal{B}_3.$$

Since ϵ is arbitrary, the following statement is true;

$$F(X) \in \bar{\mathcal{B}}_2, \quad \forall X \in \bar{\mathcal{B}}_2.$$

Therefore from the contraction mapping theorem F has the unique fixed point in $\bar{\mathcal{B}}_2$.

Suppose that, for $X_0 \in \mathcal{B}_1$, $\{X_k\}$ is generated by (2.3) and $\delta(k) := \|X_k - S\|$. Then we have

$$\begin{aligned}\|X_{k+1} - S\| &\le \bar{\mu}(\delta(k))\|X_k - S\| \\ &\le \bar{\mu}(\delta(k)) \cdot \bar{\mu}(\delta(k-1))\|X_{k-1} - S\| \\ &\le \left(\prod_{i=1}^{k} \bar{\mu}(\delta(i))\right) \|X_0 - S\|.\end{aligned}$$

Since $\delta(k) \to 0$ as $k \to \infty$ and $\bar{\mu}$ is continuous at 0, $\bar{\mu}(\delta(k))$ is converges to α ([18, Thm. 3.21]). Therefore from the properties of the limit superimum ([18, Sec. 2.5]) (2.4) is easily proved. □

3. Local convergence for $\mathcal{F}_1$

Since $\mathcal{F}_1$ in (1.3) is well-defined on $\mathbb{R}^{m\times m}$, from the properties of matrix norms we have

$$\|\mathcal{F}_1(Y) - \mathcal{F}_1(X)\| \le \left(\|B^{-1}AY\| + \|B^{-1}A\|\|X\|\right) \|Y - X\|. \tag{3.1}$$

Define

$$\Gamma_1(X, Y) := \|B^{-1}AY\| + \|B^{-1}A\|\|X\|, \tag{3.2}$$

then we get the following lemmas for $\mathcal{F}_1$.

Lemma 3.1. *Let $S \in \mathbb{R}^{m\times m}$ be a solvent of QME* (1.1)*, and suppose that B is nonsingular,*

$$\|B^{-1}A\| \le a_1,\ \|B^{-1}AS\| \le b_1 \quad \text{and} \quad \|S\| \le \beta,$$

where a_1, b_1 and β are some positive constants. Then if

$$b_1 + a_1\beta < 1,$$

then

$$\Gamma_1(X, S) \le \bar{\mu}_1\left(\|X - S\|\right) < 1, \quad \forall\ X \in \mathcal{B}_1 := \mathcal{B}(S, \delta_1),$$

where Γ_1 is in (3.2)*,*

$$\delta_1 := \frac{1 - b_1 - a_1\beta}{a_1} > 0 \tag{3.3}$$

and $\bar{\mu}_1(x) := b_1 + a_1\beta + a_1 x$.

Proof. Let $X \in \mathcal{B}_1$. Then we have

$$\|X\| \le \beta + \|X - S\| < \beta + \delta_1$$

and

$$\begin{aligned}\Gamma_1(X, S) &\le b_1 + a_1\|X\| \\ &\le b_1 + a_1\left(\beta + \|X - S\|\right) \left(= \bar{\mu}_1\left(\|X - S\|\right)\right)\end{aligned}$$

$$
\begin{aligned}
&< b_1 + a_1\beta + a_1\delta_1 \, (= \bar{\mu}_1(\delta_1)) \\
&\le b_1 + a_1\beta + (1 - b_1 - a_1\beta) = 1.
\end{aligned}
$$
□

Lemma 3.2. *Under the assumption of Lemma 3.1, if*

$$b_1 + a_1\beta < 1,$$

then

$$\Gamma_1(Y, Z) \le \bar{\nu}_1 (\|Y - S\|, \|Z - S\|) < 1, \quad \forall\, Y, Z \in \bar{\mathcal{B}}_2 := \bar{\mathcal{B}}\,(S, \delta_1'),$$

where Γ_1 *is in* (3.2),

$$\delta_1' \in \left(0,\ \frac{1 - b_1 - a_1\beta}{2a_1}\right), \tag{3.4}$$

and $\bar{\nu}_1(y, z) := b_1 + a_1\beta + a_1(y + z)$.

Proof. Let $Y, Z \in \bar{\mathcal{B}}_2$. Then from

$$\|Y - S\| \le \delta_1' \quad \text{and} \quad \|B^{-1}AS - B^{-1}AZ\| \le \|B^{-1}A\|\|Z - S\| \le a_1\delta_1',$$

we have

$$\|Y\| \le \beta + \delta_1' \quad \text{and} \quad \|B^{-1}AZ\| \le b_1 + a_1\delta_1'.$$

The reminder of the proof is similar to Lemma 3.1. □

Remark 3.3. In Lemmas 3.1 and 3.2, if $\|\cdot\|_p$ denotes either the 1-, 2- or ∞-matrix norms then, from the property of the Kronecker product [13, p. 439], the bound condition $b_1 + a_1\beta < 1$ implies

$$
\begin{aligned}
1 > b_1 + a_1\beta &= \|B^{-1}AS\|_p + \|S\|_p\|B^{-1}A\|_p \\
&= \left\|I_{m^2} \otimes B^{-1}AS\right\|_p + \left\|S^T \otimes B^{-1}A\right\|_p \\
&\ge \left\|I_{m^2} \otimes B^{-1}AS + S^T \otimes B^{-1}A\right\|_p \\
&\ge \rho\left(\mathcal{F}_1'[S]\right),
\end{aligned}
$$

where $\mathcal{F}_1'[S]$ is the Frechet derivative of $\mathcal{F}_1$ at S.

From Theorem 2.7, Lemma 3.1 and Lemma 3.2 the local convergence theorem for $\mathcal{F}_1$ follows.

Theorem 3.4. *Under the assumption of Lemma 3.1, if*

$$b_1 + a_1\beta < 1,$$

then for any $X_0 \in \mathcal{B}(S, \delta_1)$*, the iterates generated by* (1.2) *of* $i = 1$ *converges to* S*. Moreover,* S *is the unique solvent of QME* (1.1) *in* $\bar{\mathcal{B}}\,(S, \delta_1')$ *and*

$$\limsup_{k \to \infty} \sqrt[k]{\|X_k - S\|} \le b_1 + a_1\beta,$$

where δ_1 *and* δ_1' *are in* (3.3) *of Lemma 3.1 and in* (3.4) *in Lemma 3.2 respectively.*

Theorem 3.4 asserts that unless $\|B^{-1}A\|$ is much less than 1 we can not assure the convergence to a solvent confidently even though a initial matrix is properly closed to the solvent.

4. Local convergence for $\mathcal{F}_2$

Since $\mathcal{F}_2$ (1.3) and $\mathcal{F}_3$ (1.4) have the matrix inverse operations, we need the following perturbation lemma.

Lemma 4.1. *For $L, M, N \in \mathbb{C}^{m\times m}$, if L is nonsingular and $\|L^{-1}\|\|M\| < 1$, then $L - M$ is also nonsingular and*

$$\|(L-M)^{-1}N\| \le \frac{\|L^{-1}N\|}{1-\|L^{-1}\|\|M\|} \quad \textit{and} \quad \|N(L-M)^{-1}\| \le \frac{\|NL^{-1}\|}{1-\|M\|\|L^{-1}\|}.$$

For $X \in \mathbb{R}^{m\times m}$ if $\mathcal{F}_2(X)$ (1.3) is well-defined, then from the Neumann lemma for a sufficiently small perturbation matrix $H \in \mathbb{R}^{m\times m}$ we have

$$\begin{aligned}\mathcal{F}_2(X+H) &= ((B-AX)-AH)^{-1}C \\ &= (B-AX)^{-1}C + (B-AX)^{-1}AH((B-AX)-AH)^{-1}C \\ &= \mathcal{F}_2(X) + (B-AX)^{-1}AH(B-AX-AH)^{-1}C.\end{aligned}$$

From the above expression $H := Y - X$ yields

$$\mathcal{F}_2(Y) - \mathcal{F}_2(X) = (B-AX)^{-1}A(Y-X)(B-AY)^{-1}C$$

and

$$\begin{aligned}\|\mathcal{F}_2(Y) - \mathcal{F}_2(X)\| &= \|(B-AX)^{-1}A(Y-X)(B-AY)^{-1}C\| \\ &\le \|(B-AX)^{-1}A\|\|(B-AY)^{-1}C\|\,\|Y-X\|.\end{aligned}$$

Define

$$\Gamma_2(X,Y) := \|(B-AX)^{-1}A\|\|(B-AY)^{-1}C\|, \tag{4.1}$$

then we have following lemmas.

Lemma 4.2. *Let $S \in \mathbb{R}^{m\times m}$ be a solvent of QME* (1.1)*, and suppose that $B - AS$ is nonsingular,*

$$\|(B-AS)^{-1}\|\|A\| \le a_2, \ \|S\| \le \beta,$$

where a_2 and β are positive constants. Then, for a positive number $\delta > 0$ such that $\delta < a_2^{-1}$, $\mathcal{F}_2$ is well defined on $\bar{\mathcal{B}} := \bar{\mathcal{B}}(S,\delta)$

$$\Gamma_2(X,S) \le \bar{\mu}_2(\|X-S\|) < \bar{\mu}_2(\delta), \qquad \forall X \in \mathcal{B}$$

and

$$\Gamma_2(Y,Z) \le \bar{\nu}_2(\|Y-S\|, \|Z-S\|) \le \bar{\nu}_2(\delta,\delta), \qquad \forall Y, Z \in \bar{\mathcal{B}},$$

where Γ_2 is in (4.1),

$$\bar{\mu}_2(x) = \frac{a_2\beta}{(1-a_2x)} \quad \textit{and} \quad \bar{\nu}_2(y,z) = \frac{a_2\beta}{(1-a_2y)(1-a_2z)}. \tag{4.2}$$

Proof. From the assumption we have

$$S = (B - AS)^{-1}C.$$

Let $X \in \mathcal{B}$. Then since

$$\|(B - AS)^{-1}A(X - S)\| \le \|(B - AS)^{-1}\|\|A\|\|X - S\| < a_2\delta < 1,$$

from

$$B - AX = (B - AS) - A(X - S)$$

and Lemma 4.1, $B - AX$ is nonsingular and

$$\left\|(B - AX)^{-1} A\right\| \le \frac{a_2}{1 - a_2\|X - S\|} < \frac{a_2}{1 - a_2\delta}.$$

Therefore we have

$$\begin{aligned}\Gamma_2(X, S) &= \|(B - AX)^{-1}A\|\|S\| \\ &\le \frac{a_2\beta}{1 - a_2\|X - S\|} \left(= \bar{\mu}_2(\|X - S\|)\right) \\ &< \frac{a_2\beta}{1 - a_2\delta} \left(= \bar{\mu}_2(\delta)\right).\end{aligned}$$

It is easy to prove the remaining result. □

Lemma 4.3. *Under the assumption of Lemma 4.2, if*

$$a_2\beta < 1,$$

then

$$\Gamma_2(X, S) \le \bar{\mu}_2(\|X - S\|) < 1, \quad \forall\ X \in \mathcal{B} := \mathcal{B}(S, \delta_2),$$

where Γ_2 *and* $\bar{\mu}_2$ *are in* (4.1) *and* (4.2) *of Lemma 4.2 respectively, and*

$$\delta_2 := \frac{1 - a_2\beta}{a_2}. \tag{4.3}$$

Proof. Since $1 - a_2\beta > 0$ and $\|X - S\| \le \delta_2$, clearly we have

$$\bar{\mu}_2(\|X - S\|) = \frac{a_2\beta}{1 - a_2\|X - S\|} \le \frac{a_2\beta}{1 - a_2\delta_2} \le 1.$$

Therefore from Lemma 4.1 we have required result. □

Lemma 4.4. *Under the assumption of Lemma 4.2, if*

$$a_2\beta < 1,$$

then

$$\Gamma_2(Y, Z) \le \bar{\nu}_2(\|Y - S\|, \|Z - S\|) < 1, \quad \forall\ Y, Z \in \bar{\mathcal{B}} := \bar{\mathcal{B}}(S, \delta_2'),$$

where Γ_2 *and* $\bar{\nu}_2$ *are in* (4.1) *and* (4.2) *of Lemma 4.2 respectively, and*

$$\delta_2' \in \left(0, \frac{1 - \sqrt{a_2\beta}}{a_2}\right). \tag{4.4}$$

Proof. If we define the scalar quadratic polynomial such as

$$f(x) = a_2^2 x^2 - 2a_2 x + 1 - a_2\beta,$$

then from

$$\begin{cases} f(0) = 1 - a_2\beta > 0 \\ f\left(a_2^{-1}\right) = 1 - 2 + 1 - a_2\beta < 0 \end{cases}$$

we acquire

$$f(x) > 0, \quad \forall x \in \left(0,\ \frac{1 - \sqrt{a_2\beta}}{a_2}\right).$$

This implies that

$$\begin{aligned} & a_2^2 \left(\delta_2'\right)^2 - 2a_2 \left(\delta_2'\right) + 1 - a_2\beta > 0 \\ \Leftrightarrow & \left(1 - a_2 \left(\delta_2'\right)\right)^2 - a_2\beta > 0 \\ \Leftrightarrow & \left(1 - a_2 \left(\delta_2'\right)\right)^2 > a_2\beta \\ \Leftrightarrow & 1 > \frac{a_2\beta}{\left(1 - a_2 \left(\delta_2'\right)\right)^2}. \end{aligned} \tag{4.5}$$

Let $Y, Z \in \bar{\mathcal{B}}$. Then since $\delta_2' < \alpha_2^{-1}$, from Lemma 4.2 we have

$$\Gamma_2(Y, Z) \le \bar{\nu}_2(\|X - S\|, \|Y - S\|) \le \bar{\nu}_2(\delta_2', \delta_2'). \tag{4.6}$$

Also from (4.5) we get

$$\bar{\nu}_2(\delta_2', \delta_2') = \frac{a_2\beta}{\left(1 - \beta\delta_2'\right)^2} < 1. \tag{4.7}$$

Therefore from (4.6) and (4.7) we have required result. □

Remark 4.5. In Lemmas 4.3 and 4.4 if $\|\cdot\|_p$ denotes either the 1-, 2- or ∞-matrix norms, then the bound condition $a_2\beta < 1$ implies

$$\begin{aligned} 1 > a_2\beta &= \|\left(B - AS\right)^{-1} A\|_p \|S\|_p \\ &= \left\|S^T \otimes (B - AS)^{-1} A\right\|_p \\ &\ge \rho\left(\mathcal{F}_2'[S]\right), \end{aligned}$$

where $\mathcal{F}_2'[S]$ is the Frechet derivative of $\mathcal{F}_2$ at S.

Theorem 2.7, Lemma 4.2 and Lemma 4.4 lead the following theorem.

Theorem 4.6. *Under the assumption of Lemma 4.2, if*

$$a_2\beta < 1,$$

then, for any $X_0 \in \mathcal{B} := \mathcal{B}(S, \delta_2)$*, the iterations generated by* (1.2) *of* $i = 2$ *converges to* S*. Moreover,* S *is the unique solvent of QME* (1.1) *in* $\bar{\mathcal{B}}(S, \delta_2')$ *and*

$$\limsup_{k \to \infty} \sqrt[k]{\|X_k - S\|} \le a_2\beta,$$

where δ_2 *and* δ_2' *are in* (4.3) *of Lemma 4.3 and* (4.4) *in Lemma 4.4 respectively.*

5. Local convergence for $\mathcal{F}_3$

For $X \in \mathbb{R}^{m\times m}$ if $\mathcal{F}_3(X)$ (1.5) is well-defined, then by the Neumann lemma for a sufficiently small matrix $H \in \mathbb{R}^{m\times m}$ we have

$$\begin{aligned}\mathcal{F}_3(X+H) &= A^{-1}\left(B - C(X+H)^{-1}\right)\\ &= A^{-1}\left(B - C\left(X^{-1} - X^{-1}H(X+H)^{-1}\right)\right)\\ &= \mathcal{F}_3(X) + A^{-1}CX^{-1}H(X+H)^{-1}.\end{aligned}$$

Let $H = Y - X$. Then from the above equality we have

$$\mathcal{F}_3(Y) - \mathcal{F}_3(X) = A^{-1}CX^{-1}(Y-X)Y^{-1}$$

and

$$\begin{aligned}\|\mathcal{F}_3(Y) - \mathcal{F}_3(X)\| &= \left\|A^{-1}CX^{-1}(Y-X)Y^{-1}\right\|\\ &\le \left\|A^{-1}CX^{-1}\right\|\left\|Y^{-1}\right\|\|Y-X\|.\end{aligned}$$

Define

$$\Gamma_3(X,Y) := \left\|A^{-1}CY^{-1}\right\|\left\|X^{-1}\right\| \tag{5.1}$$

then we have the following lemma.

Lemma 5.1. *Let $S \in \mathbb{R}^{m\times m}$ be a solvent of QME* (1.1) *and suppose that A and S are nonsingular,*

$$\|A^{-1}CS^{-1}\| \le a_3,\ \|S^{-1}\| \le \beta',$$

where a_3 and β' are positive constants. Then, for a positive number $\delta > 0$ such that $\delta < \beta_3$, $\mathcal{F}_3$ is well defined on $\bar{\mathcal{B}} := \bar{\mathcal{B}}(S,\delta)$,

$$\Gamma_3(X,S) \le \bar{\mu}_3(\|X-S\|) < \bar{\mu}_3(\delta), \qquad \forall X \in \mathcal{B}$$

and

$$\Gamma_3(Y,Z) \le \bar{\nu}_3(\|Y-S\|, \|Z-S\|) \le \bar{\nu}_3(\delta,\delta), \qquad \forall Y,Z \in \bar{\mathcal{B}},$$

where Γ_3 is in (5.1),

$$\bar{\mu}_3(x) = \frac{a_3\beta'}{(1-\beta' x)} \quad \text{and} \quad \bar{\nu}_3(y,z) = \frac{a_3\beta'}{(1-\beta' y)(1-\beta' z)}. \tag{5.2}$$

Proof. Let $X, Y, Z \in \mathcal{B}$. Then since

$$\|S^{-1}(S-X)\| < \beta'\delta < 1$$

from $X = S - (S - X)$ and the Neumann lemma, X is nonsingular and

$$\|X^{-1}\| \le \frac{\beta'}{1-\beta'\|X-S\|} < \frac{\beta'}{1-\beta'\delta}$$

and

$$\|Y^{-1}\| \le \frac{\beta'}{1-\beta'\|Y-S\|} \le \frac{\beta'}{1-\beta'\delta}.$$

From Lemma 2.2 also we have

$$\|A^{-1}CZ^{-1}\| \le \frac{a_3}{1-\beta'\|Z-S\|} \le \frac{a_3}{1-\beta'\delta}.$$

Therefore we have required results. □

Lemma 5.2. *Under the assumption of Lemma 5.1, if*

$$a_3\beta' < 1,$$

then

$$\Gamma_3(X,S) \le \bar{\mu}_3\left(\|X-S\|\right) < 1, \quad \forall\ X \in \mathcal{B} := \mathcal{B}(S,\delta_3)$$

and

$$\Gamma_3(Y,Z) \le \bar{\nu}_3\left(\|Y-S\|, \|Z-S\|\right) < 1, \quad \forall\ X \in \bar{\mathcal{B}} := \bar{\mathcal{B}}(S,\delta_3'),$$

where Γ_3 *is in* (5.1), $\bar{\mu}_3$ *and* $\bar{\nu}_3$ *are in* (5.1) *of Lemma 5.1, and*

$$\delta_3 := \frac{1-a_3\beta'}{\beta'} > 0 \text{ and } \delta_3' \in \left(0,\ \frac{1-\sqrt{a_3\beta'}}{\beta'}\right). \tag{5.3}$$

Proof. The proof is similar to those of Lemma 4.3 and Lemma 4.4. □

Remark 5.3. In Lemma 5.2 if $\|\cdot\|_p$ denotes either the 1-, 2- or ∞- matrix norms, then the bound condition $a_3\beta' < 1$ implies

$$\begin{aligned} 1 > a_3\beta' &= \|A^{-1}CS^{-1}\|_p\|S^{-1}\|_p \\ &= \left\|\left(S^{-1}\right)^T \otimes A^{-1}CS^{-1}\right\|_p \ge \rho\left(\mathcal{F}_3'[S]\right), \end{aligned}$$

where $\mathcal{F}_3'[S]$ is the Frechet derivative of $\mathcal{F}_3$ at S.

From Theorem 2.7 and Lemma 5.2 we have the following theorem.

Theorem 5.4. *Under the assumption of Lemma 5.1, if*

$$a_3\beta' < 1,$$

then, for any $X_0 \in \mathcal{B} := \mathcal{B}(S,\delta_3)$, *the iterates generated by* (1.2) *of* $i = 3$ *converges to* S. *Moreover,* S *is the unique solvent of QME* (1.1) *in* $\bar{\mathcal{B}}\left(S,\delta_3'\right)$ *and*

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - S\|} \le a_3\beta',$$

where δ_3 *and* δ_3' *are in* (5.2) *of Lemma 5.1.*

6. Numerical experiments

In this section, we will consider the local convergence for $\mathcal{F}_1$. The solvent S of the quadratic matrix equation derived by the Newton's method. We calculate some constants and δ, the radius of convergence derived by the function in this paper.

Example 6.1. In this example, we will consider the quadratic matrix equation $AX^2 + BX + C = 0$ where coefficient matrices are following matrices.

$$A = I,\ B = \begin{bmatrix} 20 & -10 & & & \\ -10 & 30 & & & \\ & & \ddots & \ddots & \\ & & \ddots & 30 & -10 \\ & & & -10 & 20 \end{bmatrix},$$

$$C = \begin{bmatrix} 15 & -5 & & & \\ -5 & 15 & & & \\ & & \ddots & \ddots & \\ & & \ddots & 15 & -5 \\ & & & -5 & 15 \end{bmatrix}.$$

This is a well-known example for finding the non-positive maximal solution. From the above matrices we can calculate constants by using matrix 2-Norm.

F_1 for Example 6.1	$n = 10$	$n = 30$	$n = 50$	$n = 70$
a_1	0.1000	0.1000	0.1000	0.1000
b_1	0.0743	0.0734	0.0734	0.0734
β	0.8848	0.8843	0.8843	0.8843
$\frac{1-b_1-a_1\beta}{a_1}$	8.3726	8.3821	8.3821	8.3821
$\frac{1-b_1-a_1\beta}{a_1}$	4.1863	4.1911	4.1911	4.1911

F_2 for Example 6.1	$n = 10$	$n = 30$	$n = 50$	$n = 70$
a_2	0.1069	0.1061	0.1060	0.1059
β	0.8848	0.8843	0.8843	0.8843
$\frac{1-a_2\beta}{a_2}$	8.4658	8.5388	8.5515	8.5557
$\frac{1-\sqrt{a_2\beta}}{a_2}$	6.4742	6.5364	6.5471	6.5507

Specially $n = 2$, we can set the coefficient matrices as follows.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 20 & -10 \\ -10 & 20 \end{bmatrix},\ C = \begin{bmatrix} 15 & -5 \\ -5 & 15 \end{bmatrix}.$$

From the above coefficient matrices, we find the solvent S and compute constants.

$$S = \begin{bmatrix} -0.9046 & -0.2224 \\ -0.2224 & -0.9046 \end{bmatrix},$$

$$V = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix},$$

$$D = \begin{bmatrix} -1.1270 & 0 \\ 0 & -0.6822 \end{bmatrix}.$$

F_1	a_1	b_1	β	$\frac{1-b_1-a_1\beta}{a_1}$	$\frac{1-b_1-a_1\beta}{2a_1}$
	0.1000	0.1127	1.1270	7.7460	1.1270

We can easily see the local convergence by graph when we set starting matrices X_0 as following form.

$$X_0 = V(D + D_0)V^{-1} \quad \text{where} \quad D_0 = \text{diag}(x, y).$$

By using $D_0 = \text{diag}(x, y)$, we graph local convergence of $\mathcal{F}_1$ in rectangular coordinate system. The origin is the solvent. x-axis and y-axis present elements of $D_0 = \text{diag}(x, y)$. On the graph, black points are interior points of the circle whose radius is δ_1. The points converge to the solution are presented by red color (inner part of rectangular except the circle) and yellow points (outer part of the inner rectangular) don't converge.

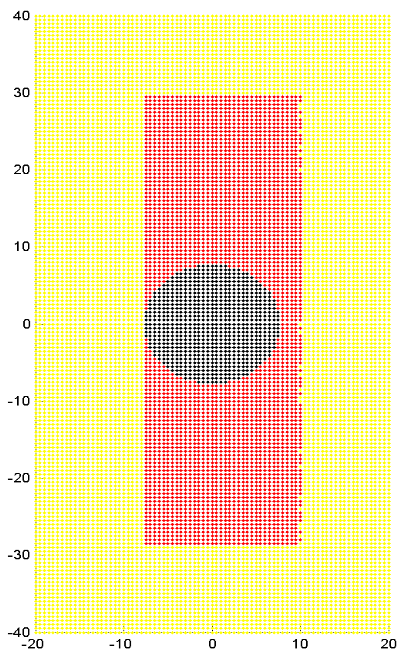

FIGURE 6.1. The local convergence of F_1 in Example 6.1

Example 6.2. In this example, we will consider the quadratic matrix equation $CX^2 + BX + A = 0$ where the coefficient matrices are in the above example with the starting matrix is the identity matrix. The inverse of the solvent of this equation is the solvent of the matrix equation $AX^2 + BX + C = 0$ in Example 6.1.

F_3 for Example 6.1	$n = 10$	$n = 30$	$n = 50$	$n = 70$
a_3	0.1118	0.1094	0.1093	0.1093
β'	0.8848	0.8843	0.8843	0.8843
$\frac{1-a_3\beta'}{\beta'}$	1.0184	1.0215	1.0215	1.0215
$\frac{1-\sqrt{a_3\beta'}}{\beta'}$	0.7747	1.0215	0.7793	0.7793

Example 6.3. Consider the following coefficient matrices and $AX^2+BX+C = 0$. It is easy to verify that $\|B^{-1}A\| + \|B^{-1}C\| < 1$ where $\alpha = 4$ and $\beta = 30$ [19].

$$A_{ij} = \begin{cases} 2/\alpha & i+j = n+1 \\ 1/\alpha & \text{others,} \end{cases} \quad B_{ij} = \begin{cases} 15 & i+j = n+1 \\ -3 & i+1 = j \\ -3 & i = j+n-2 \\ -1 & \text{others,} \end{cases} \quad C_{ij} = \begin{cases} 15/\beta & i = j \\ -1/\beta & \text{others.} \end{cases}$$

From the above matrices we can calculate constants by using matrix 2-Norm.

F_1 for Example 6.2	$n = 10$	$n = 30$	$n = 50$
a_1	0.7236	0.4825	0.3538
b_1	0.0398	0.0138	0.0110
β	0.0552	0.0386	0.0387
$\frac{1-b_1-a_1\beta}{a_1}$	1.2716	2.0053	2.7564
$\frac{1-b_1-a_1\beta}{a_1}$	0.6358	1.0026	1.3782

F_2 for Example 6.2	$n = 10$	$n = 30$	$n = 50$
a_2	0.9106	2.6591	5.6171
β	0.0552	0.0386	0.0387
$\frac{1-a_2\beta}{a_2}$	1.0429	0.3375	0.1393
$\frac{1-\sqrt{a_2\beta}}{a_2}$	0.8519	0.2556	0.0950

The local convergence of functional iterative methods can be guaranteed by using the Lipschitz condition. It is important to find a set which makes the methods satisfy the Lipschitz condition. In this paper, we suggest a procedure which make easy to find the set. In specially, we show the local convergence of fixed point methods for a quadratic matrix equation.

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