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BJÖRLING FORMULA FOR MEAN CURVATURE ONE SURFACES IN HYPERBOLIC THREE-SPACE AND IN DE SITTER THREE-SPACE

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Abstract. We solve the Björling problem for constant mean curvature one surfaces in hyperbolic three-space and in de Sitter three-space. That is, we show that for any regular, analytic (and spacelike in the case of de Sitter three-space) curve γ and an analytic (timelike in the case of de Sitter three-space) unit vector field N along and orthogonal to γ, there exists a unique (spacelike in the case of de Sitter three-space) surface of constant mean curvature 1 which contains γ and the unit normal of which on γ is N. Some of the consequences are the planar reflection principles, and a classification of rotationally invariant CMC 1 surfaces.

1. Introduction

It is interesting that some surfaces in different space forms share similar properties. In particular, minimal surfaces in $\mathbb{E}^3$, maximal surfaces in $\mathbb{L}^3$, CMC 1 surfaces in $\mathbb{H}^3(-1)$, and CMC 1 surfaces in de Sitter three-space $\mathbb{S}^3_1(1)$ have representation formulae in terms of a meromorphic function and a holomorphic one-form.

Another common character of the four kinds of surfaces is that they admit Björling representation formula. Even though the Björling representation formula is derived from the Weierstrass representation formula, it is useful since it provides a simple way to derive examples with prescribed geometric data.

Recently, there have risen strong interests in Björling representation formula for various surfaces. In particular, Alías, Chaves, and Mira studied the Björling representation formula for maximal surfaces in $\mathbb{L}^3$ [4]. Gálvez and Mira studied the Björling problem for CMC 1 surfaces in $\mathbb{H}^3(-1)$ [17]. For more results, see also [7], [9], [10], [23].

Our focus is for CMC 1 surfaces in $\mathbb{S}^3_1(1)$ since the Björling formula for CMC 1 surfaces in $\mathbb{H}^3(-1)$ has been already known by Gálvez and Mira [17]. But, since our techniques are completely different from theirs and our techniques for

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CMC 1 surfaces in $\mathbb{H}^3(-1)$ and for CMC 1 surfaces in $\mathbb{S}^3_1(1)$ are basically the same, we record briefly our results for CMC 1 surfaces in $\mathbb{H}^3(-1)$ also.

Since we study only the local nature of surfaces under consideration, we restrict our attention to a simply connected domain $\mathcal{U}$ in $\mathbb{C}$ equipped with the standard coordinate $z = u + iv$ throughout this article unless specified otherwise.

Our main results are the following:

Theorem A. *Given a regular analytic curve $\gamma : I \subset \mathbb{R} \to \mathbb{H}^3(-1)$ and an analytic unit vector field $N : I \to T_\gamma\mathbb{H}^3(-1)$ perpendicular to γ, there exists a unique constant mean curvature 1 surface which contains the image of γ and whose normal at $\gamma(u)$ is $N(u)$. When $\mathbf{\Omega}$ is the unique analytic extension of $\frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du$, a conformal immersion of the surface is given by*

$$X = FF^* : \mathcal{U} \subset \mathbb{C} \to \mathbb{H}^3(-1),$$

where $F : \mathcal{U} \to SL(2,\mathbb{C})$ is the unique, up to $SU(2)$, solution of

$$dFF^{-1} = \mathbf{\Omega}, \quad F(u_0)F^*(u_0) = \gamma(u_0) \text{ for some } u_0 \in I.$$

Theorem B. *Given a regular, analytic and spacelike curve $\gamma : I \subset \mathbb{R} \to \mathbb{S}^3_1(1)$ and a timelike analytic unit vector field $N : I \to T_\gamma\mathbb{S}^3_1(1)$ perpendicular to γ, there exists a unique spacelike surface of constant mean curvature* 1 *which contains the image of γ and whose normal at $\gamma(u)$ is $N(u)$. When $\mathbf{\Omega}$ is the unique analytic extension of $\frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du$, a conformal immersion of the surface is given by*

$$X = Fe_3F^* : \mathcal{U} \subset \mathbb{C} \to \mathbb{S}^3_1(1),$$

where e_3 is as in Definition 2.1 and $F : \mathcal{U} \to SL(2,\mathbb{C})$ is the unique, up to $SU(1,1)$, solution of

$$dFF^{-1} = \mathbf{\Omega}, \quad F(u_0)e_3F(u_0)^* = \gamma(u_0) \quad \text{for some } u_0 \in I.$$

The reason why dFF^{-1}, rather than $F^{-1}dF$, is used in this article is because of the fact that if F is holomorphic and $X := FF^*$ or $X := Fe_3F^*$, then $X_zX^{-1} = F_zF^{-1}$ while $X^{-1}X_z \neq F^{-1}F_z$.

As consequences of Theorem B, we classify the rotationally invariant spacelike CMC 1 surfaces of $\mathbb{S}^3_1(1)$, and prove the planar reflection principle for CMC 1 surfaces in $\mathbb{S}^3_1(1)$.

Some properties of minimal or maximal surfaces do not hold for CMC 1 surfaces in $\mathbb{H}^3(-1)$ or in $\mathbb{S}^3_1(1)$. For example, nonorientable minimal surfaces may be constructed by Björling formula [24] but there do not exist nonorientable CMC 1 surfaces. The geodesic reflection principle do not hold for CMC 1 surfaces in $\mathbb{H}^3(-1)$ and in $\mathbb{S}^3_1(1)$. In fact, there are many differences as well as many similarities between the various surfaces. Björling formula shows both characters at once.

It should be remarked that, unlike the Riemannian counterparts, the maximal surfaces in $\mathbb{L}^3$ and the CMC 1 surfaces in $\mathbb{S}^3_1(1)$ admit *singular* Björling representation formula [19, 31]. It roughly says that given an analytic *null* curve and null directions on it perpendicular to the curve, there exists a unique maximal or CMC 1 surface which contains the given null curve as a *singular* curve and the null directions as the normal directions of the surface.

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2. Preliminaries

2.1. Hermitian model of $\mathbb{L}^4$

$\mathbb{L}^4$ is the set of quadruples of real numbers (x_0, x_1, x_2, x_3) equipped with the metric $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. As in [5, 8, 28], we identity $\mathbb{L}^4$ with $\mathcal{H}erm(2)$, the set of all 2×2 Hermitian matrices, via the correspondence

$$(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 \leftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \mathcal{H}erm(2). \tag{2.1}$$

The metric is given by $\langle v, v\rangle = -\det v$. The Hermitian model enables us to use the matrix multiplication, which is essential in our derivation of the Björling formula. We use a dot to represent the matrix multiplication, but usually omit it if there is no danger of confusion.

Einstein's summation convention is employed whenever necessary.

Definition 2.1.

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that, for $j, k = 1, 2, 3$,

$$e_1e_2 = -ie_3, \quad e_2e_3 = -ie_1, \quad e_3e_1 = -ie_2, \quad e_je_k = -e_ke_j, \quad e_j^2 = e_0.$$

$SL(2, \mathbb{C})$ acts isometrically on $\mathbb{L}^4$ via

$$SL(2, \mathbb{C}) \times \mathbb{L}^4 \to \mathbb{L}^4, \quad (\sigma, v) \mapsto \sigma v \sigma^*, \quad v^* := \bar{v}^t. \tag{2.2}$$

This action induces a double covering of the identity component of the special Lorentz group $SO(3, 1)$.

2.2. Holomorphic null curves in $SL(2, \mathbb{C})$

A holomorphic $F : \mathcal{U} \subset \mathbb{C} \to SL(2, \mathbb{C})$ is called null if $\det(F_z) = 0$. See [8]. If F is holomorphic and null, then

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -1 \end{pmatrix} \omega, \quad dFF^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \Omega \tag{2.3}$$

for some meromorphic functions g, G, which are called the secondary Gauss map, the hyperbolic Gauss map of F, respectively.

2.3. Hyperbolic three-space

The hyperbolic three-space is identified with

$$\mathbb{H}^3(-1) = \{v \in \mathcal{H}erm(2) : \det v = 1 \text{ and } \operatorname{tr} v > 0\} = \{\sigma\sigma^* : \sigma \in SL(2,\mathbb{C})\}.$$

2.3.1. *Cross product structure.* As we will see later, we use the cross product defined in $T_p\mathbb{H}^3(-1) \cdot p^{-1} := \{vp^{-1} : v \in T_p\mathbb{H}^3(-1)\}$. Note that in general $T_p\mathbb{H}^3(-1) \cdot p^{-1} \not\subset \mathbb{L}^4$. It is a subspace of $sl(2,\mathbb{C}) \subset \mathbb{C}^4_1$ of real dimension 3.

Definition 2.2. for any $U, V \in T_p\mathbb{H}^3(-1) \cdot p^{-1}$,

$$U \times V := i\, UV - i\langle U, V\rangle_{\mathbb{C}^4_1}\, e_0. \tag{2.4}$$

Here, $\langle z^\alpha e_\alpha, w^\beta e_\beta\rangle_{\mathbb{C}^4_1} := -z^0w^0 + z^1w^1 + z^2w^2 + z^3w^3$ for $z^\alpha, w^\beta \in \mathbb{C}$.

Following the frame methods as in [8] for example, we first observe that $p = FF^*$ for some $F \in SL(2,\mathbb{C})$. Then, a basis of $T_p\mathbb{H}^3(-1) \cdot p^{-1}$ consists of

$$\hat{e_0} := Fe_1F^{-1}, \quad \hat{e_1} := Fe_2F^{-1}, \quad \hat{e_2} := Fe_3F^{-1}.$$

Note that $\hat{e_1}, \hat{e_2}, \hat{e_3} \notin \mathbb{L}^4$ in general.

Lemma 2.3. *If $U, V \in T_p\mathbb{H}^3(-1) \cdot p^{-1}$, then $U \times V \in T_p\mathbb{H}^3(-1) \cdot p^{-1}$. Furthermore, for $i, j, k, \ell, m = 1, 2, 3$,*

$$\hat{e_1} \times \hat{e_2} = \hat{e_3}, \quad \hat{e_2} \times \hat{e_3} = \hat{e_1}, \quad \hat{e_3} \times \hat{e_1} = \hat{e_2}, \quad \hat{e_i} \times \hat{e_j} = -\hat{e_j} \times \hat{e_i},$$

$$a^k\hat{e_k} \times b^\ell\hat{e_\ell} = \begin{vmatrix} \hat{e_1} & \hat{e_2} & \hat{e_3} \\ a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \end{vmatrix}, \quad \langle a^k\hat{e_k} \times b^\ell\hat{e_\ell}, c^m\hat{e_m}\rangle = \begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{vmatrix}.$$

Proof. The result follows easily from the following formula:

$$(a^kFe_kF^{-1}) \times (b^lFe_lF^{-1}) = iF(a^ke_k)(b^le_l)F^{-1} - i(a_1b_1 + a_2b_2 + a_3b_3)e_0.$$

$\square$

Definition 2.4. We define that the ordered triple $U, V, W \in T_p\mathbb{H}^3(-1) \cdot p^{-1}$ is positively oriented if $\langle U \times V, W\rangle$ is positive.

According to this, e_1, e_2, e_3 are positively oriented. This cross product structure is a complete analogue of the usual cross product of $\mathbb{E}^3$.

2.3.2. *Weierstrass-Bryant type representation theorem.* Our derivation of the Björling formula crucially depends upon the following fact.

Theorem 2.5 ([8], [30]). *If $F : \mathcal{U} \subset \mathbb{C} \to SL(2,\mathbb{C})$ is holomorphic and null, then $X := FF^* : \mathcal{U} \to \mathbb{H}^3(-1)$ is a smooth conformal immersion of CMC 1, possibly with isolated singular points.*

Conversely, given a simply connected domain $\mathcal{U} \subset \mathbb{C}$ and a conformal immersion $X : \mathcal{U} \to \mathbb{H}^3(-1)$ with CMC 1, there exists a holomorphic null map $F : \mathcal{U} \to SL(2,\mathbb{C})$ such that $X = FF^*$.

2.4. de Sitter three-space

The de Sitter three-space is identified with

$$\mathbb{S}^3_1(1) = \{v \in \mathcal{H}erm(2) : \det v = -1\} = \{\sigma e_3 \sigma^* : \sigma \in SL(2,\mathbb{C})\}.$$

It is a pseudo-Riemannian manifold of constant sectional curvature 1.

2.4.1. *Cross product structure.* As we will see later, we use cross product structure in the vector space $T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3 := \{vp^{-1}e_3 : v \in T_p\mathbb{S}^3_1(1)\}$, which is a subspace of $sl(2,\mathbb{C})\cdot e_3 := \{ve_3 : v \in sl(2,\mathbb{C})\}$ of real dimension 3.

Definition 2.6. For any $U, V \in T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$,

$$U \times V := iUe_3V + i\langle U, V\rangle_{\mathbb{C}^4_1}\, e_3. \tag{2.5}$$

It is clear that $U \times V \in T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$.

Any $p \in \mathbb{S}^3_1(1)$ may be written as $p = Fe_3F^*$ for some $F \in SL(2,\mathbb{C})$, where F is unique up to $SU(1,1)$. A basis of $T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$ consists of

$$\tilde{e_0} := Fe_0e_3F^{-1}e_3, \quad \tilde{e_1} := Fe_1e_3F^{-1}e_3, \quad \tilde{e_2} := Fe_2e_3F^{-1}e_3.$$

Note that $\tilde{e_0}, \tilde{e_1}, \tilde{e_2} \notin \mathbb{L}^4$ in general.

Lemma 2.7. *If $U, V \in T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$, then $U \times V \in T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$. Furthermore, for $i, j, k, \ell, m = 0, 1, 2$,*

$$\tilde{e_0} \times \tilde{e_1} = \tilde{e_2}, \quad \tilde{e_1} \times \tilde{e_2} = -\tilde{e_0}, \quad \tilde{e_2} \times \tilde{e_0} = \tilde{e_1}, \quad \tilde{e_i} \times \tilde{e_j} = -\tilde{e_j} \times \tilde{e_i},$$

$$a^k\tilde{e_k} \times b^\ell\tilde{e_\ell} = \begin{vmatrix} \tilde{e_1} & \tilde{e_2} & -\tilde{e_0} \\ a^1 & a^2 & a^0 \\ b^1 & b^2 & b^0 \end{vmatrix}, \quad \langle a^k\tilde{e_k} \times b^\ell\tilde{e_\ell}, c^m\tilde{e_m}\rangle = \begin{vmatrix} a^1 & a^2 & a^0 \\ b^1 & b^2 & b^0 \\ c^1 & c^2 & c^0 \end{vmatrix}.$$

Proof. It follows easily from the definitions. □

Definition 2.8. We define that the ordered triple $U, V, W \in T_p\mathbb{S}^3_1(1)\cdot p^{-1}\cdot e_3$ is positively oriented if $\langle U \times V, W\rangle$ is positive.

According to this, $\tilde{e_1}, \tilde{e_2}, \tilde{e_0}$ are positively oriented, though $\tilde{e_1} \times \tilde{e_2} = -\tilde{e_0}$. This cross product structure is a complete analogue of the usual cross product of $\mathbb{L}^3$ [4].

2.4.2. *Weierstrass-Bryant type representation theorem for constant mean curvature one surfaces in de Sitter three-space.*

Theorem 2.9 ([1])**.** *If $F : \mathcal{U} \subset \mathbb{C} \to SL(2,\mathbb{C})$ is holomorphic and null, and in addition the secondary Gauss map g of F satisfies $|g| \neq 1$, then $X := Fe_3F^*$ is a space-like conformal CMC 1 surface of $\mathbb{S}^3_1(1)$, possibly with isolated singular points.*

Conversely, if $X : \mathcal{U} \subset \mathbb{C} \to \mathbb{S}^3_1(1)$ is a regular spacelike conformal immersion of CMC 1, then there exists a holomorphic null map $F : \mathcal{U} \subset \mathbb{C} \to SL(2,\mathbb{C})$, unique up to $SU(1,1)$, such that $X = Fe_3F^*$.

The induced metric and the second fundamental form of X are

$$I = (1 - |g|^2)^2|\omega|^2, \quad II = I + \omega dg + \overline{\omega dg}.$$

2.4.3. *The hollow ball model of* $\mathbb{S}^3_1(1)$. For visualization purposes, we let

$$y_k = \frac{e^{\tan^{-1} x_0}}{\sqrt{1+x_0^2}} x_k, \quad k = 1,2,3, \quad \text{for a given } \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$

$\mathbb{S}^3_1(1)$ is identified with a hollow ball $e^{-\pi/2} < \sqrt{y_1^2 + y_2^2 + y_3^2} < e^{\pi/2}$ in the $y_1 y_2 y_3$-space. If $y_1 = \rho \sin\phi \cos\theta,\ y_2 = \rho \sin\phi \sin\theta,\ y_3 = \rho\cos\phi$, then

$$ds^2_{\mathbb{S}^3_1(1)} = \rho^{-2} \sec^2(2\ln\rho)\left(-d\rho^2 + \rho^2(d\phi^2 + \sin^2\phi\, d\theta^2)\right).$$

Out of this, we see that ∂_ρ is a future-pointing timelike vector field. See [11].

3. Björling formula for CMC 1 surfaces in $\mathbb{H}^3(-1)$

3.1. Proof of Theorem A

The following proof is motivated by [24].

Proof. We first show the uniqueness assuming the existence. Suppose that $M \subset \mathbb{H}^3(-1)$ is a CMC 1 surface with unit normal $\mathcal{N}$ and that $\gamma(I) \subset M$ and $\mathcal{N}|_I = N$. By standard theory, there is a conformal immersion $X : \mathcal{U} \to \mathbb{H}^3(-1)$ whose image is M. We may assume without loss of generality that $X_u, X_v, \mathcal{N}$ are positively oriented. Then by Theorem 2.5, there is a null holomorphic $F : \mathcal{U} \to SL(2,\mathbb{C})$ such that $X := FF^*$. It is easy to see

$$dFF^{-1} = F_z F^{-1} dz = X_z X^{-1} dz = \frac{1}{2}(X_u X^{-1} - iX_v X^{-1})dz. \tag{3.1}$$

Since X is conformal and $X_u, X_v, \mathcal{N}$ are positively oriented, we have

$$X_v X^{-1} = (\mathcal{N}X^{-1}) \times (X_u X^{-1}) = i(\mathcal{N}X^{-1})(X_u X^{-1}). \tag{3.2}$$

Therefore, $X_v X^{-1}|_I = iN\gamma^{-1}\gamma_u\gamma^{-1}$ and

$$dFF^{-1}|_I = \frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1} du.$$

This implies that if F exists, then dFF^{-1} must be the (unique) analytic extension of $\frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du$.

Now we turn to prove the existence. First, we define a one-form

$$\mathbf{\Omega} := \text{ the unique analytic extension of } \frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du. \tag{3.3}$$

Then, it is easy to see that

(a) from (2.4), we see $N\gamma^{-1}\gamma_u\gamma^{-1} = (-i)N\gamma^{-1}\times\gamma_u\gamma^{-1}$, which is in $sl(2,\mathbb{C})$ by Lemma 2.3. So, $\mathbf{\Omega}$ is $sl(2,\mathbb{C})$-valued on I, hence by the monodromy principle, everywhere.
(b) $d\mathbf{\Omega} = \mathbf{\Omega} \wedge \mathbf{\Omega}$ because $\mathbf{\Omega}$ is holomorphic.
(c) $\det \mathbf{\Omega} = 0$. (Since $N\gamma^{-1} \in sl(2,\mathbb{C})$, we may write $N\gamma^{-1} = N^k e_k$ for some $N^k \in \mathbb{C}$. Since $\det N\gamma^{-1} = \det N = -\langle N, N\rangle = -1$, we have $\sum (N^k)^2 = 1$, which implies $\det(I + N\gamma^{-1}) = 0$ on I. By the monodromy principle again, it is so everywhere.)

Therefore, by standard theory, there exists a null holomorphic F, unique up to $SU(2)$, such that $dFF^{-1} = \mathbf{\Omega}$. By Theorem 2.5, $X := FF^*$ is a conformal CMC 1 immersion, whose unit normal is denoted by $\mathcal{N}$. We choose $\mathcal{N}$ such that $X_u, X_v, \mathcal{N}$ are positively oriented.

We may assume without loss of generality that $X(u_0) = \gamma(u_0)$ for some $u_0 \in I$. Hence (3.1) and (3.2) are valid.

We proceed to show that $X|_I = \gamma$. We first observe that

$$\gamma_u \gamma^{-1} du = (\mathbf{\Omega} + \gamma \mathbf{\Omega}^* \gamma^{-1})|_I$$

since γ, N are Hermitian. On the other hand, since $X = FF^*$ and F is holomorphic, we have $X_{\bar{z}} = F(F^*)_{\bar{z}} = F(F_z)^*$, hence

$$\frac{1}{2}(X_u X^{-1} + i X_v X^{-1}) d\bar{z} = X_{\bar{z}} X^{-1} d\bar{z} = FF^*(F_z F^{-1} dz)^* (FF^*)^{-1} = X \mathbf{\Omega}^* X^{-1}.$$

Restricted to the real interval I and combined with (3.1), this yields

$$X_u X^{-1} du|_I = (X_z X^{-1} dz + X_{\bar{z}} X^{-1} d\bar{z})|_I = (\mathbf{\Omega} + X \mathbf{\Omega}^* X^{-1})|_I.$$

By the uniqueness of the solution of this system of ODE, we conclude that $X|_I = \gamma$.

Next, we proceed to show that $\mathcal{N}|_I = N$. From (3.1) and (3.3) and the fact that $dFF^{-1} = \mathbf{\Omega}$ and $X|_I = \gamma$, we conclude

$$X_v X^{-1}|_I = i N \gamma^{-1} \gamma_u \gamma^{-1}.$$

By comparing this with (3.2), we conclude that $\mathcal{N}|_I = N$. (γ_u is invertible since γ is regular.) □

3.2. Examples and further results

The biggest difference from the viewpoint of the Björling construction for minimal surfaces in $\mathbb{E}^3$ and CMC 1 surfaces in $\mathbb{H}^3(-1)$ is that γ, N and $\gamma, -N$ produce the same minimal surface in $\mathbb{E}^3$, while they produce different CMC 1 surfaces in $\mathbb{H}^3(-1)$. Compare the examples with $t = 0$ and $t = \pi$ in Example 3.1. This happens because changing the normal of a surface by its negative changes the mean curvature by its negative. A simple consequence is that there is no nonorientable CMC 1 surface. So, while the Björling formula can be used in the construction for a nonorientable minimal surface in $\mathbb{E}^3$ [24], it can not be used for constructing nonorientable CMC 1 surfaces in $\mathbb{H}^3(-1)$.

It is known that catenoid cousins and horospheres are the only rotationally invariant CMC 1 surfaces [8, 29]. Björling formula can also be used to give a clear geometric classification of this fact.

Example 3.1 (Rotationally invariant CMC 1 surfaces)**.** Fix $r \in \mathbb{R}^+$, $t \in [0, 2\pi)$, and define $\gamma : \mathbb{R} \to \mathbb{H}^3(-1)$ and N by

$$\gamma(u) := \begin{pmatrix} \cosh r & e^{iu} \sinh r \\ e^{-iu} \sinh r & \cosh r \end{pmatrix},$$

$$N(u) := \cos t \begin{pmatrix} \sinh r & e^{iu}\cosh r \\ e^{-iu}\cosh r & \sinh r \end{pmatrix} + \sin t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

γ is a circle in the plane $x_3 = 0$, of radius r centered at $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is easy to see that $N(u) \in T_{\gamma(u)}\mathbb{H}^3(-1)$, $\langle N,\, \gamma_u\rangle = 0$, $\langle N,\, N\rangle = 0$, and that it is invariant under the rotation around the geodesic $x_1 = x_2 = 0$. These data produces all the rotationally invariant CMC 1 surfaces in $\mathbb{H}^3(-1)$ up to congruency. If $\sinh r + \cos t \cosh r = 0$ we obtain horospheres. Otherwise, we obtain two-ended surfaces.

Example 3.2 (Helicoidal CMC 1 surfaces)**.** Take

$$\gamma(u) := \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}, \quad N(u) := \begin{pmatrix} 0 & e^{f(u)i} \\ e^{-f(u)i} & 0 \end{pmatrix},$$

where $f : \mathbb{R} \to \mathbb{R}$ is analytic. γ is a geodesic. We easily compute

$$\frac{1}{2}\left(I + N\gamma^{-1}\right)\gamma_u\gamma^{-1}du = \frac{1}{2}\begin{pmatrix} 1 & -e^{u+f(u)i} \\ e^{-u-f(u)i} & -1 \end{pmatrix} du.$$

If we take $f(u) = \theta u$ for some $\theta \in \mathbb{R}^+$, then $G = e^{(1+i\theta)z}$, $\Omega = \frac{1}{2}e^{-(1+i\theta)z}dz$, where G, Ω are as in (2.3). By substituting $(1 + i\theta)z$ by w, we obtain $G = e^w$, $\Omega = \frac{1}{2(1+i\theta)}e^{-w}dw$. Compare these data with those of the Euclidean helicoid. We refer interested readers to [27, Example 1.8] for further analysis of this example.

We now introduce some properties of CMC 1 surfaces in $\mathbb{H}^3(-1)$ which can easily be derived from the Björling formula.

Lemma 3.3. *Given a regular and analytic curve γ of unit speed and nonzero curvature in $\mathbb{H}^3(-1)$, there exists a CMC 1 surface which contains γ as a geodesic, but there can not be more than two.*

Proof. Let N be the principal normal of γ. By solving the Björling problem with N or $-N$ the existence is proved. On the other hand, N must be the unit normal or the negative unit normal of the surface which contains γ as a geodesic. So there cannot be more than two such surfaces. $\square$

Surfaces in Example 3.1 with $t = 0, \pi$ are CMC 1 surfaces with a circle as a geodesic. It would be interesting to find γ which has only one CMC 1 surface with γ as a geodesic.

Sa Earp and Toubiana proved in [27] that CMC 1 surfaces in $\mathbb{H}^3(-1)$ satisfy the planar reflection, but not the geodesic reflection. Björling formula can provide another proof for the planar reflection.

4. Björling formula for CMC 1 surfaces in $\mathbb{S}^3_1(1)$

4.1. Proof of Theorem B

The following proof is basically the same as the proof of Theorem A, but we present the details to make this proof self contained.

Proof. We first show the uniqueness assuming the existence. Suppose that $M \subset \mathbb{S}^3_1(1)$ is a spacelike CMC 1 surface with timelike unit normal $\mathcal{N}$ and that $\gamma(I) \subset M$ and $\mathcal{N}|_I = N$. By Theorem 2.9, for any $p \in \gamma(I)$, there is a null holomorphic $F : \mathcal{U} \to SL(2,\mathbb{C})$ such that a neighborhood of p in M is the image of $X := Fe_3F^*$. We obtain

$$\begin{aligned} dFF^{-1} &= F_zF^{-1}dz = F_ze_3F^*(F^*)^{-1}e_3^{-1}F^{-1}dz \\ &= (Fe_3F^*)_z(Fe_3F^*)^{-1}dz = X_zX^{-1}dz \\ &= \frac{1}{2}(X_uX^{-1} - iX_vX^{-1})dz. \end{aligned} \tag{4.1}$$

Since X is conformal and $X_u, X_v, \mathcal{N}$ are positively oriented, we have

$$X_vX^{-1} = (\mathcal{N}X^{-1}) \times (X_uX^{-1}) = i(\mathcal{N}X^{-1})(X_uX^{-1}). \tag{4.2}$$

Therefore, $X_vX^{-1}|_I = iN\gamma^{-1}\gamma_u\gamma^{-1}$, and (4.1), (4.2) imply that

$$dFF^{-1}|_I = \frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du.$$

This implies that if F exists, then dFF^{-1} must be the (unique) analytic extension of $\frac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du$.

Now we turn to prove the existence. First recall that $sl(2,\mathbb{C})$ is spanned by e_1, e_2, e_3 over $\mathbb{C}$. We define a one-form

$$\mathbf{\Omega} := \text{ the unique analytic extension of } \tfrac{1}{2}(I + N\gamma^{-1})\gamma_u\gamma^{-1}du. \tag{4.3}$$

Then, it is easy to see that

(a) From (2.5), we see $N\gamma^{-1}\gamma_u\gamma^{-1} = -i(N\gamma^{-1}e_3) \times (\gamma_u\gamma^{-1}e_3)$, which is in $sl(2,\mathbb{C})$ by Lemma 2.7. So, $\mathbf{\Omega}$ is $sl(2,\mathbb{C})$-valued on I, hence by the monodromy principle, everywhere.
(b) $d\mathbf{\Omega} = \mathbf{\Omega} \wedge \mathbf{\Omega}$ because $\mathbf{\Omega}$ is holomorphic.
(c) $\det \mathbf{\Omega} = 0$. (Since $N\gamma^{-1} \in sl(2,\mathbb{C})$, we may write $N\gamma^{-1} = \sum_{k=1}^3 N^ke_k$ for some $N^k \in \mathbb{C}$. Since $\det N\gamma^{-1} = -\det N = \langle N, N\rangle = -1$, we have $\sum(N^k)^2 = 1$, which implies $\det(I + N\gamma^{-1}) = 0$ on I. By the monodromy principle again, it is so everywhere.)

Therefore, by standard theory, there exists a null holomorphic F such that $dFF^{-1} = \mathbf{\Omega}$. By Theorem 2.9, $X := Fe_3F^*$ is a conformal CMC 1 immersion, whose unit (timelike) normal is denoted by $\mathcal{N}$. We may assume without loss of generality that $X(u_0) = \gamma(u_0)$ for some $u_0 \in I$ by multiplying F by an element of $SL(2,\mathbb{C})$ if necessary, and that $X_uX^{-1}e_3$, $X_vX^{-1}e_3$, $\mathcal{N}X^{-1}e_3$ are positively oriented.

We proceed to show that $X|_I = \gamma$. We first observe that

$$\gamma_u\gamma^{-1}du = (\mathbf{\Omega} + \gamma\mathbf{\Omega}^*\gamma^{-1})|_I$$

since γ, N are Hermitian and $\langle N, \gamma_u \rangle = 0$. On the other hand, since $X = Fe_3F^*$ and F is holomorphic, we have $X_{\bar{z}} = Fe_3(F^*)_{\bar{z}} = Fe_3(F_z)^*$, hence

$$\frac{1}{2}(X_uX^{-1} + iX_vX^{-1})d\bar{z} = X_{\bar{z}}X^{-1}d\bar{z} = (Fe_3F^*)(F_zF^{-1}dz)^*(Fe_3F^*)^{-1}$$
$$= X\mathbf{\Omega}^*X^{-1}.$$

Restricted to the real interval I and combined with (4.1), this yields

$$X_uX^{-1}du|_I = (X_zX^{-1}dz + X_{\bar{z}}X^{-1}d\bar{z})|_I = (\mathbf{\Omega} + X\mathbf{\Omega}^*X^{-1})|_I.$$

By the uniqueness of the solution of this system of ODE, we conclude that $X|_I = \gamma$.

Next, we proceed to show that $\mathcal{N}|_I = N$. Note that F and X satisfy (4.1) and (4.2), respectively. From (4.1) and (4.3) and the fact that $dFF^{-1} = \mathbf{\Omega}$ and $X|_I = \gamma$, we conclude

$$X_vX^{-1}|_I = iN\gamma^{-1}\gamma_u\gamma^{-1}.$$

By comparing this with (4.2), we conclude that $\mathcal{N}|_I = N$. (γ_u is invertible since γ is regular.) $\square$

Remark 4.1. Our Björling formula is essentially the same as the Björling formula for maximal surfaces in $\mathbb{L}^3$ [4], which is obtained by taking the real part of the integral of the analytic extension of $\gamma_u - iN \times \gamma$.

4.2. Examples and further results

As an application of the Björling formula, we provide an explicit description of all the rotational CMC 1 surfaces of $\mathbb{S}^3_1(1)$. It should remarked that rotationally invariant linear Weingarten surfaces of $\mathbb{S}^3_1(1)$ are classified in [22].

Recall that the following curves

$$\gamma_T(t) := \begin{pmatrix} e^t & 0 \\ 0 & -e^{-t} \end{pmatrix}, \quad \gamma_S(t) := \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}, \quad \gamma_L(t) := \begin{pmatrix} t+1 & t \\ t & t-1 \end{pmatrix}$$

are timelike, spacelike, lightlike geodesics of speed $1, 1, 0$, respectively, passing through e_3 when $t = 0$ and that the rotations of $\mathbb{S}^3_1(1)$ around these geodesics are represented by

$$\mathcal{R}_T(\varphi) := \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix},$$
$$\mathcal{R}_S(\varphi) := \begin{pmatrix} \cosh\frac{\varphi}{2} & i\sinh\frac{\varphi}{2} \\ -i\sinh\frac{\varphi}{2} & \cosh\frac{\varphi}{2} \end{pmatrix},$$
$$\mathcal{R}_L(\varphi) := \begin{pmatrix} 1 - i\frac{\varphi}{2} & i\frac{\varphi}{2} \\ -i\frac{\varphi}{2} & 1 + i\frac{\varphi}{2} \end{pmatrix}.$$

That is, given $p \in \mathbb{S}^3_1(1)$, the map $\varphi \mapsto \mathcal{R}_\star(\varphi)\cdot p \cdot \mathcal{R}_\star(\varphi)^*$ rotates p around $\gamma_\star$ for each $\star = T, S, L$. The translations along $\gamma_T, \gamma_L, \gamma_S$ are represented by

$$\mathcal{T}_T(s) := \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}, \quad \mathcal{T}_L(s) := \begin{pmatrix} 1+s & -s \\ s & 1-s \end{pmatrix}, \quad \mathcal{T}_S(s) := \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}.$$

In particular, $\mathcal{T}_\star(t/2)e_3\mathcal{T}_\star(t/2)^* = \gamma_\star(t)$.

We now present explicit description of the standard rotational CMC 1 surfaces.

Lemma 4.2. *Let* $S_L := \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{i} & -\sqrt{-i} \\ \sqrt{i} & \sqrt{-i} \end{pmatrix}$, $S_S := \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$,

$$\Phi_p(c,z) := \begin{cases} \frac{1}{\sqrt{\mu}}\begin{pmatrix} \frac{-1-\mu}{2} z^{(1-\mu)/2} & \frac{1-\mu}{2} z^{(1+\mu)/2} \\ \frac{1-\mu}{2} z^{(-1-\mu)/2} & \frac{-1-\mu}{2} z^{(\mu-1)/2} \end{pmatrix} & \text{if } \mu := \sqrt{1-4c} \neq 0, \\ \frac{1}{2\sqrt{z}}\begin{pmatrix} -z & z(-2+\log z) \\ 1 & -2-\log z \end{pmatrix} & \text{if } c = \frac{1}{4}, \end{cases}$$

and

$$\begin{aligned} F_T(\alpha,z) &:= \Phi_p(\tfrac{\alpha}{2}, e^{-iz})\Phi_p(\tfrac{\alpha}{2},1)^{-1}, \\ F_L(\alpha,z) &:= S_L\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{\alpha}\sin(\sqrt{\alpha}z) & \cos(\sqrt{\alpha}z) \\ \cos(\sqrt{\alpha}z) & -\frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}z) \end{pmatrix} S_L^{-1}, \\ F_S(\alpha,z) &:= S_S\Phi_p(\tfrac{\alpha}{2}, -e^{-z})\Phi_p(\tfrac{\alpha}{2},-1)^{-1}S_S^{-1}. \end{aligned}$$

Then, $dF_\star(\alpha,z)F_\star(\alpha,z)^{-1} = -\frac{i}{2}\alpha\tilde{\mathbf{\Omega}}_\star(z)$ *and* $F_\star(\alpha,0) = e_3$*, where*

$$\tilde{\mathbf{\Omega}}_T(z) := \begin{pmatrix} 1 & -e^{-iz} \\ e^{iz} & -1 \end{pmatrix} dz, \quad \tilde{\mathbf{\Omega}}_S(z) := \begin{pmatrix} \cosh z & -(1+i\sinh z) \\ 1-i\sinh z & -\cosh z \end{pmatrix} dz,$$

$$\tilde{\mathbf{\Omega}}_L(z) := \begin{pmatrix} z^2+1 & -(z^2-1+2iz) \\ z^2-1-2iz & -(z^2+1) \end{pmatrix} dz.$$

Furthermore, for $\alpha \in \mathbb{R}\setminus\{0\}$, $X_\star(\alpha,z) := F_\star(\alpha,z)e_3F_\star(\alpha,z)^*$ *are rotationally invariant around* $\gamma_\star$. X_T *has closing period and has two ends, which are elliptic or parabolic or hyperbolic if* $1+2\alpha >$ *or* $=$ *or* < 0*, respectively.* (*See* [15] *for the definition of elliptic, parabolic, hyperbolic ends.*) *For* $\star = T, L, S$*, the surfaces with different* α *are not congruent.*

Proof. It is computed by Fujimori in [15] that $\Phi_p(c,z)$ is a particular solution of

$$\Phi_z\Phi^{-1} = c\begin{pmatrix} 1/z & -1 \\ 1/z^2 & -1/z \end{pmatrix}, \quad c \in \mathbb{C}.$$

The rest of the proof follows from direct calculations. □

Now we construct arbitrary rotational CMC 1 surfaces around $\gamma_\star$. First, choose three pairs of a point in $\mathbb{S}^3_1(1)\cap\{x_2=0\}$ and a timelike vector of length 1 in $T_p\mathbb{S}^3_1(1)\cap\{x_2=0\}$ as follows, where $t\in\mathbb{R}$ and $k\in\mathbb{R}\setminus\{0\}$.

(T) $p_T := \gamma_S(t)$ and $V_T := \frac{k+k^{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{k-k^{-1}}{2}\begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix}$.

(S) $p_S := \gamma_T(t)$ and $V_S := \frac{k+k^{-1}}{2}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} + \frac{k-k^{-1}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$.

(L) $p_L := \mathcal{R}_T(\pi)\gamma_L(t)\mathcal{R}_T(\pi)^*$, and

$$V_L := \frac{1}{2k}\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix} + \frac{k}{2}\begin{pmatrix} (1+t)^2 & 1-t^2 \\ 1-t^2 & (1-t)^2\end{pmatrix}.$$

By rotating $p_\star$ and $V_\star$ around the geodesic $\gamma_\star$, we obtain the Björling data we want:

$$c_\star(u) := \mathcal{R}_\star(u)\cdot p_\star \cdot \mathcal{R}^*_\star(u), \quad \mathcal{V}_\star(u) := \mathcal{R}_\star(u)\cdot V_\star \cdot \mathcal{R}_\star(u)^* \tag{4.4}$$

are circles around $\gamma_\star$ and a timelike unit vector field on it, respectively. Now we apply Theorem B with $c_\star$ and $\mathcal{V}_\star$ to obtain $\Omega_\star$ and $F_\star$, the analysis of which becomes easy by the use of the following lemma.

Lemma 4.3. (1) *For $\star = T, L, S$ and any $v_0 \in \mathbb{R}, z \in \mathbb{Z}$,*

$$\mathcal{T}_\star(v_0)\tilde{\mathbf{\Omega}}_\star(z)\mathcal{T}_\star(v_0)^{-1} = \tilde{\mathbf{\Omega}}_\star(z+2iv_0).$$

(2) *Suppose that $P\tilde{\mathbf{\Omega}}(z)P^{-1} = \tilde{\mathbf{\Omega}}(z+iv_0)$ for some $v_0 \in \mathbb{R}$ and $P \in SL(2,\mathbb{C})$ and that $G_z(z)G^{-1}(z) = \tilde{\mathbf{\Omega}}(z)$. Then $F(z) := G(z+2iv_0)$ satisfies $F_z(z)F^{-1}(z) = P\tilde{\mathbf{\Omega}}(z)P^{-1}$.*

Proof. The proof is easy and left to the reader. □

In the following three examples, note that P is some translation matrix $\mathcal{T}_\star$.

Example 4.4 (Rotationally invariant around the timelike geodesic γ_T)**.** Consider the circle $c_T(u)$ and the vector field $\mathcal{V}_T(u)$. If $(k^2-1)\cos t + 2k\sin t \neq 0$, then $\beta := \sqrt{\frac{(k^2-1)\cos t+2k\sin t}{1+k^2-2k\cos t+(k^2-1)\sin t}} \neq 0$ and Theorem B yields $\mathbf{\Omega} = P\mathbf{\Omega}_T P^{-1}$, where

$$\mathbf{\Omega}_T = -\frac{i}{2}\alpha\tilde{\mathbf{\Omega}}_T,\ \alpha = \frac{\sin t}{2k}((k^2-1)\cos t + 2k\sin t),\ P = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1}\end{pmatrix}.$$

Note that P is a translation matrix along γ_T even if $\beta^2 < 0$. These Weierstrass data have appeared in [27].

Suppose $\beta^2 > 0$. Then, $F(z) := \mathcal{T}_T(v_1)F_T(z+2v_0 i)$, where v_0, v_1 are such that

$$\mathcal{T}_T(v_0+v_1) = P, \quad \mathcal{T}_T(v_1)F_T(2iv_0)e_3F_T(2iv_0)^*\mathcal{T}_T(v_1)^* = c_T(0)$$

is a solution to $dF(z)F^{-1}(z) = \mathbf{\Omega}(z)$ and $F(0)e_3F(0)^* = c_T(0)$. The case for $\beta^2 < 0$ may be argued similarly.

On the other hand, suppose $(k^2-1)\cos t + 2k\sin t = 0$. Then, $\mathbf{\Omega} = -i\tan t\left(\begin{smallmatrix} 0 & e^{iz} \\ 0 & 0\end{smallmatrix}\right)dz$ if in addition $|t| < \pi/2$, but $\mathbf{\Omega} = -i\tan t\left(\begin{smallmatrix} 0 & 0 \\ e^{-iz} & 0\end{smallmatrix}\right)dz$ if in addition $\pi/2 < |t| < \pi$. In both of the cases, we get a complete CMC 1 surface in $\mathbb{S}^3_1(1)$, which is unique up to congruency and has only one end.

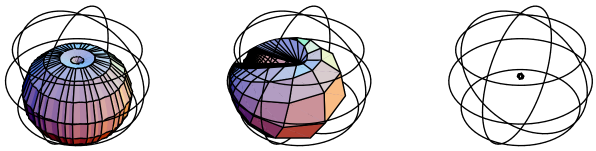

FIGURE 1. The rotational surfaces in Examples 4.4, 4.5, 4.6, respectively.

Example 4.5 (Rotationally invariant around the lightlike geodesic γ_L)**.** Consider the circle $c_L(u)$ and the vector field $\mathcal{V}_L(u)$. If $1+tk \neq 0$, then Theorem B yields $\mathbf{\Omega} = P\mathbf{\Omega}_L P^{-1}$ where

$$\mathbf{\Omega}_L = -\frac{i}{2}\alpha\tilde{\mathbf{\Omega}}_L, \quad \alpha = t(1+tk)^2/k, \quad P := \begin{pmatrix} 1+\frac{k}{2(1+tk)} & -\frac{k}{2(1+tk)} \\ \frac{k}{2(1+tk)} & 1-\frac{k}{2(1+tk)} \end{pmatrix}.$$

Then, $F(z) := \mathcal{T}_L(v_1)F_L(z+2v_0 i)$, where v_0, v_1 are such that

$$\mathcal{T}_L(v_0+v_1) = P, \quad \mathcal{T}_L(v_1)F_L(2iv_0)e_3F_L(2iv_0)^*\mathcal{T}_L(v_1)^* = c_L(0),$$

is a solution to $dF(z)F^{-1}(z) = \mathbf{\Omega}(z)$ and $F(0)e_3F(0)^* = c_L(0)$. The resulting surface X is well defined on $\mathbb{C}$.

If $1+tk=0$, then we again obtain the complete CMC 1 surface.

Example 4.6 (Rotationally invariant around a spacelike geodesic γ_S)**.** For the circle $c_S(u)$ and the vector field $\mathcal{V}_S(u)$, Theorem B yields $\mathbf{\Omega} = P\mathbf{\Omega}_S P^{-1}$ where

$$\mathbf{\Omega}_S := -\frac{i}{2}\alpha\tilde{\mathbf{\Omega}}_S, \quad \alpha := \frac{e^{-2t}-1}{8k}\left(e^{2t}(k+1)^2+(k-1)^2\right),$$

$$P := \frac{1}{\sqrt{2}\sqrt{e^{2t}(k+1)^2+(k-1)^2}}\begin{pmatrix} e^t(k+1)+k-1 & e^t(k+1)-k+1 \\ -(e^t(k+1)-k+1) & e^t(k+1)+k-1 \end{pmatrix}.$$

Then, $F(z) := \mathcal{T}_S(v_1)F_S(z+2v_0 i)$, where v_0, v_1 are such that

$$\mathcal{T}_S(v_0+v_1) = P, \quad \mathcal{T}_S(v_1)F_S(2iv_0)e_3F_S(2iv_0)^*\mathcal{T}_S(v_1)^* = c_S(0),$$

is a solution to $dF(z)F^{-1}(z) = \mathbf{\Omega}(z)$ and $F(0)e_3F(0)^* = c_S(0)$.

Note that $\tilde{\mathbf{\Omega}}_\star e_3$ for $\star \in \{T, S, L\}$ on the u-axis are

$$(1, \cos u, -\sin u, 0)du, \quad (u^2+1, u^2-1, 2u, 0)du, \quad (\cosh u, 1, \sinh u, 0)du,$$

respectively, and that they represent the ellipse, parabola, hyperbola in the lightcone $0 = -x_0^2+x_1^2+x_2^2, x_3 = 0$. Why the one-forms are related to the conics in this way is explained in the *singular* Björling formula for CMC 1 surfaces in $\mathbb{S}^3_1(1)$ [31].

Lemma 4.7 (Rotationally invariant spacelike CMC 1 surfaces)**.** *Any rotationally invariant spacelike CMC* 1 *surface of revolution is congruent to* (*a piece of*) *one of the four kinds of surfaces described in the above three examples.*

Proof. By applying isometries if necessary, we may assume that any rotationally invariant spacelike CMC 1 surface of revolution contains a circle and unit normal as in (4.4). Then the uniqueness of the solution to the Björling problem implies the lemma. □

Remark 4.8. The rotationally invariant surfaces in $\mathbb{L}^3$ are called elliptic, parabolic, or hyperbolic catenoids depending upon the causal character of the axis. However, for CMC 1 surfaces in $\mathbb{S}^3_1(1)$ those names have been already taken up to denote the two-ended surfaces with elliptic, parabolic, or hyperbolic monodromy. See [15].

Lemma 4.9. *Let γ be a spacelike and analytic curve in $\mathbb{S}^3_1(1)$ of unit speed. If $\nabla_{\dot\gamma}\dot\gamma$ is timelike and nonzero everywhere, where ∇ is the Levi-Civita connection of $\mathbb{S}^3_1(1)$, then there exists a unique spacelike surface, which contains γ as a geodesic, of CMC 1 with respect to its future pointing unit normal.*

Proof. Apply Theorem B by taking $N = \pm\frac{\nabla_{\dot\gamma}\dot\gamma}{|\nabla_{\dot\gamma}\dot\gamma|}$. (We must take $-$ if $\nabla_{\dot\gamma}\dot\gamma$ is past pointing.) □

Now, we study the reflection principles of the spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$. In the following lemmas, a plane means a surface in $\mathbb{S}^3_1(1)$ which is congruent to $\mathbb{S}^3_1(1) \cap \{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 : x_2 = 0\}$.

Lemma 4.10. *In addition to the hypotheses of Theorem B, suppose that both γ and N are planar. Then the Björling solution is symmetric with respect to the plane. In particular, it is perpendicular to the plane.*

Proof. We may assume without loss of generality that the plane is $x_2 = 0$. Let $X : D \supset I \to \mathbb{S}^3_1(1)$ be the unique solution of the Björling problem with γ and N. Let $\tilde D := \{(u,v) \in \mathbb{C} : (u,-v) \in D\}$, and define $\tilde X : \tilde D \to \mathbb{S}^3_1(1)$ by $\tilde X(u,v)$= the complex conjugate of $X(u,-v)$. Then, the image of $\tilde X$ is the reflection of the image of X with respect to the plane, and both X and $\tilde X$ are the solution of the Björling problem with γ and N. By the uniqueness of the solution, we have $\tilde X(u,v) = X(u,v)$ on $D \cap \tilde D$. □

Corollary 4.11 (Planar reflection)**.** *If a plane intersects a CMC 1 surface orthogonally, then the surface is symmetric with respect to the plane.*

Proof. It follows because the surface is the solution to Björling problem with the intersection of the surface and the plane and the future pointing unit normal. □

Remark 4.12. The geodesic reflection principle does not hold in general for CMC 1 surfaces in $\mathbb{S}^3_1(1)$. For example, consider

$$\gamma(u) = \begin{pmatrix} 0 & e^{iu} \\ e^{-iu} & 0 \end{pmatrix}, \quad N(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -\infty < u < \infty.$$

Applying Theorem B, we obtain the equation $dFF^{-1} = \begin{pmatrix} 1 & -e^{iz} \\ e^{-iz} & -1 \end{pmatrix} \frac{i}{2}dz$, whose general solution is

$$F(z) = \begin{pmatrix} e^{iz/2} & 0 \\ 0 & e^{-iz/2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} e^{-z/2} & 0 \\ 0 & e^{z/2} \end{pmatrix} R, \quad R \in SL(2, \mathbb{C}).$$

It is easy to see that $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ makes $F(t)e_3F^*(t) = \gamma(t)$ for $t \in \mathbb{R}$.

Now we consider the reflection through the geodesic $\gamma(t) = (0, \cos t, \sin t, 0)$. If $V \in T_{\gamma(t)}\mathbb{S}^3_1(1)$ is perpendicular to γ, then $\langle V, \gamma(t)\rangle_{\mathbb{S}^3_1(1)} = \langle V, \dot{\gamma}(t)\rangle_{\mathbb{S}^3_1(1)} = 0$, hence V is necessarily of the form $V = (v_0, 0, 0, v_3)$. The reflection we consider is equal to the transformation $(x_0, x_1, x_2, x_3) \to (-x_0, x_1, x_2, -x_3)$. Now it is a trivial matter to check that the surface we obtained is not invariant under this transformation.

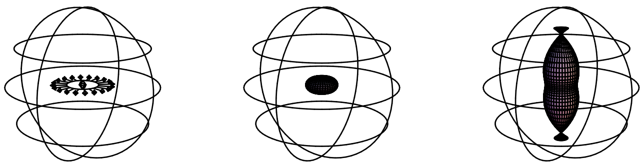

FIGURE 2. The first are γ and N. The second is the solution of the Björling problem with γ, N, the reflection of which with respect to γ is the third.

The following result is motivated by [27, Remark 3.4], which is about CMC surfaces in $\mathbb{H}^3(-1)$. The proof is basically the same, but we include it here for the sake of completeness of this article.

Lemma 4.13. *Let $S \subset \mathbb{S}^3_1(1)$ be a surface of constant mean curvature $H \in \mathbb{R}$, bounded by a piece of a spacelike geodesic L of $\mathbb{S}^3_1(1)$. Then, S can be extended to a constant mean curvature surface that is symmetric with respect to L if and only if $H = 0$, that is, if and only if S is a maximal surface of $\mathbb{S}^3_1(1)$.*

Proof. Let p be a point in $L \cap S$, and P be the geodesic plane through p orthogonal to L. Let $\gamma = S \cap P$ and let $\tilde{\gamma} = \gamma \cup R(\gamma)$, where $R : P \to P$ is the reflection with respect to p. That is, $\tilde{\gamma}$ is obtained by symmetrizing the intersection curve of P and S with respect to p. $\tilde{\gamma}$ is a C^2 curve whose curvature at p is 0. Varying p on $L \cap S$, we obtain a C^2 surface. If we think about the mean curvature of this surface at p, we see that there are two orthogonal directions in which the curvature is 0, that is the direction of L and the direction of γ, hence the mean curvature of the surface is 0 along $L \cap S$. □

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