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THREE NONTRIVIAL NONNEGATIVE SOLUTIONS FOR SOME CRITICAL p-LAPLACIAN SYSTEMS WITH LOWER-ORDER NEGATIVE PERTURBATIONS

Chang-Mu Chu, Chun-Yu Lei, Jiao-Jiao Sun, and Hong-Min Suo

Abstract. Three nontrivial nonnegative solutions for some critical quasilinear elliptic systems with lower-order negative perturbations are obtained by using the Ekeland's variational principle and the mountain pass theorem.

1. Introduction and main results

Let $N > p^2$, $1 < r < q < p$, $p^* = \frac{Np}{N-p}$. We are concerned with the following problems

$$(1) \qquad \begin{cases} -\triangle_p u = F_u(u,v) + \lambda G_u(u,v) - \mu H_u(u,v), & \text{in } \Omega, \\ -\triangle_p v = F_v(u,v) + \lambda G_v(u,v) - \mu H_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; λ and μ are positive parameters; $\triangle_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$ denotes the p-Laplacian operator; $F, G, H \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ are homogeneous of degree p^*, q and r, respectively. We recall that a function $\Gamma : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is homogeneous of degree k when $\Gamma(tz) = t^k\Gamma(z)$ for any $t \geq 0$ and $z \in (\mathbb{R}^+)^2$.

In recent years, more and more attention have been paid to the existence and multiplicity of nonnegative or positive solutions for the elliptic problems involving concave terms and critical Sobolev exponent. Results relating to these problems can be found in [1], [2], [4, 5, 12, 13], [7, 8, 9], [11, 14, 15, 16, 17, 18, 19, 20, 21], and the references therein. By the results of the above papers we know that the number of nontrivial solutions for problem (1) is affected by the concave-convex nonlinearities. Applying the strong maximum principle, it is easy to obtain the positive solutions for problem (1) when $\mu = 0$. However,

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if the concave terms of problem (1) are negative or local negative in Ω as $|z|$ near origin, then the strong maximum principle can not be applied (see [4] and [21]).

When F, G, H depends only on the first variable, problem (1) reduces to the following Dirichlet problem

$$\begin{cases} -\triangle_p u = u^{p^*-1} + \lambda u^{q-1} - \mu u^{r-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $1 < r < q < p < p^*$. Anello in [4] proved that problem (2) has at least two nontrivial nonnegative solutions for $\lambda > 0$ and $\mu > 0$ small enough by truncation techniques and variational methods. Anello also considered the subcritical growth case and obtain three nontrivial nonnegative solutions of the related problems (see Theorem 1 in [4]). The purpose of this paper is to apply the ideas of Theorem 1 in [4] to the critical growth case to obtain more than two solutions.

In particular, using the Ekeland's variational principle and the mountain pass theorem, we will prove problem (1) has at least three nontrivial nonnegative solutions.

Before stating our results, we introduce the following notations: we consider the space $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ equipped norm $\|z\|_E = (\|u\|^p + \|v\|^p)^{\frac{1}{p}}$, where $z = (u,v) \in E$ and $\|w\| := \left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}$ is the standard norm in $W_0^{1,p}(\Omega)$. Moreover, we denote by $\|w\|_s := \left(\int_\Omega |w|^s dx\right)^{\frac{1}{s}}$ $(1 < s < \infty)$ the norm of $L^s(\Omega)$, and by $\|w\|_\infty = ess\sup_\Omega |w|$ the norm of $L^\infty(\Omega)$. In addition, we denote positive constants by $C, C_1, C_2, \ldots$. The main result of this paper is the following theorem.

Theorem 1. *Let $N > p^2$, $1 < r \leq \frac{N(p-1)}{N-p} < q < p$, $F, G, H \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ be homogeneous functions of degree p^*, q and r, respectively. Assume that $m_G > 0$, $m_H > 0$ and $F_u(u,0) = F_u(0,v) = F_v(u,0) = F_v(0,v) = G_u(0,v) = G_v(u,0) = H_u(0,v) = H_v(u,0) = 0$ for all $u, v \in R^+$, then there exists $\Lambda > 0$ with the following property: for every $\lambda \in (0,\Lambda)$ there exists $\mu_\lambda > 0$ such that problem (1) for all $\mu \in (0,\mu_\lambda)$ has at least three solutions $z_i = (u_i, v_i)$ satisfies that $u_i \geq 0$, $v_i \geq 0$ in Ω and $u_i \not\equiv 0$, $v_i \not\equiv 0$ $(i = 1,2,3)$.*

Remark 1. We are not aware of any results in the literature on multiplicity of nontrivial nonnegative solutions for problem (1). There are many homogeneous functions satisfying the conditions of our theorem. Some classical examples are:

(i) $F(z) = \sum_j a_j u^{\alpha_j} v^{\beta_j}$;

(ii) $G(z) = |z|_s^q$, $H(z) = |z|_s^r$,

where $a_j > 0$, $\alpha_j > 1$, $\beta_j > 1$, $\alpha_j + \beta_j = p^*$, $|z|_s := (|u|^s + |v|^s)^{1/s}$ with $s > 1$.

From elliptic systems reduce to elliptic equations, our Theorem 1 can be described as:

Corollary 1. *Let $N > p^2$, $1 < r \leq \frac{N(p-1)}{N-p} < q < p$. Then there exists $\Lambda > 0$ with the following property: for every $\lambda \in (0, \Lambda)$ there exists $\mu_\lambda > 0$ such that problem (2) for all $\mu \in (0, \mu_\lambda)$ has at least three nontrivial nonnegative solutions.*

Remark 2. In Corollary 1, the question of the necessity of the restrictions on the exponents p, q and r. The authors in [4] obtained two nontrivial nonnegative solutions of problem (2) in the case $1 < r < q < p$.

This paper is organized as follows. In Section 2, we give Palais-Smale condition and some preliminaries. The proof of Theorem 1 is provided in Section 3.

2. Palais-Smale condition and some preliminaries

Let $u^\pm = \max\{\pm u, 0\}$. In this section, we show that the energy functional

$$I(u,v) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)dx - \int_\Omega F(u^+, v^+)dx - \lambda \int_\Omega G(u^+, v^+)dx + \mu \int_\Omega H(u^+, v^+)dx,$$

$(u, v) \in E$, associated to problem (1) satisfies the $(PS)_c$ condition at certain energy levels. Under the hypotheses of Theorem 1, it is obvious that I is a C^1 functional. It is well known that any critical point of I in E is a weak solution of problem (1). Hence, in order to obtain the nontrivial solutions of problem (1), we only need to look for the nontrivial critical points of I in E. In addition, since $F, G, H \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ are homogeneous functions of degree p^*, q and r, respectively, we have the so-called Euler identity

$$z \cdot \nabla F(z) = p^* F(z), \quad z \cdot \nabla G(z) = qG(z), \quad z \cdot \nabla H(z) = rH(z) \tag{3}$$

and

$$\begin{aligned} &m_F|z|^{p^*} \leq F(z) \leq M_F|z|^{p^*}, \\ &m_G|z|^q \leq G(z) \leq M_G|z|^q, \\ &m_H|z|^r \leq H(z) \leq M_H|z|^r \end{aligned} \tag{4}$$

for all $z \in (\mathbb{R}^+)^2$, where $m_\Gamma = \min_{\{z\in(\mathbb{R}^+)^2:\ |z|=1\}} \Gamma(z)$, $M_\Gamma = \max_{\{z\in(R^+)^2:\ |z|=1\}} \Gamma(z)$. Now we first give some preliminaries.

Definition 1. Let $c \in \mathbb{R}$, and let E^* be the dual space of the Banach space E.

(i) A sequence $\{z_n\} \subset E$ is called a $(PS)_c$ sequence of I if $I(z_n) \to c$ and $I'(z_n) \to 0$ in E^* as $n \to \infty$;

(ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{z_n\} \subset E$ of I has a convergent subsequence.

Lemma 1. *Under the hypotheses of Theorem 1, let $\{z_n\} = \{(u_n, v_n)\} \subset E$ be a $(PS)_c$ sequence of I, then $\{z_n\}$ is bounded.*

Proof. By the Sobolev imbedding theorem, there exists $C > 0$ such that

$$\|w\|_s \leq C\|w\| \quad \text{for all} \quad w \in W_0^{1,p}(\Omega) \quad \text{and} \quad 1 \leq s \leq p^*. \tag{5}$$

For each $\varepsilon > 0$, by the Hölder inequality and the Young inequality, we infer from (4) and (5) that

$$\begin{aligned}\left|\lambda \int_\Omega G(z^+)dx\right| &\leq \lambda M_G \int_\Omega |z|^q dx \\ &\leq \lambda M_G |\Omega|^{\frac{p-q}{p}} (\|u\|_p^p + \|v\|_p^p)^{\frac{q}{p}} \\ &\leq \lambda M_G C^q |\Omega|^{\frac{p-q}{p}} \|z\|_E^q \\ &\leq \varepsilon\|z\|_E^p + |\Omega|\varepsilon^{-\frac{q}{p-q}} (\lambda M_G C^q)^{\frac{p}{p-q}} \\ &= \varepsilon\|z\|_E^p + C(\varepsilon)\lambda^{\frac{p}{p-q}} \end{aligned} \tag{6}$$

for any $z \in (\mathbb{R}^+)^2$, where $C(\varepsilon) = |\Omega|\varepsilon^{-\frac{q}{p-q}} (M_G C^q)^{\frac{p}{p-q}}$.

Let $\{z_n\}$ be a $(PS)_c$ sequence of I. Using the hypotheses that $F_u(0, v) = F_v(u, 0) = G_u(0, v) = G_v(u, 0) = H_u(0, v) = H_v(u, 0) = 0$ for all $u, v \in \mathbb{R}^+$, we derive from (3) and (6) that

$$\begin{aligned} & p^* I(z_n) - \langle I'(z_n), z_n\rangle \\ = & \frac{p^* - p}{p}\|z_n\|_E^p - \lambda(p^* - q)\int_\Omega G(z_n^+)dx + \mu(p^* - r)\int_\Omega H(z_n^+)dx \\ \geq & \left(\frac{p^* - p}{p} - (p^* - q)\varepsilon\right)\|z_n\|_E^p - (p^* - q)C(\varepsilon)\lambda^{\frac{p}{p-q}}. \end{aligned}$$

It follows that

$$\left(\frac{p^* - p}{p} - (p^* - q)\varepsilon\right)\|z_n\|_E^p \leq p^* c + (p^* - q)C(\varepsilon)\lambda^{\frac{p}{p-q}} + o(\|z_n\|_E).$$

Let $\varepsilon < \frac{p^*-p}{p(p^*-q)}$, we obtain $\{z_n\}$ is bounded in E. □

Now we introduce the following version of the Brezis-Lieb lemma (see [3] or [6]).

Lemma 2. *Assume that $\Gamma \in C^1(\mathbb{R}^2)$ with $\Gamma(0, 0) = 0$ and $|\frac{\partial\Gamma}{\partial u}(z)|$, $|\frac{\partial\Gamma}{\partial v}(z)| \leq C_1|z|^{s-1}$ for some $1 \leq s < \infty$. Let $\{z_n\}$ be a bounded sequence in $L^s(\Omega)\times L^s(\Omega)$ and such that $z_n \rightharpoonup z$ weakly in E. Then, as $n \to \infty$,*

$$\int_\Omega \Gamma(z_n)dx = \int_\Omega \Gamma(z_n - z)dx + \int_\Omega \Gamma(z)dx + o(1).$$

Let

$$S = \inf_{w\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^p dx}{(\int_\Omega |w|^{p^*} dx)^{p/p^*}}$$

denote the best Sobolev constant for the imbedding of $W_0^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$. S is achieved on $\Omega = \mathbb{R}^N$ by the function $W(x) = \frac{K}{(1+|x|^{p/(p-1)})^{(N-p)/p^2}}$, where $K = \left[N\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{(N-p)/p^2}$ (see [9] or [20]). Define

$$S_F := \inf_{(u,v)\in E}\left\{\frac{\int_\Omega(|\nabla u|^p + |\nabla v|^p)dx}{(\int_\Omega F(u^+,v^+)dx)^{p/p^*}} : \int_\Omega F(u^+,v^+)dx > 0\right\}.$$

We have the following lemmas.

Lemma 3. *Under the hypotheses of Theorem 1, let $\{z_n\}$ be a $(PS)_c$ sequence of I with $z_n \rightharpoonup z$ in E. Then, there exists a positive constant $B = B(p,q,N,S,M_G,|\Omega|)$ such that*

$$\langle I^{'}(z), z\rangle = 0 \quad \textit{and} \quad I(z) \geq -B\lambda^{\frac{p}{p-q}}.$$

Proof. Let $\{z_n\} = \{(u_n,v_n)\}$ be a $(PS)_c$ sequence of I with $z_n \rightharpoonup z = (u,v)$ in E. Then we have

$$I^{'}(z_n) \to 0, \quad \text{strongly in } E^* \text{ as } \ n\to\infty.$$

Since $\{z_n\}$ is bounded, we can obtain a subsequence still denoted by $\{z_n\}$ such that

$$\begin{cases} z_n = (u_n,v_n) \to (u,v) = z, & \text{in } \ L^s(\Omega)\times L^s(\Omega), \quad 1 < s < p^*,\\ z_n = (u_n,v_n) \to (u,v) = z, & \text{a.e. in } \ \Omega,\\ \nabla u_n \to \nabla u, \quad \nabla v_n \to \nabla v, & \text{a.e. in } \ \Omega. \end{cases}$$

Consequently, passing to the limit in $\langle I^{'}(z_n),(\varphi,\psi)\rangle$ as $n\to\infty$, and using the hypotheses of our Lemma 3, we have

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx - \int_\Omega F_u(u^+,v^+)\varphi dx\\ &- \lambda\int_\Omega G_u(u^+,v^+)\varphi dx + \mu\int_\Omega H_u(u^+,v^+)\varphi dx = 0,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |\nabla v|^{p-2}\nabla v\cdot\nabla\psi dx - \int_\Omega F_v(u^+,v^+)\psi dx\\ &- \lambda\int_\Omega G_v(u^+,v^+)\psi dx + \mu\int_\Omega H_v(u^+,v^+)\psi dx = 0\end{aligned}$$

for all $(\varphi,\ \psi)\in E$, that is, $I^{'}(z) = 0$.

In particular, we have $\langle I^{'}(z), z\rangle = 0$, which implies from (3) that

$$\|z\|_E^p = p^*\int_\Omega F(z^+)dx + q\lambda\int_\Omega G(z^+)dx - r\mu\int_\Omega H(z^+)dx.$$

It follows that

$$I(z) = \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda}{p^*}\int_\Omega G(z^+)dx + \frac{(p^*-r)\mu}{p^*}\int_\Omega H(z^+)dx.$$

Using the Hölder inequality, the Young inequality and the Sobolev imbedding theorem, one has

$$\begin{aligned}
I(z) &\geq \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda}{p^*}\int_\Omega G(z^+)dx \\
&\geq \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda M_G}{p^*}\int_\Omega |z|^q dx \\
&\geq \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda M_G}{p^*}|\Omega|^{\frac{p^*-q}{p^*}}\left(\int_\Omega |z|^{p^*}dx\right)^{\frac{q}{p^*}} \\
&\geq \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda M_G}{p^*}2^{\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\left(\int_\Omega (|u|^{p^*}+|v|^{p^*})dx\right)^{\frac{q}{p^*}} \\
&\geq \frac{1}{N}\|z\|_E^p - \frac{(p^*-q)\lambda M_G}{p^*}\left(\frac{2}{S}\right)^{\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\|z\|_E^q \\
&\geq \frac{1}{N}\|z\|_E^p - \left(\frac{1}{N}\|z\|_E^p + \left(\frac{2N}{S}\right)^{\frac{q}{p-q}}\left[\frac{(p^*-q)\lambda M_G}{p^*}\right]^{\frac{p}{p-q}}|\Omega|^{\frac{p(p^*-q)}{p^*(p-q)}}\right) \\
&= -B\lambda^{\frac{p}{p-q}},
\end{aligned}$$

where $B = \left(\frac{2N}{S}\right)^{\frac{q}{p-q}}\left[\frac{(p^*-q)M_G}{p^*}\right]^{\frac{p}{p-q}}|\Omega|^{\frac{p(p^*-q)}{p^*(p-q)}} > 0$. □

Lemma 4. *Under the hypotheses of Theorem* 1, *I satisfies the $(PS)_c$ condition with c satisfying*

$$c < \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}},$$

where B is the positive constant given in Lemma 3.

Proof. Let $\{z_n = (u_n, v_n)\} \subset E$ be a $(PS)_c$ sequence of I with $c < \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}$. By Lemma 1, we know that $\{z_n\}$ is bounded. Up to a subsequence, we may assume that

$$\begin{cases} z_n = (u_n, v_n) \rightharpoonup (u, v) = z, & \text{in } E, \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{a.e. on } \Omega, \\ z_n = (u_n, v_n) \to (u, v) = z, & \text{in } L^s(\Omega)\times L^s(\Omega), \quad 1 < s < p^*. \end{cases}$$

From Lemma 3, we have that $\langle I'(z), z\rangle = 0$. Let $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$, where $\widetilde{u}_n = u_n - u$, $\widetilde{v}_n = v_n - v$. Using the hypotheses of Theorem 1, we infer from Lemma 2 that

$$\|\widetilde{z}_n\|_E^p = \|z_n\|_E^p - \|z\|_E^p + o(1),$$

$$\int_\Omega F((\widetilde{z}_n)^+)dx = \int_\Omega F(z_n^+)dx - \int_\Omega F(z^+)dx + o(1),$$

$$\int_\Omega G((\widetilde{z}_n)^+)dx = \int_\Omega G(z_n^+)dx - \int_\Omega G(z^+)dx + o(1),$$

and

$$\int_\Omega H((\widetilde{z}_n)^+)dx = \int_\Omega H(z_n^+)dx - \int_\Omega H(z^+)dx + o(1).$$

Since $I(z_n) = c + o(1)$ and $\langle I'(z_n), z_n\rangle = o(1)$, we obtain

$$\frac{1}{p}\|\widetilde{z}_n\|_E^p - \int_\Omega F((\widetilde{z}_n)^+)dx = c - I(z) + o(1) \tag{7}$$

and

$$\|\widetilde{z}_n\|_E^p - p^* \int_\Omega F((\widetilde{z}_n)^+)dx = o(1). \tag{8}$$

From (8), we may assume that

$$\|\widetilde{z}_n\|_E^p \to p^* l, \qquad \int_\Omega F((\widetilde{z}_n)^+)dx \to l.$$

Assume that $l > 0$, by the definition of S_F, we have

$$\|\widetilde{z}_n\|_E^p \geq S_F \left(\int_\Omega F((\widetilde{z}_n)^+)dx\right)^{\frac{p}{p^*}}.$$

As $n \to \infty$, we deduce that

$$l \geq \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}}.$$

It follows from (7) and Lemma 3 that

$$c = \left(\frac{p^*}{p} - 1\right) l + I(z) \geq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}},$$

which contradicts the fact $c < \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}$. Therefore, we have $l = 0$, which implies that

$$z_n \to z \quad \text{in} \quad E.$$

Hence I satisfies the $(PS)_c$ condition with $c < \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}$. □

Lemma 5. *Under the hypotheses of Theorem* 1, 0 *is a local minimum of* I *for any* $\lambda > 0$ *and* $\mu > 0$.

Proof. For each $\tau > 0$, by the Young inequality, we have

$$\int_\Omega |z^+|^q dx = \int_\Omega |z^+|^{\frac{(p^*-q)r}{p^*-r}} |z^+|^{\frac{p^*(q-r)}{p^*-r}} dx \leq \tau \int_\Omega |z^+|^r dx + \tau^{-\frac{p^*-q}{q-r}} \int_\Omega |z^+|^{p^*} dx.$$

Using the hypotheses of Theorem 1, we infer from (4) that

$$I(z) \geq \frac{1}{p}\|z\|_E^p + \mu m_H \int_\Omega |z^+|^r dx - \lambda M_G \int_\Omega |z^+|^q dx - M_F \int_\Omega |z^+|^{p^*} dx$$

$$\geq \frac{1}{p}\|z\|_E^p + (\mu m_H - \tau\lambda M_G)\int_\Omega |z^+|^r dx$$
$$- \left(M_F + \tau^{-\frac{p^*-q}{q-r}}\lambda M_G\right)\int_\Omega |z^+|^{p^*}dx.$$

Set $\tau = \frac{\mu m_H}{\lambda M_G}$. By the Sobolev imbedding theorem, one has

$$I(z) \geq \frac{1}{p}\|z\|_E^p - \left[M_F + (\lambda M_G)^{\frac{p^*-r}{q-r}}(\mu m_H)^{-\frac{p^*-q}{q-r}}\right]\int_\Omega |z^+|^{p^*}dx$$
$$\geq \frac{1}{p}\|z\|_E^p - \left[M_F + (\lambda M_G)^{\frac{p^*-r}{q-r}}(\mu m_H)^{-\frac{p^*-q}{q-r}}\right]\left(\frac{2}{S}\right)^{\frac{p^*}{p}}\|z\|_E^{p^*}.$$

Hence, for any $\lambda > 0$ and $\mu > 0$, we can find $\rho_1 > 0$ such that

$$I(z) > 0 \quad \text{if} \quad \|z\|_E = \rho_1 \quad \text{and} \quad I(z) \geq 0 = I(0) \quad \text{if} \quad \|z\|_E \leq \rho_1.$$

Therefore, 0 is a local minimum of I in E. □

In order to obtain a negative local minimal value of I in E, we consider the following system

$$(E_\lambda) \qquad \begin{cases} -\triangle_p u = \lambda G_u(u,v), & \text{in } \Omega, \\ -\triangle_p v = \lambda G_v(u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p \in (1, p^*)$, $\lambda > 0$. The corresponding functional of equation (E_λ) is

$$\Phi_\lambda(z) = \frac{1}{p}\|z\|_E^p - \lambda\int_\Omega G(z^+)dx.$$

We have the following lemma.

Lemma 6. *Let $G \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ be a homogeneous function of degree q. Assume that $m_G > 0$ and $G_u(0,v) = G_v(u,0) = 0$ for all $u, v \in \mathbb{R}^+$. Then, for equation (E_λ) there exists a nontrivial nonnegative solution z_λ such that $\Phi_\lambda(z_\lambda) < 0$ for any $\lambda > 0$.*

Proof. From inequality (6), with $\varepsilon = \frac{1}{2p}$, we obtain

$$\Phi_\lambda(z) \geq \frac{1}{2p}\|z\|_E^p - |\Omega|(2p)^{\frac{q}{p-q}}(\lambda M_G C^q)^{\frac{p}{p-q}}.$$

Hence, for any $\lambda > 0$, we can find $\rho_2 > 0$ such that

$$\Phi_\lambda(z) > 0 \ \text{ if } \ \|z\|_E = \rho_2 \quad \text{and} \quad \Phi_\lambda(z) \geq -C(\lambda) \quad \text{if} \quad \|z\|_E \leq \rho_2,$$

where $C(\lambda) = |\Omega|(2p)^{\frac{q}{p-q}}(\lambda M_G C^q)^{\frac{p}{p-q}} > 0$.

From $m_G > 0$, there exists $z_0 \in E$ such that $G(z_0^+) > 0$. Thus, for $k_0 > 0$ small enough, one has

$$\Phi_\lambda(k_0 z_0) = \frac{1}{p}k_0^p\|z_0\|_E^p - \lambda k_0^q\int_\Omega G(z_0^+)dx < 0,$$

which implies that

$$\alpha_\lambda = \inf_{z\in B_{\rho_2}(0)} \Phi_\lambda(z) < 0 < \inf_{z\in\partial B_{\rho_2}(0)} \Phi_\lambda(z).$$

By applying the Ekeland's variational principle (see [10]) in $\overline{B_{\rho_2}(0)}$, there is a minimizing sequence $\{\overline{z}_n\} = \{(\overline{u}_n, \overline{v}_n)\} \subset \overline{B_{\rho_2}(0)}$ such that

$$\Phi_\lambda(\overline{z}_n) \le \alpha_\lambda + \frac{1}{n},$$

and

$$\Phi_\lambda(z) \ge \Phi_\lambda(\overline{z}_n) - \frac{1}{n}\|z - \overline{z}_n\|_E, \qquad \forall\, z \in \overline{B_{\rho_2}(0)}.$$

Therefore, for any $\varphi,\ \psi \in W_0^{1,p}(\Omega)$, we have

$$\langle \Phi'_\lambda(\overline{z}_n), (\varphi,\psi)\rangle \to 0 \quad \text{and} \quad \Phi_\lambda(\overline{z}_n) \to \alpha_\lambda \quad \text{as } \ n\to\infty.$$

Since $\{\overline{z}_n\}$ is bounded and $\overline{B_{\rho_2}(0)}$ is a closed convex set, there exist $z_\lambda = (u_\lambda, v_\lambda) \in \overline{B_{\rho_2}(0)} \subset E$ and a subsequence still denoted by $\{\overline{z}_n\}$ such that

$$\begin{cases} \overline{z}_n \rightharpoonup z_\lambda, & \text{in } \ E,\\ \overline{z}_n \to z_\lambda, & \text{a.e. in } \ \Omega,\\ \overline{z}_n \to z_\lambda, & \text{in } \ L^s(\Omega)\times L^s(\Omega), \quad 1\le s < p^*.\end{cases}$$

Consequently, passing to the limit in $\langle \Phi'_\lambda(\overline{z}_n), (\varphi,\psi)\rangle$ as $n\to\infty$ and noticing that $G_u(0,v) = G_v(u,0) = 0$ for all $u, v \in R^+$, we have

$$\int_\Omega |\nabla(u_\lambda)|^{p-2}\nabla(u_\lambda)\cdot\nabla\varphi dx - \lambda\int_\Omega G_u(u_\lambda^+, v_\lambda^+)\varphi dx = 0$$

and

$$\int_\Omega |\nabla(v_\lambda)|^{p-2}\nabla(v_\lambda)\cdot\nabla\psi dx - \lambda\int_\Omega G_v(u_\lambda^+, v_\lambda^+)\psi dx = 0$$

for all $(\varphi,\psi) \in E$, that is, $\langle \Phi'_\lambda(z_\lambda), (\varphi,\psi)\rangle = 0$. Thus z_λ is a critical point of the functional Φ_λ. Since $\Phi_\lambda(0) = 0$ and any critical point of Φ_λ in E is nonnegative, we obtain that z_λ is a nontrivial nonnegative solution of equation (E_λ). In particular, we have $\langle \Phi'_\lambda(z_\lambda), z_\lambda\rangle = 0$, that is

$$\|z_\lambda\|_E^p = q\lambda\int_\Omega G(z_\lambda)dx. \tag{9}$$

Therefore, we obtain

$$\Phi_\lambda(z_\lambda) = \frac{1}{p}\|z_\lambda\|_E^p - \lambda\int_\Omega G(z_\lambda)dx = \left(\frac{1}{p} - \frac{1}{q}\right)\|z_\lambda\|_E^p < 0.$$

□

Lemma 7. *Under the hypotheses of Theorem* 1*, there exist a nonnegative function* $z \in E$*,* $\Lambda^* > 0$*, for all* $\lambda \in (0,\Lambda^*)$ *there exists* $\mu_\lambda^* > 0$ *such that for any* $\mu \in (0, \mu_\lambda^*)$

$$\sup_{t\ge 0} I(tz) < \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}},$$

where B is the positive constant given in Lemma 3.

Proof. For convenience, we consider the functional $J : E \to \mathbb{R}$ defined by

$$J(z) = \frac{1}{p}\|z\|_E^p - \int_\Omega F(z^+)dx + \mu \int_\Omega H(z^+)dx \quad \text{for all } \ z = (u, v) \in E.$$

Since system (1) is autonomous, without loss of generality we may assume $0 \in \Omega$. Let $\delta_0 > 0$ be such that $B(0, 2\delta_0) \subset \Omega$. Define a cut-off function $\phi(x) \in C_0^\infty(\Omega)$ that $\phi(x) = 1$ for $|x| < \delta_0$, $\phi(x) = 0$ for $x > 2\delta_0$, $0 \le \phi(x) \le 1$ and $|\nabla\phi| \le C_2$, where $C_2 > 0$ is a positive constant. Let

$$u_\eta(x) = \frac{\eta^{\frac{N-p}{p(p-1)}}\phi(x)}{\left(\eta^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}.$$

We have the following estimate (as $\eta \to 0$)

$$\frac{\int_\Omega |\nabla u_\eta|^p dx}{\left(\int_\Omega |u_\eta|^{p^*} dx\right)^{p/p^*}} = S + O\left(\eta^{\frac{N-p}{p-1}}\right). \tag{10}$$

Indeed, one have

$$\nabla u_\eta(x) = \eta^{\frac{N-p}{p(p-1)}}\left(\frac{\nabla\phi(x)}{\left(\eta^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}} - \frac{N-p}{p-1}\frac{\phi(x)|x|^{\frac{2-p}{p-1}}x}{\left(\eta^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{N}{p}}}\right).$$

Moreover, since $\phi(x) \equiv 1$ for $|x| < \delta_0$ and $|\nabla\phi| \le C_2$, one has

$$\begin{aligned}
\int_\Omega |\nabla u_\eta|^p dx &= \eta^{\frac{N-p}{p-1}}\int_\Omega \frac{|x|^{p/(p-1)}}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^N}dx + O\left(\eta^{\frac{N-p}{p-1}}\right)\\
&= \eta^{\frac{N-p}{p-1}}\int_{R^N} \frac{|x|^{p/(p-1)}}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^N}dx + O\left(\eta^{\frac{N-p}{p-1}}\right)\\
&= \int_{R^N} \frac{|y|^{p/(p-1)}}{\left(1 + |y|^{p/(p-1)}\right)^N}dy + O\left(\eta^{\frac{N-p}{p-1}}\right)\\
&= \|\nabla U\|_{L^p(R^N)}^p + O\left(\eta^{\frac{N-p}{p-1}}\right) \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |u_\eta|^{p^*} dx &= \eta^{\frac{N}{p-1}}\int_\Omega \frac{\phi^{p^*}dx}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^N}\\
&= \eta^{\frac{N}{p-1}}\int_{B(0,\delta_0)} \frac{dx}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right)\\
&= \eta^{\frac{N}{p-1}}\int_{R^N} \frac{dx}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right)
\end{aligned}$$

$$= \int_{R^N} \frac{dy}{\left(1+|y|^{p/(p-1)}\right)^N} + O\left(\eta^{\frac{N}{p-1}}\right)$$

$$= \|U\|^{p^*}_{L^{p^*}(R^N)} + O\left(\eta^{\frac{N}{p-1}}\right), \tag{12}$$

where $U(x) = \left(1+|x|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}} \in W^{1,p}(\mathbb{R}^N)$ satisfies

$$\frac{\|\nabla U\|^p_{L^p(\mathbb{R}^N)}}{\|U\|^p_{L^{p^*}(\mathbb{R}^N)}} = S = \inf_{w \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla w\|^p_{L^p(\mathbb{R}^N)}}{\|w\|^p_{L^{p^*}(\mathbb{R}^N)}}.$$

Combining (11) with (12), we deduce that (10) holds.

It follows from $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ and (4) that there exists $(e_1, e_2) \in \{z \in (\mathbb{R}^+)^2 : |z| = 1\}$ such that $F(e_1, e_2) = M_F$, which implies that

$$J(te_1u_\eta, te_2u_\eta) = \frac{1}{p}t^p \int_\Omega |\nabla u_\eta|^p dx - M_F t^{p^*} \int_\Omega |u_\eta|^{p^*} dx$$

$$+ \mu t^r \int_\Omega H(e_1, e_2)|u_\eta|^r dx$$

$$\leq \frac{1}{p}t^p \int_\Omega |\nabla u_\eta|^p dx - M_F t^{p^*} \int_\Omega |u_\eta|^{p^*} dx + \mu M_H t^r \int_\Omega |u_\eta|^r dx. \tag{13}$$

Define

$$\varphi(t) = \frac{1}{p}t^p \int_\Omega |\nabla u_\eta|^p dx - M_F t^{p^*} \int_\Omega |u_\eta|^{p^*} dx + \mu M_H t^r \int_\Omega |u_\eta|^r dx$$

and

$$\psi(t) = \frac{1}{p}t^p \int_\Omega |\nabla u_\eta|^p dx - M_F t^{p^*} \int_\Omega |u_\eta|^{p^*} dx$$

for all $t \geq 0$. Observe that the function φ attains its maximum in $[0, +\infty)$ at a point $t_\eta > 0$. Clearly, one has

$$0 = \varphi'(t_\eta)$$

$$= t_\eta^{p-1} \int_\Omega |\nabla u_\eta|^p dx - p^* M_F t_\eta^{p^*-1} \int_\Omega |u_\eta|^{p^*} dx + r\mu M_H t_\eta^{r-1} \int_\Omega |u_\eta|^r dx. \tag{14}$$

From (11), (12) and (14), there is $\eta_1 > 0$ such that

$$t_\eta \geq \left(\frac{\int_\Omega |\nabla u_\eta|^p dx}{p^* M_F \int_\Omega |u_\eta|^{p^*} dx}\right)^{\frac{1}{p^*-p}} \geq \left(\frac{|\nabla U|^p_{L^p(R^N)}}{4p^* M_F |U|^{p^*}_{L^{p^*}(R^N)}}\right)^{\frac{1}{p^*-p}} \doteq C_3 \tag{15}$$

for all $0 < \eta < \eta_1$. According to $1 < r \leq \frac{N(p-1)}{N-p}$, we have

$$\int_\Omega |u_\eta|^r dx = \eta^{\frac{r(N-p)}{p(p-1)}} \int_\Omega \frac{\phi^r dx}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^{\frac{r(N-p)}{p}}}$$

$$\leq \eta^{\frac{r(N-p)}{p(p-1)}} \int_{R^N} \frac{dx}{\left(\eta^{p/(p-1)} + |x|^{p/(p-1)}\right)^{\frac{r(N-p)}{p}}}$$

$$= \eta^{\frac{pN+pr-rN}{p}} \int_{R^N} \frac{dy}{\left(1+|y|^{p/(p-1)}\right)^{\frac{r(N-p)}{p}}}$$
$$= \eta^{\frac{pN+pr-rN}{p}} |U|^r_{L^r(R^N)}$$
$$\le C_4 \eta^{\frac{N}{p}}. \tag{16}$$

By (11), (12), (14), (15) and (16), there exist $\mu_1 > 0$ and η_2 satisfies that $0 < \eta_2 \le \eta_1$ such that

$$t_\eta \le \left(\frac{\int_\Omega |\nabla u_\eta|^p dx + r\mu M_H C_2^{r-p} \int_\Omega |u_\eta|^r dx}{p^* M_F \int_\Omega |u_\eta|^{p^*} dx} \right)^{\frac{1}{p^*-p}}$$
$$\le \left(\frac{4|\nabla U|^p_{L^p(R^N)}}{p^* M_F |U|^{p^*}_{L^{p^*}(R^N)}} \right)^{\frac{1}{p^*-p}} \doteq C_5 \tag{17}$$

for all $0 < \mu < \mu_1$ and $0 < \eta < \eta_2$.

According to (4) and the Minkowski inequality, we have

$$\left(\int_\Omega F(z^+) dx \right)^{\frac{p}{p^*}} \le (M_F)^{\frac{p}{p^*}} \left(\int_\Omega |z|^{p^*} dx \right)^{\frac{p}{p^*}}$$
$$\le (M_F)^{\frac{p}{p^*}} \left[\left(\int_\Omega |u|^{p^*} dx \right)^{\frac{p}{p^*}} + \left(\int_\Omega |v|^{p^*} dx \right)^{\frac{p}{p^*}} \right]$$
$$\le (M_F)^{\frac{p}{p^*}} \frac{1}{S} \int_\Omega (|\nabla u|^p + |\nabla v|^p) dx$$

for any $z = (u, v) \in E$. It implies that

$$S_F \ge S(M_F)^{-\frac{p}{p^*}} > 0. \tag{18}$$

After a direct calculation, we deduce from (10) and (18) that

$$\max_{t\ge 0} \psi(t) = \frac{1}{N} (p^* M_F)^{-\frac{N-p}{p}} \left[\frac{\int_\Omega |\nabla u_\eta|^p dx}{\left(\int_\Omega |u_\eta|^{p^*} dx\right)^{p/p^*}} \right]^{\frac{N}{p}}$$
$$= \frac{1}{N} (p^* M_F)^{-\frac{N-p}{p}} S^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}})$$
$$\le \frac{p}{N-p} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}). \tag{19}$$

According to (13), (16), (17) and (19), we have

$$\sup_{t\ge 0} J(te_1 u_\eta, te_2 u_\eta) \le \psi(t_\eta) + \mu M_H t_\eta^r \int_\Omega |u_\eta|^r dx$$
$$\le \frac{p}{N-p} \left(\frac{S_F}{p^*} \right)^{\frac{N}{p}} + C_6 \mu \eta^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}})$$

$$\leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) \tag{20}$$

for any $0<\eta<\eta_2$ and $\mu<\mu_\eta = \min\left\{\mu_1, \eta^{\frac{N-p^2}{p(p-1)}}\right\}$.

Noticing that $S_F>0$, we can choose $\delta_1>0$ such that

$$\frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}} > 0, \quad \forall\ \lambda\in(0,\delta_1).$$

Since

$$I(te_1u_\eta, te_2u_\eta) \leq \frac{1}{p}t^p\int_\Omega |\nabla u_\eta|^p dx + \mu M_H t^r\int_\Omega |u_\eta|^r dx,$$

it follows from (11) and (16) that there exist $T\in(0,1)$ and $\eta_3\in(0,\eta_2)$ such that

$$\sup_{0\leq t\leq T} I(te_1u_\eta, te_2u_\eta) \leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}} \tag{21}$$

for all $0<\lambda<\delta_1$, $0<\eta<\eta_3$ and $0<\mu<\mu_\eta$. Moreover, using the definitions of I and u_η, it follows from (4) and (20) that

$$\begin{aligned}\sup_{t\geq T} I(te_1u_\eta, te_2u_\eta) &= \sup_{t\geq T}\left(J(te_1u_\eta, te_2u_\eta) - \lambda t^q\int_\Omega G(e_1,e_2)|u_\eta|^q dx\right)\\ &\leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O\left(\eta^{\frac{N-p}{p-1}}\right) - \lambda m_G T^q\int_{B(0,\delta_0)} |u_\eta|^q dx\end{aligned}$$

for any $0<\eta<\eta_2$ and $\mu<\mu_\eta$. By Lemma $A5$ of [12], it implies from $\frac{N(p-1)}{N-p}<q<p<p^*$ that there exists $C_7>0$ such that

$$\int_{B(0,\delta_0)} |u_\eta|^q dx \geq C_7\eta^{\frac{N(p-q)+pq}{p}}.$$

By the above two inequalities, we have

$$\begin{aligned}&\sup_{t\geq T} I(te_1u_\eta, te_2u_\eta)\\ &\leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} + O(\eta^{\frac{N-p}{p-1}}) - C_7 m_G T^q\lambda\eta^{\frac{N(p-q)+pq}{p}}\end{aligned} \tag{22}$$

for any $0<\eta<\eta_2$ and $\mu<\mu_\eta$.

By the hypothesis $\frac{N(p-1)}{N-p}<q<p$, we obtain $\frac{(N-p)(p-q)}{p(N-Np+Nq-pq)}>0$. For some positive constants C_8 and C_9, let $\eta=\lambda^{\frac{p(p-1)}{(p-q)(N-p)}}$ and $\lambda<\left(\frac{C_9}{B+C_8}\right)^{\frac{(N-p)(p-q)}{p(N-Np+Nq-pq)}}$, we have

$$C_8\eta^{\frac{N-p}{p-1}} - C_9\lambda\eta^{\frac{N(p-q)+pq}{p}} = C_8\lambda^{\frac{p}{p-q}} - C_9\lambda^{\frac{(Np-Nq+pq-p)p}{(N-p)(p-q)}} < -B\lambda^{\frac{p}{p-q}},$$

which implies that there exists $\delta_2 > 0$ such that for all $\eta = \lambda^{\frac{p(p-1)}{(p-q)(N-p)}}$ and $0 < \lambda < \delta_2$

$$O(\eta^{\frac{N-p}{p-1}}) - C_7 m_G T^q \lambda \eta^{\frac{N(p-q)+pq}{p}} < -B\lambda^{\frac{p}{p-q}}. \tag{23}$$

From (22) and (23), for all $\eta = \lambda^{\frac{p(p-1)}{(p-q)(N-p)}}$, $0 < \lambda < \delta_2$ and $\mu < \mu_\eta$,

$$\sup_{t\geq T} I(te_1u_\eta, te_2u_\eta) \leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}. \tag{24}$$

Set $\Lambda^* = \min\left\{\delta_1, \delta_2, \eta_3^{\frac{(N-p)(p-q)}{p(p-1)}}\right\}$. Combining (21) with (24), for all $\eta = \lambda^{\frac{p(p-1)}{(p-q)(N-p)}}$ and $\lambda \in (0, \Lambda^*)$, there exists $\mu_\lambda^* > 0$ such that for any $\mu \in (0, \mu_\lambda^*)$,

$$\sup_{t\geq 0} I(te_1u_\eta, te_2u_\eta) \leq \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}.$$

□

3. The proof of main results

In this section, we prove Theorem 1 by using the Ekeland's variational principle and the mountain pass theorem.

Proof of Theorem 1. Set $\rho = (p^*)^{1/p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p^2}}$. It follows that

$$\begin{aligned}\inf_{\|z\|_E=\rho}\left(\frac{1}{p}\|z\|_E^p - \int_\Omega F(z^+)dx\right) &\geq \inf_{\|z\|_E=\rho}\left(\frac{1}{p}\|z\|_E^p - S_F^{-\frac{p^*}{p}}\|z\|_E^{p^*}\right)\\ &= \frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}}.\end{aligned} \tag{25}$$

Consider the following function

$$\Psi(\lambda) = \inf_{\|z\|_E=\rho}\left(\frac{1}{p}\|z\|_E^p - \int_\Omega F(z^+)dx - \lambda\int_\Omega G(z^+)dx\right), \quad \lambda \in \mathbb{R}.$$

It is easy to check that Ψ is continuous in $\mathbb{R}$. Moreover, from (25) there exists $\Lambda \in (0, \Lambda^*)$ such that

$$\Psi(\lambda) > \frac{p}{2(N-p)}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}}$$

and

$$\frac{p}{N-p}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}} > \frac{p}{2(N-p)}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} > 0 \tag{26}$$

for any $\lambda \in (0, \Lambda)$. Fix $\lambda \in (0, \Lambda)$ and consider the function

$$\Psi_\lambda(\mu) = \inf_{\|z\|_E=\rho}\left(\frac{1}{p}\|z\|_E^p - \int_\Omega F(z^+)dx - \lambda\int_\Omega G(z^+)dx + \mu\int_\Omega H(z^+)dx\right), \ \mu \in \mathbb{R}.$$

Obviously,

$$\Psi_\lambda(0) = \Psi(\lambda) > \frac{p}{2(N-p)}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}}.$$

By the continuity of Ψ_λ, there is $\mu_{1,\lambda} \in (0,1)$ such that

$$\Psi_\lambda(\mu) > \frac{p}{2(N-p)}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} \quad \text{for all} \quad \mu \in (0, \mu_{1,\lambda}). \tag{27}$$

Let $z_\lambda \in E$ be the nontrivial nonnegative solution of the equation (E_λ) obtained by Lemma 6 and define

$$\chi(t) = \frac{1}{p}\|tz_\lambda\|_E^p - \lambda \int_\Omega G(tz_\lambda)dx.$$

It follows from (9) and the homogeneity of G that

$$\chi^{'}(t) = t^{p-1}\|z_\lambda\|_E^p - q\lambda t^{q-1}\int_\Omega G(z_\lambda)dx = (t^{p-1} - t^{q-1})\|z_\lambda\|_E^p < 0$$

for all $t \in (0,1)$. In particular,

$$\max_{t\in[0,1]} \chi(t) = \chi(0) = 0 \qquad \text{and} \qquad \chi(1) < 0.$$

By the nonnegativity of F, we obtain that

$$\max_{t\in[0,1]}\left(\frac{1}{p}\|tz_\lambda\|_E^p - \int_\Omega F(tz_\lambda)dx - \lambda\int_\Omega G(tz_\lambda)dx\right) = 0 \tag{28}$$

and

$$\frac{1}{p}\|z_\lambda\|_E^p - \int_\Omega F(z_\lambda)dx - \lambda\int_\Omega G(z_\lambda)dx < 0. \tag{29}$$

Now we consider that the function

$$\begin{aligned} &K_\lambda(\mu)\\ &= \max_{t\in[0,1]}\left(\frac{1}{p}\|tz_\lambda\|_E^p - \int_\Omega F(tz_\lambda)dx - \lambda\int_\Omega G(tz_\lambda)dx + \mu\int_\Omega H(tz_\lambda)dx\right), \ \mu \in \mathbb{R}. \end{aligned}$$

It follows from (28) that $K_\lambda(0) = 0$. Using the continuity of K_λ, there exists $\mu_{2,\lambda} \in (0, \mu_{1,\lambda})$ such that

$$K_\lambda(\mu) < \frac{p}{2(N-p)}\left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} \quad \text{for any } \mu \in (0, \mu_{2,\lambda}). \tag{30}$$

Moreover, according to (29), there is $\mu_\lambda \in (0, \mu_{2,\lambda})$ such that

$$I(z_\lambda) = \frac{1}{p}\|z_\lambda\|_E^p - \int_\Omega F(z_\lambda)dx - \lambda\int_\Omega G(z_\lambda)dx + \mu\int_\Omega H(z_\lambda)dx < 0 \tag{31}$$

for any $\mu \in (0, \mu_\lambda)$. At this point, fix $\mu \in (0, \mu_\lambda)$. Combining (27) with (30), we obtain $\|z_\lambda\|_E < \rho$. Thus we deduce from (31) that

$$c_1 = \inf_{z\in B_\rho(0)} I(z) < 0 < \inf_{z\in\partial B_\rho(0)} I(z).$$

By applying the Ekeland's variational principle in $\overline{B_\rho(0)}$, we obtain that there exists a $(PS)_{c_1}$ sequence $\{z_n\} = \{(u_n, v_n)\} \subset \overline{B_\rho(0)}$. It follows from (26) and Lemma 4 that I satisfies the $(PS)_{c_1}$ condition. Therefore, one has a subsequence still denoted by $\{z_n\}$ and $z_1 = (u_1, v_1) \in E$ such that $z_n \to z_1$ in E and

$$I(z_1) = c_1 < 0, \quad I^{'}(z_1) = 0,$$

which implies that $z_1 \neq 0$ is a solution of problem (1).

Applying Lemma 5, we know that 0 is a local minimum for I. Define

$$\Gamma_1 = \{\gamma \in C([0,1], E) |\ \gamma(0) = 0,\ \gamma(1) = z_\lambda\}, \quad c_2 = \inf_{\gamma\in\Gamma_1} \max_{t\in[0,1]} I(\gamma(t)).$$

It follows from (26) and (30) that

$$c_2 \leq \max_{t\in[0,1]} I(tz_\lambda) < \frac{p}{2(N-p)} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} < \frac{p}{N-p} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}. \tag{32}$$

Applying Lemma 4, we know that I satisfies the $(PS)_{c_2}$ condition. By the mountain pass theorem (see [2]), we obtain that problem (1) has the second solution $z_2 = (u_2, v_2)$ with $I(z_2) = c_2 > 0$.

Let $\bar{z} = (e_1, e_2)$ satisfy $F(\bar{z}) = M_F$. From (6), we have

$$I(t\bar{z}) \leq \left(\frac{1}{p} + \varepsilon\right) t^p \|\bar{z}\|_E^p - M_F|\Omega|t^{p^*} + M_H|\Omega|t^r + C(\varepsilon)\lambda^{\frac{p}{p-q}},$$

which implies that

$$I(t\bar{z}) \to -\infty \quad \text{as } t \to +\infty.$$

Hence, there exists a positive number t_0 such that $\|t_0\bar{z}\|_E > \rho$ and $I(t_0\bar{z}) < 0$ for any $\lambda \in (0, \Lambda)$. Therefore, the functional I has the mountain pass geometry. Define

$$\Gamma_2 = \{\gamma \in C([0,1], E) |\ \gamma(0) = 0,\ \gamma(1) = t_0\bar{z}\}, \quad c_3 = \inf_{\gamma\in\Gamma_2} \max_{t\in[0,1]} I(\gamma(t)).$$

From Lemma 7, we have

$$c_3 < \frac{p}{N-p} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} - B\lambda^{\frac{p}{p-q}}.$$

According to Lemma 4, we know that I satisfies the $(PS)_{c_3}$ condition. By using the mountain pass theorem, we obtain that problem (1) has the third solution $z_3 = (u_3, v_3)$ with $I(z_3) = c_3$. Combining (27) with (32), we have

$$I(z_3) > \frac{p}{2(N-p)} \left(\frac{S_F}{p^*}\right)^{\frac{N}{p}} > I(z_2) > 0 > I(z_1),$$

which implies that z_1, z_2 and z_3 are distinct.

Now we show that any critical point of I in E is nonnegative. In fact, let $z = (u, v)$ be any critical point of I in E. Using the hypothesis that

$F_u(0,v^+) = F_v(u^+,0) = G_u(0,v^+) = G_v(u^+,0) = H_u(0,v^+) = H_v(u^+,0) = 0$ for any $(u,v) \in E$, after a direct calculation, we derive that

$$\|u^-\|^p = \langle I_u^{'}(u,v), -u^-\rangle = 0, \quad \text{and} \quad \|v^-\|^p = \langle I_v^{'}(u,v), -v^-\rangle = 0,$$

which implies that $u^- = 0$ and $v^- = 0$. Hence we have $u \geq 0$ and $v \geq 0$. Therefore, z_1, z_2 and z_3 are three nontrivial nonnegative solutions of problem (1).

Next, we show that $u_i \neq 0$ and $v_i \neq 0$ $(i = 1,2,3)$. Since $I(z_i) \neq 0 = I(0,0)$, we have $u_i \neq 0$ or $v_i \neq 0$. Without loss of generality, we may assume that $u_i \neq 0$ and $v_i = 0$. Using the hypothesis that $F_u(u,0) = F_v(0,v) = 0$ for all $u, v \in \mathbb{R}^+$, it is easy to obtain u_i $(i = 1,2,3)$ satisfies that

$$\begin{cases} -\triangle_p u_i = \lambda G_u(u_i,0) - \mu H_u(u_i,0), & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Acting on (33) with $u_i \in W_0^{1,p}(\Omega)$, it follows from (3) that

$$\int_\Omega |\nabla u_i|^p dx = q\lambda \int_\Omega G(u_i,0)dx - r\mu \int_\Omega H(u_i,0)dx.$$

Since $G, H \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$, we have

$$\begin{aligned} I(u_i,0) &= \frac{1}{p}\int_\Omega |\nabla u_i|^p dx - \lambda \int_\Omega G(u_i,0)dx + \mu \int_\Omega H(u_i,0)dx \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega |\nabla u_i|^p dx + \left(\frac{q}{r} - 1\right)\lambda \int_\Omega G(u_i,0)dx \\ &\leq \left(\frac{1}{p} - \frac{1}{r}\right) q\lambda \int_\Omega G(u_i,0)dx + \left(\frac{q}{r} - 1\right)\lambda \int_\Omega G(u_i,0)dx \\ &= \left(\frac{q}{p} - 1\right)\lambda \int_\Omega G(u_i,0)dx \leq 0. \end{aligned} \tag{34}$$

Which is a contradiction with $I(u_i,0) = I(z_i) > 0$ $(i = 2,3)$. Therefore, we have $u_i \neq 0$ and $v_i = 0$ $(i = 2,3)$ are not established. Similarly, we obtain $u_i = 0$ and $v_i \neq 0$ $(i = 2,3)$ are impossible. Hence we have $u_2 \neq 0$, $v_2 \neq 0$, $u_3 \neq 0$ and $v_3 \neq 0$.

Lastly, we demonstrate that $u_1 \neq 0$ and $v_1 \neq 0$. We may assume that $u_1 \neq 0$ and $v_1 = 0$. It follows from $m_G > 0$ that there exists $v_0 > 0$ such that

$$G(u_1, v_0) > G(u_1, 0). \tag{35}$$

In fact, according to $G \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$, we have $G(u,v) \geq G(u,0)$ for any $u > 0$ and $v > 0$. Assume that $G(u,v) = G(u,0)$ for all $u > 0$ and $v > 0$, we have

$$G(tu, v) = G(tu, tv) = t^q G(u,v).$$

It implies that $G(0,v) = 0$ for any $v > 0$. Which is a contradiction with $m_G > 0$. Therefore, we have (35) hold. Then, for any $\lambda \in (0, \Lambda)$, we have

$$I(tu_1, tv_0) = \frac{1}{p}t^p \int_\Omega |\nabla u_1|^p dx - \lambda t^q \int_\Omega G(u_1,0)dx + \mu t^q \int_\Omega H(u_1,0)dx$$

$$+ \frac{1}{p}t^p \int_\Omega |\nabla v_0|^p dx - \lambda t^q \int_\Omega (G(u_1, v_0) - G(u_1, 0))dx$$

$$- \mu t^q \int_\Omega H(u_1, 0)dx + \mu t^r \int_\Omega H(u_1, v_0)dx. \tag{36}$$

It follows from (34) that

$$\begin{aligned} &\frac{1}{p}t^p \int_\Omega |\nabla u_1|^p dx - \lambda t^q \int_\Omega G(u_1, 0)dx + \mu t^q \int_\Omega H(u_1, 0)dx \\ &= t^q \left(\frac{1}{p}t^{p-q} \int_\Omega |\nabla u_1|^p dx - \lambda \int_\Omega G(u_1, 0)dx + \mu \int_\Omega H(u_1, 0)dx\right) \\ &< \frac{1}{p} \int_\Omega |\nabla u_1|^p dx - \lambda \int_\Omega G(u_1, 0)dx + \mu \int_\Omega H(u_1, 0)dx \\ &= I(u_1, 0) = c_1 \end{aligned} \tag{37}$$

for any $0 < t < 1$.

Let $\mu = t^{2q-r}$, according to $H \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ and (35), there exists $t_1 > 0$ such that for any $t \in (0, t_1)$

$$\begin{aligned} &\frac{1}{p}t^p \int_\Omega |\nabla v_0|^p dx - \lambda t^q \int_\Omega (G(u_1, v_0) - G(u_1, 0))dx \\ &- \mu t^q \int_\Omega H(u_1, 0)dx + \mu t^r \int_\Omega H(u_1, v_0)dx \\ &\le \frac{1}{p}t^p \int_\Omega |\nabla v_0|^p dx - \lambda t^q \int_\Omega (G(u_1, v_0) - G(u_1, 0))dx \\ &+ t^{2q} \int_\Omega H(u_1, v_0)dx \\ &< 0. \end{aligned} \tag{38}$$

Set $t_2 = \min\{1, t_1, \mu_\lambda^{r-2q}\}$, from (36)-(38), for any $t \in (0, t_2)$, we obtain that

$$I(tu_1, tv_0) < c_1.$$

Thus, we can choose t so small that $(tu_1, tv_0) \in B_\rho(0)$. Hence we obtain

$$\inf_{z \in B_\rho(0)} I(z) < c_1,$$

which is a contradiction with $c_1 = \inf_{z \in B_\rho(0)} I(z)$. Therefore, we have $u_1 \neq 0$ and $v_1 = 0$ are not established. Similarly, we obtain $u_1 = 0$ and $v_1 \neq 0$ are impossible. Hence we have $u_1 \neq 0$ and $v_1 \neq 0$. The proof of Theorem 1 is completed. □

References

[1] A. Ambrosetti, H. Brezis, and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 347–381.

[3] P. Amster, P. De Nápoli, and M. C. Mariani, *Cristina existence of solutions for elliptic systems with critical Sobolev exponent*, Electron J. Differential Equations **2002** (2002), no. 49, 13 pp.
[4] G. Anello, *Multiple nonnegative solutions for an elliptic boundary value problem involving combined nonlinearities*, Math. Comput. Modelling **52** (2010), no. 1-2, 400–408.
[5] ———, *Multiplicity and asymptotic behavior of nonnegative solutions for elliptic problems involving nonlinearities indefinite in sign*, Nonlinear Anal. **75** (2012), no. 8, 3618–3628.
[6] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functional*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490.
[7] C. M. Chu and C. L. Tang, *Existence and multiplicity of positive solutions for semilinear elliptic systems with Sobolev critical exponents*, Nonlinear Anal. **71** (2009), no. 11, 5118–5130.
[8] Q. Y. Dai and L. H. Peng, *Necessary and sufficient conditions for the existence of nonnegative solutions of inhomogeneous p-Laplace equation*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), no. 1, 34–56.
[9] H. Egnell, *Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal. **104** (1988), no. 1, 57–77.
[10] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.
[11] D. G. de Figueiredo, J. P. Gossez, and P. Ubilla, *Local "superlinearity" and "sublinearity" for the p-Laplacian*, J. Funct. Anal. **3** (2009), no. 3, 721–752.
[12] J. García Azorero and I. P. Alonso, *Some results about the existence of a second positive solution in a quasilinear critical problem*, Indiana Univ. Math. J. **43** (1994), no. 3, 941–957.
[13] J. García Azorero, I. P. Alonso, and J. J. Manfredi, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math. **2** (2000), no. 3, 385–404.
[14] P. G. Han, *Multiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents*, Nonlinear Anal. **64** (2006), no. 4, 869–886.
[15] T. S. Hsu, *Multiplicity results for p-Laplacian with critical nonlinearity of concave-convex type and sign-changing weight functions*, Abstr. Appl. Anal. **2009** (2009), Art. ID 652109, 24 pp.
[16] T. S. Hsu and H. L. Lin, *Multiple positive solutions for a critical elliptic system with concave-convex nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **139** (2009), no. 6, 1163–1177.
[17] T. X. Li and T. F. Wu, *Multiple positive solutions for a Dirichlet problem involving critical Sobolev exponent*, J. Math. Anal. Appl. **369** (2010), no. 1, 245–257.
[18] D. C. de Morais Filho and M. A. S. Souto, *Systems of p-Laplacean equations involving homogeneous nonlinearities with critical Sobolev exponent degrees*, Comm. Partial Differential Equations **24** (1999), no. 7-8, 1537–1553.
[19] Y. Shen and J. H. Zhang, *Multiplicity of positive solutions for semilinear p-Laplacian system with Sobolev critical exponent*, Nonlinear Anal. **74** (2011), 1019–1030.
[20] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372.
[21] T. F. Wu, *On semilinear elliptic equations involving critical Sobolev exponents and sign-changing weight function*, Commun. Pure Appl. Anal. **7** (2008), no. 2, 383–405.

Chang-Mu Chu
College of Science
Guizhou Minzu University
Guiyang Guizhou 550025, P. R. China
E-mail address: gzmychuchangmu@sina.com

Chun-Yu Lei
College of Science
Guizhou Minzu University
Guiyang Guizhou 550025, P. R. China
E-mail address: 969290985@qq.com

Jiao-Jiao Sun
College of Science
Guizhou Minzu University
Guiyang Guizhou 550025, P. R. China
E-mail address: 295284483@qq.com

Hong-Min Suo
College of Science
Guizhou Minzu University
Guiyang Guizhou 550025, P. R. China
E-mail address: gzmysxx@sina.com