Bull. Korean Math. Soc. **54** (2017), No. 1, pp. 187–198
https://doi.org/10.4134/BKMS.b160014
pISSN: 1015-8634 / eISSN: 2234-3016

ESSENTIAL NORM OF THE COMPOSITION OPERATORS BETWEEN BERGMAN SPACES OF LOGARITHMIC WEIGHTS

Ern Gun Kwon and Jinkee Lee

Abstract. We obtain some necessary and sufficient conditions for the boundedness of the composition operators between weighted Bergman spaces of logarithmic weights. In terms of the conditions for the boundedness, we compute the essential norm of the composition operators.

1. Introduction

1.1. Logarithmic weights and modified counting functions

For $-1 < \gamma < \infty$, $\delta \leq 0$ and $0 < p < \infty$, we define the weighted Bergman space $A^p_{\omega_{\gamma,\delta}}$ as consisting of holomorphic functions f on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$ for which

$$\Big|\Big|f\Big|\Big|^p_{A^p_{\omega_{\gamma,\delta}}} := \int_{\mathbb{D}} |f(z)|^p \omega_{\gamma,\delta}(z)\ dA(z) < \infty,$$

where the weight is defined by

$$\omega_{\gamma,\delta}(z) = \Big(\log\frac{1}{|z|}\Big)^{\gamma}\Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{\delta}$$

and dA is the Lebesgue measure on $\mathbb{D}$ normalized to be $A(\mathbb{D}) = 1$. It is same as the space of holomorphic functions f satisfying

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\gamma}\Big(\log\frac{1}{1-|z|}\Big)^{\delta}\ dA(z) < \infty.$$

When $\gamma = 0$, $\delta = 0$ the space becomes the Bergman space A^p, and when $\delta = 0$ it is the weighted Bergman space A^p_γ.

Received January 9, 2016; Revised January 21, 2016.
2010 *Mathematics Subject Classification.* Primary 47B33, 32A36, 46E15.
Key words and phrases. composition operator, logarithmic weighted, Bergman space, essential norm.
This work was financially supported by NRF 2010-0021986.

For holomorphic self-maps φ of $\mathbb{D}$, $0 \le r < 1$, $0 \le \gamma < \infty$, $\delta \le 0$ and $a \in \mathbb{D} \setminus \{\varphi(0)\}$, we define $N_{\varphi,\gamma,\delta}$ as

$$N_{\varphi,\gamma,\delta}(r,a) := \sum_{z_j(a)\in\varphi^{-1}(a)} \Big(\log \frac{r}{|z_j(a)|}\Big)^{\gamma} \Big[\log\Big(1 + \frac{1}{\log \frac{r}{|z_j(a)|}}\Big)\Big]^{\delta}$$

with $|z_j(a)| < r$, counting multiplicities, and

$$N_{\varphi,\gamma,\delta}(a) = N_{\varphi,\gamma,\delta}(1,a) := \sum_{z_j(a)\in\varphi^{-1}(a)} \omega_{\gamma,\delta}(z_j(a)).$$

We consider $N_{\varphi,\gamma,\delta}(r,a)$ to be defined on the space $\mathbb{D}\backslash\{\varphi(0)\}$ and $N_{\varphi,\gamma,\delta}(r,a) = 0$ if a is not in $\varphi(r\mathbb{D})$ where $r\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$.

When $\delta = 0$, $N_{\varphi,\gamma,\delta}$ coincides with the generalized Nevanlinna counting function $N_{\varphi,\gamma}$ introduced by J. H. Shapiro ([5]) as, for $a \in \mathbb{D}\backslash\{\varphi(0)\}$, $0 \le r < 1$ and $\gamma \ge 0$,

$$N_{\varphi,\gamma}(r,a) := \sum_{z\in\varphi^{-1}(a),\ |z|<r} \Big(\log \frac{r}{|z|}\Big)^{\gamma},$$

$$N_{\varphi,\gamma}(a) = N_{\varphi,\gamma}(1,a) = \sum_{z\in\varphi^{-1}(a)} \Big(\log \frac{1}{|z|}\Big)^{\gamma}.$$

1.2. Boundedeness and essential norm of composition operator

Any holomorphic self-map φ of $\mathbb{D}$ induces the composition operator $\mathcal{C}_\varphi$ on holomorphic function spaces as $\mathcal{C}_\varphi f(z) = f(\varphi(z))$, $z \in \mathbb{D}$. The linear operator $\mathcal{C}_\varphi$ is bounded on A^p by Littlewood's Subordination Theorem (see [1]). Concerning $\mathcal{C}_\varphi$ between different Bergman spaces, W. Smith ([6]) characterized the condition on φ that makes $\mathcal{C}_\varphi$ between weighted Bergman spaces bounded:

Theorem 1.1 ([6], Theorem 3.1 and Theorem 4.3)**.** *Let $0 < p \le q$, $-1 < \alpha, \beta < \infty$. Then $\mathcal{C}_\varphi : A^p_\alpha \to A^q_\beta$ is bounded if and only if*

$$N_{\varphi,\beta+2}(a) = O\Big(\Big[\log \frac{1}{|a|}\Big]^{(\alpha+2)q/p}\Big) \qquad (|a| \to 1^-).$$

Later, F. Pérez-González, J. Rättyä and D. Vukotić established more equivalences.

Theorem 1.2 ([4], Theorem 1)**.** *For $0 < p \le q < \infty$, $-1 \le \alpha < \infty$ and $-1 < \beta < \infty$. Then the following statements are equivalent: for $0 \le s < \infty$*

(1) $C_\varphi : A^p_\alpha \to A^q_\beta$ *is bounded;*

(2) $N_{\varphi,\beta+2}(z) = O\Big(\log \frac{1}{|z|}\Big)^{\frac{(\alpha+2)q}{p}} \qquad (|z| \to 1^-)$;

(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{q(2+\alpha)}{p}+s} |\varphi'(z)|^s (1-|z|^2)^{s+\beta} dA(z) < \infty$;

(4) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{q(2+\alpha)}{p}+s} (1-|\varphi(z)|^2)^s (1-|z|^2)^{\beta} dA(z) < \infty$,

where φ_a *is the Möbius transformation*: $\varphi_a(z) = \frac{a-z}{1-\bar{a}z},\ z \in \mathbb{D}$.

We recall that the essential norm of an operator T is the distance from T to the compact operators; that is,

$$||T||_e := \inf\{||T - K|| : K \text{ is compact}\},$$

where $||\cdot||$ denotes the usual operator norm. Shapiro ([5]) expressed the essential norm of the composition operator on $A^2_\alpha(\mathbb{D})$ in terms of the generalized Nevanlinna counting function. Furthermore, F. Pérez-González, J. Rättyä and D. Vukotić gave several quantities for the essential norm $||C_\varphi||_e$.

Theorem 1.3 ([4], Theorem 5)**.** *Let* $1 < p \leq q < \infty$, $-1 \leq \alpha < \infty$ *and* $-1 < \beta < \infty$. *If* $C_\varphi : A^p_\alpha \to A^q_\beta$ *is bounded, then the following quantities are comparable*: *for* $0 \leq s < \infty$

$$A = ||C_\varphi||^q_e;$$

$$B = \limsup_{|z|\to 1} \frac{N_{\varphi,\beta+2}(z)}{\left(\log \frac{1}{|z|}\right)^{q(\alpha+2)/p}};$$

$$C = \limsup_{|a|\to 1} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{q(2+\alpha)}{p}+s} |\varphi'(z)|^s (1-|z|^2)^{s+\beta} dA(z);$$

$$D = \limsup_{|a|\to 1} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{q(2+\alpha)}{p}+s} (1-|\varphi(z)|^2)^s (1-|z|^2)^{\beta} dA(z).$$

1.3. Main results of this paper

Our purpose of this paper is to extend the results of Theorem 1.2 and Theorem 1.3 up to $A^p_{\omega_{\gamma,\delta}}$. That is, we give equivalent characterizations that provide the boundedness of composition operator $\mathcal{C}_\varphi$ from one $A^p_{\omega_{\gamma,\delta}}$ to another, and obtain parallel equivalences for the essential norm of the composition operator. The followings are our main results.

Theorem 1.4. *Let* $0 < p \leq q < \infty$, $-1 < \gamma_1, \gamma_2 < \infty$ *and* $\delta_1, \delta_2 \leq 0$. *Then there exists* $s = s(\omega_{\gamma_1,\delta_1}) > 1$ *with* $2 + \gamma_1 - \delta_1 < s < \infty$ *such that the following conditions are equivalent*:

(1) C_φ *maps* $A^p_{\omega_{\gamma_1,\delta_1}}$ *boundedly into* $A^q_{\omega_{\gamma_2,\delta_2}}$;

(2) $N_{\varphi,\gamma_2+2,\delta_2}(a) = O\Big(\Big[\Big(\log \dfrac{1}{|a|}\Big)^2 \omega_{\gamma_1,\delta_1}(a)\Big]^{q/p}\Big) \qquad (|a| \to 1^-)$;

(3) $$\sup_{a\in\mathbb{D}} \frac{1}{\left[\left(\log \frac{1}{|a|}\right)^2 \omega_{\gamma_1,\delta_1}(a)\right]^{q/p}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{sq/p}}{|1-\bar{a}z|^{sq/p+2}} N_{\varphi,\gamma_2+2,\delta_2}(z) dA(z) < \infty;$$

(4) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^{\frac{(2+\gamma_1)q}{p}+2} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\frac{\delta_1 q}{p}} N_{\varphi,\gamma_2+2,\delta_2}(z) dA(z) < \infty;$$

(5) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{(2+\gamma_1)q}{p}+2} |\varphi'(z)|^2 \Big[\log\Big(1 - \frac{1}{\log|\varphi(z)|}\Big)\Big]^{-\frac{\delta_1 q}{p}} \omega_{\gamma_2+2,\delta_2}(z) dA(z) < \infty.$$

Theorem 1.5. *For $1 < p \le q < \infty$, $-1 < \gamma_1, \gamma_2 < \infty$ and $\delta_1, \delta_2 \le 0$, if $C_\varphi : A^p_{\omega_{\gamma_1,\delta_1}} \to A^q_{\omega_{\gamma_2,\delta_2}}$ is bounded, then there exists $s = s(\omega_{\gamma_1,\delta_1}) > 1$ with $2 + \gamma_1 - \delta_1 < s < \infty$ such that the following quantities are comparable*:

$$A = ||C_\varphi||_e^q;$$

$$B = \limsup_{|z|\to 1} \frac{N_{\varphi,\gamma_2+2,\delta_2}(z)}{\left[\left(\log\frac{1}{|z|}\right)^2 \omega_{\gamma_1,\delta_1}(z)\right]^{q/p}};$$

$$C = \limsup_{|a|\to 1} \frac{1}{\left[\left(\log\frac{1}{|a|}\right)^2 \omega_{\gamma_1,\delta_1}(a)\right]^{q/p}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{sq/p}}{|1-\bar{a}z|^{sq/p+2}} N_{\varphi,\gamma_2+2,\delta_2}(z) dA(z);$$

$$D = \limsup_{|a|\to 1} \int_{\mathbb{D}} |\varphi_a'(z)|^{\frac{(2+\gamma_1)q}{p}+2} \left[\log\left(1 - \frac{1}{\log|z|}\right)\right]^{-\frac{\delta_1 q}{p}} N_{\varphi,\gamma_2+2,\delta_2}(z) dA(z);$$

$$E = \limsup_{|a|\to 1} \int_{\mathbb{D}} |\varphi_a'(\varphi(z))|^{\frac{(2+\gamma_1)q}{p}+2} |\varphi'(z)|^2 \left[\log\left(1 - \frac{1}{\log|\varphi(z)|}\right)\right]^{-\frac{\delta_1 q}{p}} \omega_{\gamma_2+2,\delta_2}(z) dA(z).$$

We may compare the case of $\delta_1 = 0$, $\delta_2 = 0$ and $\gamma_1 = \alpha$, $\gamma_2 = \beta$ in Theorem 1.4 and Theorem 1.5 to Theorem 1.2 and Theorem 1.3, respectively. The authors recently find a nice work of J. A. Peláez and J. Rättyä ([3]) wherein parts of Theorem 1.4 and Theorem 1.5 are included under a wide scope and different approach.

1.4. Contents of this paper

In Section 2, we introduce some properties for the modified Nevanlinna counting function and the weighted Bergman space of logarithmic weight. In Section 3, we prove some necessary and sufficient conditions for the boundedness of the composition operator. In Section 4, we compute the essential norm. All the functions f under consideration are assumed to be holomorphic on $\mathbb{D}$. Moreover, φ always denotes a holomorphic self map of $\mathbb{D}$. Also throughout this paper, the symbols " $\lesssim$ " means that the left hand side is bounded above by a constant multiple of the right hand side, where the constant is positive and independent of f. " $\gtrsim$ " means analogously. The symbol " $\approx$ " means " $\lesssim$ " and " $\gtrsim$ " simultaneously. We are to abbreviate $\omega_{\gamma,\delta}$ as ω, $\omega_{\gamma_1,\delta_1}$ as ω_1, and $\omega_{\gamma_2,\delta_2}$ as ω_2.

2. Background contents for $N_{\varphi,\gamma,\delta}$ and A^p_ω

In this section we introduce some useful tools for our main theorems. See [2], for proofs.

2.1. Subharmonic mean value property

For the generalized counting function $N_{\varphi,\gamma}$, the subharmonic mean value property appeared in [5]. Similar result holds for $N_{\varphi,\gamma,\delta}$.

Lemma A ([2], Theorem 2.1)**.** *Let $1 \le \gamma < \infty$ and $\delta \le 0$. If φ is a holomorphic self-map of $\mathbb{D}$ and $\triangle$ is a disc in $\mathbb{D}$ not containing $\varphi(0)$ with center a, then*

$$N_{\varphi,\gamma,\delta}(a) \le \frac{1}{|\triangle|}\int_{\triangle} N_{\varphi,\gamma,\delta}(u)\ dA(u),$$

where $|\triangle|$ is the normalized area measure of $\triangle : |\triangle| = \int \chi_{\triangle}(z)dA(z)$.

2.2. Change of a variable formula

Lemma B ([2], Lemma 2.3)**.** *If g is a non-negative measurable function on $\mathbb{D}$, then*

$$\int_{\mathbb{D}} (g\circ\varphi)(z)|\varphi'(z)|^2\omega(z)dA(z) = \int_{\mathbb{D}} g(u)N_{\varphi,\gamma,\delta}(u)dA(u).$$

Lemma C ([2], Lemma 2.4)**.** *For a holomorphic self-map φ of $\mathbb{D}$ and $a \in \mathbb{D}$ we have*

$$(N_{\varphi,\gamma,\delta})\circ\varphi_a = N_{\varphi_a\circ\varphi,\gamma,\delta}.$$

2.3. Quantities compared to the norm

Lemma D ([2], Lemma 3.2)**.** *For a fixed $r_0 \in [0,1)$,*

$$||f||^p_{A^p_\omega} \approx \int_{\mathbb{D}\smallsetminus r_0\mathbb{D}} |f(z)|^p\omega(z)\ dA(z).$$

Lemma E ([2], Lemma 3.3)**.** *Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \le 0$. If $f \in A^p_\omega$, then*

$$|f(z)| \lesssim \Big[\Big(\log\frac{1}{|z|}\Big)^2\omega(z)\Big]^{-\frac{1}{p}}||f||_{A^p_\omega}$$

for $z \in \mathbb{D}$ with $|z| \ge \frac{1}{2}$.

Lemma F ([2], Lemma 3.4)**.** *Let $\delta \le 0$, $-1 < \gamma < \infty$, and $\beta > \gamma - \delta$. Then for $a \in \mathbb{D}$ with $|a| \ge \frac{1}{2}$,*

$$\int_{\mathbb{D}} \frac{1}{|1-\bar{a}z|^{2+\beta}}\omega(z)\ dA(z) \lesssim \frac{1}{(1-|a|)^{\beta-\gamma}}\Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{\delta}.$$

Lemma G ([2], Theorem 3.6)**.** *Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \le 0$. Then for $r_0 \in [0,1)$, $f \in A^p_\omega$ if and only if*

$$\int_{\mathbb{D}\smallsetminus r_0\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\Big(\log\frac{1}{|z|}\Big)^2\omega(z)\ dA(z) < \infty.$$

Lemma H. *Let $0 < p < \infty$, $-1 < \gamma < \infty$ and $\delta \le 0$. Then*

$$||f||^p_{A^p_\omega} \approx |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\Big(\log\frac{1}{|z|}\Big)^2\omega(z)\ dA(z).$$

Proof. See the proof of Lemma G in [2]. □

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let $D(\lambda, \delta) = \{w : |\varphi_\lambda(w)| < \delta\}$ be the pseudohyperbolic disk with center λ and radius δ.

Lemma 3.1.

$$\log\Big(1 - \frac{1}{\log x}\Big) \approx \log\frac{1}{1-x}, \ \ \frac{1}{2} \le x < 1.$$

Proof. From

$$1 - x \le \log\frac{1}{x} \le (\log 4)(1-x), \quad \frac{1}{2} \le x < 1,$$

by letting $c = \log 4$, we have

$$c\log\Big(1 - \frac{1}{\log x}\Big) \ge \log\frac{c}{\log\frac{1}{x}} \ge \log\frac{1}{1-x}.$$

On the other hand,

$$\log\Big(1 - \frac{1}{\log x}\Big) \le \log\Big(1 + \frac{1}{1-x}\Big) \le \log\frac{1}{1-x} + \log 2 \le 2\log\frac{1}{1-x}. \qquad \square$$

Lemma 3.2. *Let* $|a| > \frac{1}{2}$. *Then*

$$\log\Big(1 - \frac{1}{\log|w|}\Big) \approx \log\Big(1 - \frac{1}{\log|a|}\Big), \quad w \in D(a, 1/2). \tag{1}$$

Proof. Let $w \in D(a, \frac{1}{2})$ be $w = \frac{a-z}{1-\bar{a}z}$ with $|z| < \frac{1}{2}$. Then

$$\frac{2|a|-1}{2-|a|} \le |w| \le \frac{1+2|a|}{2+|a|},$$

so that

$$\frac{1-|a|}{2+|a|} \le 1 - |w| \le \frac{3(1-|a|)}{2-|a|}.$$

Thus $1 - |w| \approx 1 - |a|$, whence the equivalence (1) follows from Lemma 3.1. $\square$

Lemma 3.3. *Let* $1 \le \gamma < \infty$, $\alpha, \delta \le 0$ *and* $0 < m,\ t < \infty$ *with* $m - t > -\alpha$. *Then*

$$N_{\varphi,\gamma,\delta}(a) = O\Big(\omega_{t,\alpha}(a)\Big) \quad (|a| \to 1^-) \tag{2}$$

if and only if

$$\sup_{a\in\mathbb{D}} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)\, dA(z) < \infty. \tag{3}$$

In particular,

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \approx \limsup_{|a|\to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)\, dA(z). \tag{4}$$

Proof. Suppose that (2) is satisfied. Then there exists r such that

$$N_{\varphi,\gamma,\delta}(a) \lesssim \omega_{t,\alpha}(a) \quad \text{for} \quad r \le |a| < 1.$$

Thus, taking $m - t > -\alpha$, by Lemma F

$$\begin{aligned}
(5) \qquad & \int_{\mathbb{D}\smallsetminus r\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \\
&\lesssim \int_{\mathbb{D}\smallsetminus r\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} \omega_{t,\alpha}(z)dA(z) \\
&\lesssim (1-|a|^2)^m \int_{\mathbb{D}} \frac{1}{|1-\bar{a}z|^{m+2}} \omega_{t,\alpha}(a)dA(z) \\
&\lesssim (1-|a|^2)^m \frac{1}{(1-|a|^2)^{m-t}} \Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{\alpha} \\
&= \omega_{t,\alpha}(a).
\end{aligned}$$

On the other hand

$$\begin{aligned}
(6) \qquad & \frac{1}{\omega_{t,\alpha}(a)} \int_{r\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \\
&= (1-|a|^2)^{m-t}\Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{-\alpha} \int_{r\mathbb{D}} \frac{N_{\varphi,\gamma,\delta}(z)}{|1-\bar{a}z|^{m+2}} dA(z) < \infty.
\end{aligned}$$

Therefore

$$\sup_{a\in\mathbb{D}} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) < \infty.$$

Conversely, suppose that (3) is satisfied. Then by Lemmas A and C

$$\begin{aligned}
(7) \qquad N_{\varphi,\gamma,\delta}(a) = N_{\varphi_a\circ\varphi,\gamma,\delta}(0) &\le 4\int_{\frac{1}{2}\mathbb{D}} N_{\varphi_a\circ\varphi,\gamma,\delta}(u)dA(u) \\
&= 4\int_{D(a,1/2)} N_{\varphi_a\circ\varphi,\gamma,\delta}(\varphi_a(z))|\varphi_a'(z)|^2 dA(z) \\
&= 4\int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z)|\varphi_a'(z)|^2 dA(z) \\
&= 4\int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z)\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z),
\end{aligned}$$

so that by the fact $1-|a| \approx |1-\bar{a}z|$ for a in the pseudohyperbolic disk $D(z,1/2)$, we have

$$\begin{aligned}
(8) \qquad N_{\varphi,\gamma,\delta}(a) &\lesssim \int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z)\frac{(1-|a|^2)^{2+m-2}}{|1-\bar{a}z|^{4+m-2}} dA(z) \\
&\lesssim \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \\
&\lesssim \omega_{t,\alpha}(a) \quad \textit{as} \quad |a| \to 1^-.
\end{aligned}$$

In particular, by (8), we have

$$\limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \lesssim \limsup_{|a|\to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z).$$

To prove the inverse inequality, putting

$$B = \limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)},$$

then given $\varepsilon > 0$, there exists $r_\varepsilon \in (0,1)$ such that $\frac{N_{\varphi,\gamma,\delta}(z)}{\omega_{t,\alpha}(z)} \leq B+\varepsilon$ for all $|z| \geq r_\varepsilon$. Therefore, by (5)

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \\
&= \int_{r_\varepsilon\mathbb{D}} + \int_{\mathbb{D}\setminus r_\varepsilon\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \\
&\lesssim \frac{(1-|a|^2)^m}{(1-r_\varepsilon)^{m+2}} \int_{r_\varepsilon\mathbb{D}} N_{\varphi,\gamma,\delta}(z)dA(z) + (B+\varepsilon)\int_{\mathbb{D}\setminus r_\varepsilon\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} \omega_{t,\alpha}(z)dA(z) \\
&\lesssim \frac{(1-|a|^2)^m}{(1-r_\varepsilon)^{m+2}} \int_{r_\varepsilon\mathbb{D}} N_{\varphi,\gamma,\delta}(z)dA(z) + (B+\varepsilon)\omega_{t,\alpha}(a),
\end{aligned}$$

and it follows that by (6)

$$\limsup_{|a|\to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_{\mathbb{D}} \frac{(1-|a|^2)^m}{|1-\bar{a}z|^{m+2}} N_{\varphi,\gamma,\delta}(z)dA(z) \lesssim \limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)}.$$

The proof is complete. $\square$

Lemma 3.4. *Let* $1 \leq \gamma < \infty$, $\alpha, \delta \leq 0$ *and* $-1 < t < \infty$. *Then* (2) *is equivalent to*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+t} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\alpha} N_{\varphi,\gamma,\delta}(z)dA(z) < \infty. \tag{9}$$

Furthermore,

$$\begin{aligned}
&\limsup_{|a|\to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \\
&\approx \limsup_{|a|\to 1} \int_{\mathbb{D}} |\varphi_a'(z)|^{2+t} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\alpha} N_{\varphi,\gamma,\delta}(z)dA(z).
\end{aligned} \tag{10}$$

Proof. The proof uses arguments similar to those in Lemma 3.3. By (2), there exists r such that

$$N_{\varphi,\gamma,\delta}(a) \lesssim \omega_{t,\alpha}(a) \quad \text{for} \quad r \leq |a| < 1.$$

Thus, we have

$$\int_{\mathbb{D}\setminus r\mathbb{D}} |\varphi_a'(z)|^{2+t} \Big[\log\Big(1 - \frac{1}{\log|z|}\Big)\Big]^{-\alpha} N_{\varphi,\gamma,\delta}(z)dA(z)$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{D}\setminus r\mathbb{D}} |\varphi_a'(z)|^{2+t}\Big[\log\Big(1-\frac{1}{\log|z|}\Big)\Big]^{-\alpha}\omega_{t,\alpha}(z)dA(z)\\
&= \int_{\mathbb{D}\setminus r\mathbb{D}} \frac{(1-|a|^2)^{2+t}}{|1-\bar{a}z|^{4+2t}}\Big(\log\frac{1}{|z|}\Big)^t\, dA(z)\\
&\lesssim \int_{\mathbb{D}} \frac{(1-|a|^2)^{2+t}}{|1-\bar{a}z|^{4+2t}}(1-|z|)^t\, dA(z) \lesssim 1,
\end{aligned}$$

and obviously

$$\int_{r\mathbb{D}} |\varphi_a'(z)|^{2+t}\Big[\log\Big(1-\frac{1}{\log|z|}\Big)\Big]^{-\alpha} N_{\varphi,\gamma,\delta}(z)dA(z) < \infty,$$

so that (9) is satisfied.

Conversely, by (7)

$$\begin{aligned}
N_{\varphi,\gamma,\delta}(a) &\lesssim 4\int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z)|\varphi_a'(z)|^{2+t}\Big(\frac{|1-\bar{a}z|^2}{1-|a|^2}\Big)^t dA(z)\\
&\lesssim (1-|a|^2)^t \int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z)|\varphi_a'(z)|^{2+t} dA(z).
\end{aligned}$$

Hence, for $|a| > \frac{1}{2}$, (1) and the condition (9) yield

$$\begin{aligned}
&\frac{N_{\varphi,\gamma,\delta}(a)}{\Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{\alpha}}\\
&\lesssim (1-|a|^2)^t \int_{D(a,1/2)} |\varphi_a'(z)|^{2+t}\Big[\log\Big(1-\frac{1}{\log|z|}\Big)\Big]^{-\alpha} N_{\varphi,\gamma,\delta}(z)dA(z)\\
&\lesssim (1-|a|^2)^t,
\end{aligned}$$

thus (2) is satisfied. The proof of (10) follows from a similar approach used in Lemma 3.3. The proof is complete. □

Proof of Theorem 1.4. (1) $\iff$ (2) follows from Theorem 1.1 in [2]. When $\gamma = \gamma_2 + 2$, $\delta = \delta_2$, $t = \frac{(2+\gamma_1)q}{p}$, $\alpha = \frac{\delta_1 q}{p}$ and $m = sq/p$ with $2+\gamma_1-\delta_1 < s < \infty$, (2) $\iff$ (3) follows from Lemma 3.3. (2) $\iff$ (4) follows from Lemma 3.4. (3) $\iff$ (4) follows from the change of variables formula, Lemma B. □

4. Proof of Theorem 1.5

For the same indices as Theorem 1.4, (4) and (10) ensure that $B \approx C$ and $B \approx D$, respectively. We are enough to prove $A \approx B$. For $a \in \mathbb{D}$ with $|a| > \frac{1}{2}$, consider the test function

$$k_a(z) = \frac{(1-|a|)^{-\frac{2\delta_1}{p}}}{(1-\bar{a}z)^{\frac{\gamma_1+2-2\delta_1}{p}}}\Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{-\frac{\delta_1}{p}}, \quad z \in \mathbb{D},$$

which is by Lemma F,

$$||k_a||^p_{A^p_{\omega_1}} \lesssim 1$$

and $k_a \to 0$ uniformly in compact subsets of $\mathbb{D}$ as $|a| \to 1$. If $K : A^p_{\omega_1} \to A^q_{\omega_2}$ is compact, then by Lemma D

$$||C_\varphi - K|| \geq \limsup_{|a|\to 1} ||C_\varphi(k_a) - Kk_a||_{A^q_{\omega_2}} \geq \limsup_{|a|\to 1} ||C_\varphi(k_a)||_{A^q_{\omega_2}} - \limsup_{|a|\to 1} ||Kk_a||_{A^q_{\omega_2}} = \limsup_{|a|\to 1} ||C_\varphi(k_a)||_{A^q_{\omega_2}}. \tag{11}$$

By Lemmas H and B,

$$||C_\varphi(k_a)||_{A^q_{\omega_2}} \gtrsim \int_{\mathbb{D}} |(k_a \circ \varphi)(z)|^{q-2} |(k_a \circ \varphi)'(z)|^2 \Big(\log \frac{1}{|z|}\Big)^2 \omega_2(z)\, dA(z) = \int_{\mathbb{D}} |k_a(u)|^{q-2} |k'_a(u)|^2 N_{\varphi,\gamma_2+2,\delta_2}(u)\, dA(u).$$

Inserting the test function and its derivative, the last integral equals

$$\Big(\frac{\gamma_1+2-2\delta_1}{p}\Big)^2 |a|^2 (1-|a|^2)^{-\frac{2\delta_1 q}{p}} \Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{-\frac{\delta_1 q}{p}} \times \int_{\mathbb{D}} \frac{1}{|1-\bar{a}u|^{\frac{(\gamma_1+2-2\delta_1)q}{p}+2}} N_{\varphi,\gamma_2+2,\delta_2}(u)\, dA(u).$$

The change of variables $u = \varphi_a(z)$ gives

$$\int_{\mathbb{D}} \frac{1}{|1-\bar{a}u|^{\frac{(\gamma_1+2-2\delta_1)q}{p}+2}} N_{\varphi,\gamma_2+2,\delta_2}(u)\, dA(u) = \frac{1}{(1-|a|^2)^2} \int_{\mathbb{D}} \frac{1}{|1-\bar{a}\varphi_a(z)|^{\frac{(\gamma_1+2-2\delta_1)q}{p}-2}} N_{\varphi,\gamma_2+2,\delta_2}(\varphi_a(z))\, dA(z).$$

Since $|1-\bar{a}\varphi_a(z)| \leq 2(1-|a|^2)$ if $|z| \leq \frac{1}{2}$, we have

$$\frac{1}{(1-|a|^2)^2} \int_{\mathbb{D}} \frac{1}{|1-\bar{a}\varphi_a(z)|^{\frac{(\gamma_1+2-2\delta_1)q}{p}-2}} N_{\varphi,\gamma_2+2,\delta_2}(\varphi_a(z))\, dA(z) \gtrsim \frac{1}{(1-|a|^2)^{(\gamma_1+2-2\delta_1)q/p}} \int_{\frac{1}{2}\mathbb{D}} N_{\varphi,\gamma_2+2,\delta_2}(\varphi_a(z))\, dA(z).$$

Collecting these up, it now follows that

$$||C_\varphi(k_a)||_{A^q_{\omega_2}} \gtrsim \frac{|a|^2}{(1-|a|^2)^{(\gamma_1+2)q/p}} \Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{-\delta_1 q/p} \times \int_{\frac{1}{2}\mathbb{D}} N_{\varphi,\gamma_2+2,\delta_2}(\varphi_a(z))\, dA(z).$$

Now applying Lemmas C and A, we obtain

$$||C_\varphi(k_a)||_{A^q_{\omega_2}} \gtrsim \frac{|a|^2}{(1-|a|^2)^{(\gamma_1+2)q/p}} \Big[\log\Big(1-\frac{1}{\log|a|}\Big)\Big]^{-\delta_1 q/p}$$

$$
\begin{aligned}
&\times \int_{\frac{1}{2}\mathbb{D}} N_{\varphi_a\circ\varphi,\gamma_2+2,\delta_2}(z)\ dA(z) \\
&\gtrsim \frac{|a|^2}{(1-|a|^2)^{(\gamma_1+2)q/p}}\left[\log\left(1-\frac{1}{\log|a|}\right)\right]^{-\delta_1 q/p} N_{\varphi_a\circ\varphi,\gamma_2+2,\delta_2}(0) \\
&= \frac{|a|^2}{(1-|a|^2)^{(\gamma_1+2)q/p}}\left[\log\left(1-\frac{1}{\log|a|}\right)\right]^{-\delta_1 q/p} N_{\varphi,\gamma_2+2,\delta_2}(a) \\
&\approx |a|^2\left[\left(\log\frac{1}{|a|}\right)^2\omega_{\gamma_1,\delta_1}(a)\right]^{-q/p} N_{\varphi,\gamma_2+2,\delta_2}(a).
\end{aligned}
$$

By (11), we get

$$
||C_\varphi||_e^q \gtrsim \limsup_{|a|\to 1}\frac{N_{\varphi,\gamma_2+2,\delta_2}(a)}{\left[\left(\log\frac{1}{|a|}\right)^2\omega_{\gamma_1,\delta_1}(a)\right]^{q/p}},
$$

and this means $A \gtrsim B$.

To show $A \lesssim B$, let $C_\varphi : A^p_{\omega_1} \to A^q_{\omega_2}$ be bounded and suppose

$$
\limsup_{|z|\to 1}\frac{N_{\varphi,\gamma_2+2,\delta_2}(z)}{\left[\left(\log\frac{1}{|z|}\right)^2\omega_1(z)\right]^{q/p}} = B > 0.
$$

Then there exists $r_0 \in (0,1)$ such that

$$
\frac{N_{\varphi,\gamma_2+2,\delta_2}(z)}{\left[\left(\log\frac{1}{|z|}\right)^2\omega_1(z)\right]^{q/p}} \le 2B \tag{12}
$$

for $|z| \ge r_0$. For a holomorphic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $\mathbb{D}$, let

$$
T_n f(z) = \sum_{k=0}^{n} a_k z^k, \quad R_n f(z) = \sum_{k=n+1}^{\infty} a_k z^k.
$$

Then $T_n : A^p_{\omega_1} \to A^q_{\omega_2}$ is compact, and

$$
||C_\varphi||_e = ||C_\varphi(T_n+R_n)||_e \le ||C_\varphi T_n||_e + ||C_\varphi R_n||_e = ||C_\varphi R_n||_e \le ||C_\varphi R_n||.
$$

Thus $||C_\varphi||_e \le \liminf_{n\to\infty}||C_\varphi R_n||$. Since $(R_n f\circ\varphi)(0)\to 0$ as $n\to\infty$, hence, by Lemmas H, B and (12),

$$
\begin{aligned}
&||C_\varphi||_e^q \\
\le\ & \liminf_{n\to\infty}||C_\varphi R_n||^q = \liminf_{n\to\infty}\sup_{||f||_{A^p_{\omega_1}}\le 1}||C_\varphi R_n f||^q_{A^q_{\omega_2}} \\
\approx\ & \liminf_{n\to\infty}\sup_{||f||_{A^p_{\omega_1}}\le 1}\int_{\mathbb{D}}|(R_n f\circ\varphi)(z)|^{q-2}|(R_n f\circ\varphi)'(z)|^2\left(\log\frac{1}{|z|}\right)^2\omega_2(z)\ dA(z) \\
=\ & \liminf_{n\to\infty}\sup_{||f||_{A^p_{\omega_1}}\le 1}\int_{\mathbb{D}}|R_n f(w)|^{q-2}|R_n f'(w)|^2 N_{\varphi,2+\gamma_2,\delta_2}(w)\ dA(w)
\end{aligned}
$$

$$\lesssim B \liminf_{n\to\infty} \sup_{||f||_{A^p_{\omega_1}}\le 1} \int_{\mathbb{D}} |R_n f(w)|^{q-2} |R_n f'(w)|^2 \left[\left(\log\frac{1}{|w|}\right)^2 \omega_1(w)\right]^{q/p} dA(w).$$

From Lemma E, we have

$$\begin{aligned} |R_n f(z)| &\lesssim \left[\left(\log\frac{1}{|z|}\right)^2 \omega_1(z)\right]^{-\frac{1}{p}} ||R_n f||_{A^p_{\omega_1}} \\ &\lesssim \left[\left(\log\frac{1}{|z|}\right)^2 \omega_1(z)\right]^{-\frac{1}{p}} ||f||_{A^p_{\omega_1}}, \end{aligned}$$

and by Lemma H, we obtain

$$\begin{aligned} &||C_\varphi||_e^q \\ &\lesssim B \liminf_{n\to\infty} \sup_{||f||_{A^p_{\omega_1}}\le 1} ||f||^{q-p}_{A^p_{\omega_1}} \int_{\mathbb{D}} |R_n f(w)|^{p-2} |R_n f'(w)|^2 \left(\log\frac{1}{|w|}\right)^2 \omega_1(w)\, dA(w) \\ &\approx B \liminf_{n\to\infty} \sup_{||f||_{A^p_{\omega_1}}\le 1} ||R_n f||^q_{A^p_{\omega_1}} \\ &\lesssim B \sup_{||f||_{A^p_{\omega_1}}\le 1} ||f||^q_{A^p_{\omega_1}} = B. \end{aligned}$$

The proof is complete.

References

[1] C. C. Cowen and B. D. Maccluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.

[2] E. G. Kwon and J. Lee, *Composition operators between Bergman Spaces of logarithmic weights*, Internat. J. Math. **26** (2015), no. 9, 1550068, 14 pp.

[3] J. A. Peláez and J. Rättyä, *Trace class criteria for Toeplitz and composition operators on small Bergman spaces*, preprint (submitted), http://arxiv.org/abs/1501.00131.

[4] F. Pérez-González, J. Rättyä, and D. Vukotić, *On composition operators acting between Hardy and weighted Beregman spaces*, Expo. Math. **25** (2007), no. 4, 309–323.

[5] J. H. Shapiro, *The essential norm of a composition operator*, Ann. Math. **127** (1987), no. 2, 375–404.

[6] W. Smith, *Composition operators between Bergman and Hardy spaces*, Trans. Amer. Math. Soc. **348** (1996), no. 6, 2331–2348.

Ern Gun Kwon
Department of Mathematics Education
Andong National University
Andong 760-749, Korea
E-mail address: egkwon@anu.ac.kr

Jinkee Lee
Department of Mathematics
Pusan National University
Busan 609-735, Korea
E-mail address: jklee235@pusan.ac.kr