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STRUCTURE THEOREMS FOR SOME CLASSES OF GRADE FOUR GORENSTEIN IDEALS

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Abstract. The structure theorems [3, 6, 21] for the classes of perfect ideals of grade 3 have been generalized to the structure theorems for the classes of perfect ideals linked to almost complete intersections of grade 3 by a regular sequence [15]. In this paper we obtain structure theorems for two classes of Gorenstein ideals of grade 4 expressed as the sum of a perfect ideal of grade 3 (except a Gorenstein ideal of grade 3) and an almost complete intersection of grade 3 which are geometrically linked by a regular sequence.

1. Introduction

Structure theorem for perfect ideals in a noetherian local ring goes back to the Hilbert structure theorem for perfect ideals of grade 2 over a polynomial ring [12]. It was generalized by Burch to a local ring [4]. Using multilinear algebra and algebra structure on finite free resolutions, Buchsbaum-Eisenbud gave structure theorems for two classes of Gorenstein ideals and almost complete intersections of grade 3 [6]. Kustin and Miller introduced the numerical invariant $\lambda(I)$ to classify a class of Gorenstein ideals of grade 4 in terms of resolutions of R/I in a Gorenstein local ring R [18]. Brown and Sanchez [3, 21] described structure theorems for a class of perfect ideals of grade 3 with type 2 and $\lambda(I) > 0$ and for a class of perfect ideals of grade 3 with type 3 and $\lambda(I) \geq 2$, respectively. These perfect ideals described by Buchsbaum-Eisenbud, Brown and Sanchez are algebraically linked to an almost complete intersection of grade 3 by a regular sequence. We gave a structure theorem for some classes of perfect ideals of grade 3 which are algebraically linked to an almost complete intersection by a regular sequence [15]. It says that every perfect ideal I of grade 3 with type τ and $1 \leq \tau \leq 4$ algebraically linked to an almost complete intersection J of grade 3 with type r by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ has the form:

$$I = (\mathbf{x}, p_{11}, p_{21}, \ldots, p_{r1}),$$

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where p_{i1} is an element defined in (2.11) or (2.12). This contains some classes of perfect ideals of grade 3 with type 4. Structure theorems proved by Buchsbaum-Eisenbud, Brown and Sanchez are obtained from it. A structure theorem for complete intersections of grade $g \geq 4$ is described in [10, 16]. A structure theorem for some classes of perfect ideals of grade 3 which are algebraically linked by a regular sequence to a class of perfect ideals of grade 3 minimally generated by five elements is given in [14]. Kustin and Miller gave a structure theorem for a class of Gorenstein ideals of grade 4 mentioned above. A structure theorem for a class of Gorenstein ideals of grade 4 expressed as the sum of a Gorenstein ideal of grade 3 and an almost complete intersection of grade 3 geometrically linked by a regular sequence is studied in [7]. El Khoury, Iarrobino and Srinivasan gave a structure theorem for a class of homogeneous Gorenstein ideals $I = \oplus_{t\geq 2} I_t$ of grade 4 in $R = k[x, y, z, w]$ such that height$(I_2) = 1$ and $(I_2) = (wx, wy, wz)$ or $(I_2) = (wx, wy, w^2)$ over a field k [13, 17]. The main purpose of this paper is to give two structure theorems for some classes of Gorenstein ideals of grade 4 expressed as the sum of a perfect ideal of grade 3 with type τ $(2 \leq \tau \leq 4)$ and an almost complete intersection of grade 3 with type r geometrically linked by a regular sequence. These Gorenstein ideals of grade 4 fall into one of the following two classes:

(E) a class of Gorenstein ideals H of grade 4 expressed as the sum of a perfect ideal I of grade 3 with type τ and an almost complete intersection J of grade 3 with even type r geometrically linked by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ in $I \cap J$.

(O) a class of Gorenstein ideals H of grade 4 expressed as the sum of a perfect ideal I of grade 3 with type τ and an almost complete intersection J of grade 3 with odd type r geometrically linked by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ in $I \cap J$.

In Section 2, we review linkage theory and a structure theorem for some classes of perfect ideals of grade 3 with type τ algebraically linked to almost complete intersections of grade 3.

In Section 3, we construct the minimal free resolution of R/H, where H is a Gorenstein ideal of grade 4 in one of two cases (E) and (O). To do this we build up some matrices.

In Section 4, we give structure theorems for the two classes mentioned above. We introduced a complete matrix which plays a key role in describing a structure theorem for complete intersections of grade 4 [16].

(a) If $H = I + J$ is a Gorenstein ideal of grade 4 in class (E), then H is generated by the pfaffian of a certain alternating submatrix of the alternating matrix $\mathcal{A}(L)$ induced by a skew symmetrizable matrix L and the quotients $\bar{L}_i$ of the maximal order pfaffians of $\mathcal{A}(L)$, that is,

$$H = (\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_{r+3}, \mathcal{A}(L)_{123}).$$

(b) If $H = I + J$ is a Gorenstein ideal of grade 4 in class (O), then H is generated by the pfaffians of some alternating submatrices of the alternating matrix $T = \tilde{L}$ defined in (4.5) and elements $\bar{h}_i$ defined in Theorem 4.13 for $i = 1, 2, \ldots, r$, that is,

$$H = (T_{12}, T_{13}, T_{23}, \mathrm{Pf}(T), \bar{h}_1, \bar{h}_2, \ldots, \bar{h}_r).$$

The proofs for these theorems depend on the Bass' result [2, Proposition 2.9] and the structure theorem for some classes of perfect ideals of grade 3 with type τ algebraically linked to an almost complete intersection of grade 3 with type r by a regular sequence.

2. Structure theorems for some classes of perfect ideals and linkage

An $n\times n$ matrix $Y = (y_{ij})$ with entries in a commutative ring R is alternating if $y_{ii} = 0$ and $y_{ji} = -y_{ij}$. The determinant of this matrix is a perfect square in R, and the pfaffian of Y is defined as uniquely determined the square root of the determinant of Y and is denoted by $\mathrm{Pf}(Y)$ (see Artin [1, p. 40]). Let (i) be a multi-index $i_1, i_2, \ldots, i_s$. If $s < n$, we define $\mathrm{Pf}_{(i)}(Y)$ to be the pfaffian of the alternating submatrix of Y obtained by deleting rows and columns $i_1, i_2, \ldots, i_s$ from Y. Let $\theta(i)$ denote the sign of a permutation that rearranges (i) in an increasing order. If (i) has a repeated index, then we set $\theta(i) = 0$. Let $\tau(i)$ be the sum of the entries of (i). Define

$$Y_{(i)} = (-1)^{\tau(i)+1}\theta(i)\mathrm{Pf}_{(i)}(Y). \tag{2.1}$$

If $s = n$, let $Y_{(i)} = (-1)^{\tau(i)+1}\theta(i)$ and if $s > n$, let $Y_{(i)} = 0$. Let $\mathbf{y} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}$ be the row vector of the maximal order pfaffians of Y, signed appropriately according to the conventions described above. There is a "Laplace expansion" for developing pfaffians in terms of ones of lower order.

Lemma 2.1 ([18])**.** *Let Y be an $n \times n$ alternating matrix and j a fixed integer, $1 \leq j \leq n$. Then*

(1) *$Pf(Y) = \sum_{i=1}^{n} y_{ij} Y_{ij}$, and*

(2) $\mathbf{y}Y = 0$.

The following lemma follows from Lemma 2.1.

Lemma 2.2 ([21])**.** *Let Y be an $n \times n$ alternating matrix. Let a, b, c, d, and e be distinct integers between 1 and n. Then*

(1) $\displaystyle\sum_{i=1}^{n} y_{ik} Y_{iab} = -\delta_{ka} Y_b + \delta_{kb} Y_a,$

(2) $\displaystyle\sum_{i=1}^{n} y_{ik} Y_{iabc} = \delta_{ka} Y_{bc} - \delta_{kb} Y_{ac} + \delta_{kc} Y_{ab},$

(3) $\displaystyle\sum_{i=1}^{n} y_{ik} Y_{iabcd} = -\delta_{ka} Y_{bcd} + \delta_{kb} Y_{acd} - \delta_{kc} Y_{abd} + \delta_{kd} Y_{abc},$

$$(4)\ \sum_{i=1}^{n} y_{ik}Y_{iabcde} = \delta_{ka}Y_{bcde} - \delta_{kb}Y_{acde} + \delta_{kc}Y_{abde} - \delta_{kd}Y_{abce} + \delta_{ke}Y_{abcd},$$

where δ_{ij} is Kronecker's delta.

For further purpose, we need a lemma which follows from Lemma 2.2.

Lemma 2.3. *Let τ be an integer with $\tau \geq 4$. Let i, j, k, and l be integers with $1 \leq i, j, k, l \leq \tau$. Let $Y = (y_{ij})$ be a $\tau \times \tau$ alternating matrix. Then we have*

$$Y_{jkl}Y_i - Y_{ikl}Y_j + Y_{ijl}Y_k - Y_{ijk}Y_l = 0.$$

A Gorenstein ideal of grade 3 in a noetherian local ring is characterized in the following form.

Theorem 2.4 ([6])**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $n \geqslant 3$ be an odd integer. Let F be a free R-module with rank $F = n$. Let $f : F^* \to F$ be an alternating map whose image is contained in $\mathfrak{m}F$. Suppose that $Pf_{n-1}(f)$ has grade 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal minimally generated by n elements.*

(2) *Every Gorenstein ideal of grade 3 arises in this way.*

We notice that as in [6] or [20], in most cases, linkage is used in the case of perfect ideals in Gorenstein or Cohen-Macaulay local rings. However, the results that we use here are true for perfect ideals in any commutative ring, as shown by Golod [11].

Definition 2.5. Let I and J be perfect ideals of grade g. An ideal I is linked to J, $I \sim J$ if there exists a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$ in $I \cap J$ such that $J = (\mathbf{x}) : I$ and $I = (\mathbf{x}) : J$, and *geometrically* linked to J if $I \sim J$ and $I \cap J = (\mathbf{x})$.

A fundamental result is that the linkage is a symmetric relation on the set of perfect ideals in a noetherian ring R.

Theorem 2.6 ([20])**.** *Let R be a noetherian ring. If I is a perfect ideal of grade g and $\mathbf{x} = x_1, x_2, \ldots, x_g$ is a regular sequence in I, then $J = (\mathbf{x}) : I$ is a perfect ideal of grade g and $I = (\mathbf{x}) : J$.*

An almost complete intersection of grade g is linked to a Gorenstein ideal of grade g by a regular sequence $\mathbf{x}$.

Proposition 2.7 ([6])**.** *Let I and J be perfect ideals of the same grade g in a noetherian ring R, and suppose that I is linked to J by a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$. Then*

(1) *If I is Gorenstein, then $J = (\mathbf{x}, w)$ for some w in R and*

(2) *If J is minimally generated by $\mathbf{x}$ and w, then I is Gorenstein.*

The following theorem provides a method of constructing a Gorenstein ideal of grade $g + 1$ from perfect ideals of grade g.

Theorem 2.8 ([20])**.** *Let R be a noetherian ring. Let I and J be perfect ideals of grade g. If I and J are geometrically linked, then $H = I + J$ is a Gorenstein ideal of grade $g+1$.*

Let R be a commutative ring with identity, and let $A = (a_{ij})$ be an $r \times 3$ matrix and $Y = (y_{ij})$ an $r \times r$ alternating matrix over R, where r is a positive integer greater than 1. We define $C = (c_i), E = (e_j), S = (s_{ij})$, and $Z = (z_{ij})$ to be a 1×3 matrix, a $1 \times r$ matrix, a $3 \times r$ matrix and a 3×3 matrix, respectively, given by the following: For any two integers $m < t$ in $\{i, m, t\} = \{1, 2, 3\}$, we define

$$c_i = \begin{cases} \displaystyle\sum_{\substack{1 \le u < v \le r \\ i \ne m,t}} Y_{uv} \begin{vmatrix} a_{um} & a_{ut} \\ a_{vm} & a_{vt} \end{vmatrix} & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd,} \end{cases} \tag{2.2}$$

$$e_j = \begin{cases} \displaystyle\sum_{1 \le a < b < c \le r} -Y_{jabc} D_{abc} & \text{if } r \text{ is even} \\ Y_j & \text{if } r \text{ is odd,} \end{cases} \tag{2.3}$$

$$s_{ij} = \begin{cases} (-1)^{i+1} \displaystyle\sum_{1 \le h \le r} Y_{jh} a_{hi} & \text{if } r \text{ is even} \\ (-1)^{i+1} \displaystyle\sum_{1 \le u < v \le r} Y_{juv} \begin{vmatrix} a_{um} & a_{ut} \\ a_{vm} & a_{vt} \end{vmatrix} & \text{if } r \text{ is odd,} \end{cases} \tag{2.4}$$

$$Z = \begin{cases} \operatorname{diag}\{-\operatorname{Pf}(Y), -\operatorname{Pf}(Y), -\operatorname{Pf}(Y)\} & \text{if } r \text{ is even} \\ \begin{bmatrix} 0 & Z_3 & -Z_2 \\ -Z_3 & 0 & Z_1 \\ Z_2 & -Z_1 & 0 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} Z_1 & Z_2 & Z_3 \end{bmatrix} & \text{if } r \text{ is odd,} \end{cases} \tag{2.5}$$

where D_{abc} is the determinant of a 3×3 submatrix of A formed by three rows a, b, c of A in this order, and $Z_i = -\sum_{k=1}^{r} Y_k a_{ki}$ for $i = 1, 2, 3$.

We also define w to be an element in R as follows:

$$w = \begin{cases} \operatorname{Pf}(Y) & \text{if } r \text{ is even} \\ \displaystyle\sum_{1 \le a < b < c \le r} Y_{abc} D_{abc} & \text{if } r \text{ is odd.} \end{cases} \tag{2.6}$$

For the case that r is even, we define F to be a $3 \times r$ matrix given by

$$F = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ -a_{12} & -a_{22} & \cdots & -a_{r2} \\ a_{13} & a_{23} & \cdots & a_{r3} \end{bmatrix} = (f_{ij}), \quad \text{where} \quad f_{ij} = (-1)^{i+1} a_{ji}. \tag{2.7}$$

We give an another version of a structure theorem for almost complete intersections of grade 3.

Theorem 2.9 ([15])**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and J an almost complete intersection of grade 3 with type r. With the above notation, if r is even, then*

$$J = (c_1, c_2, c_3, w), \tag{2.8}$$

with its minimal free resolution of $R/\mathcal{K}_3(f)$

$$\mathbb{F} : 0 \longrightarrow R^r \xrightarrow{f_3} R^{r+3} \xrightarrow{f_2} R^4 \xrightarrow{f_1} R,$$

where

$$f_1 = \begin{bmatrix} C & w \end{bmatrix}, \quad f_2 = \widetilde{f} = \left[\begin{array}{c|c} Z & S \\ \hline C & E \end{array}\right], \quad f_3 = \begin{bmatrix} F \\ Y \end{bmatrix},$$

and if r is odd, then

$$J = (Z_1, Z_2, Z_3, w), \tag{2.9}$$

with its minimal free resolution of $R/\mathcal{K}_3(f)$

$$\mathbb{F} : 0 \longrightarrow R^r \xrightarrow{f_3} R^{r+3} \xrightarrow{f_2} R^4 \xrightarrow{f_1} R,$$

where

$$f_1 = \begin{bmatrix} \mathbf{z} & w \end{bmatrix}, \quad f_2 = \widetilde{f} = \left[\begin{array}{c|c} Z & S \\ \hline C & E \end{array}\right], \quad f_3 = \begin{bmatrix} A & Y \end{bmatrix}^t.$$

Proof. See [15, Theorem 4.8]. □

Let $\mathbf{x} = x_1, x_2, x_3$ be a regular sequence in an almost complete intersection $J = (t_1, t_2, t_3, t_4)$ of grade 3 with type r in (2.8) or (2.9). Then we can find a 4×3 matrix $B = (b_{ij})$ such that

$$\mathbf{x} = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix} B. \tag{2.10}$$

Let $\bar{D}_{abc}$ be the determinant of the submatrix of A formed by three rows a, b, c of B in this order. In [15] we have defined p_{k1} to be an element given by if r is even, then

$$\begin{aligned} p_{k1} &= \sum_{1 \le a < b < c \le r} -Y_{kabc} D_{abc} \bar{D}_{123} - \sum_{l=1}^{r} (a_{l1} \bar{D}_{234} + a_{l2} \bar{D}_{134} + a_{l3} \bar{D}_{124}) Y_{kl} \\ &= e_k \bar{D}_{123} - (s_{1k} \bar{D}_{234} - s_{2k} \bar{D}_{134} + s_{3k} \bar{D}_{124}), \end{aligned} \tag{2.11}$$

and if r is odd, then

$$p_{k1} = -Y_k \bar{D}_{123} + (s_{1k} \bar{D}_{234} - s_{2k} \bar{D}_{134} + s_{3k} \bar{D}_{124}). \tag{2.12}$$

Now we give a structure theorem for some classes of perfect ideals of grade 3 linked to an almost complete intersection of grade 3 by a regular sequence.

Theorem 2.10 ([15]). *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let J and B be an almost complete intersection of grade 3 and a matrix defined above, respectively. Let $\mathbf{x} = x_1, x_2, x_3$ be a regular sequence in J defined in (2.10). Let r be the type of J.*

(i) *Let r be even. Let A, Y, E, and S be matrices defined in (2.2), ..., (2.5), with entries in $\mathfrak{m}$, and p_{k1} an element defined in (2.11) for $k = 1, 2, \ldots, r$.*

(ii) *Let r be odd. Let A, S, Y, and Z be matrices defined in (2.2), ..., (2.5), with entries in $\mathfrak{m}$, and p_{k1} an element defined in (2.12) for $k = 1, 2, \ldots, r$.*

If I is an ideal generated by $x_1, x_2, x_3, p_{11}, p_{21}, \ldots, p_{r1}$, then I is a perfect ideal of grade 3 linked to J by a regular sequence $\mathbf{x}$ and is type $\mu(J/(\mathbf{x}))$.

(2) *Every perfect ideal of grade 3 linked to an almost complete intersection J of grade 3 by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ arises in the way of (1).*

For further use we give the properties of matrices defined above.

Lemma 2.11 ([15]). *With the above notation, let r be a positive integer.*

(1) *If r is even, then*
(a) $wE + CS = 0$, (b) $CF + EY = 0$, (c) $ZF + SY = 0$.

(2) *If r is odd, then*
(a) $SA = wI$, (b) $SY = ZA^T$, (c) $w\mathbf{y} = \mathbf{y}AS$.

The following example illustrates Theorem 2.10.

Example 2.12. Let $R = \mathbb{Q}[[x, y, z, t]]$ be the formal power series over the field $\mathbb{Q}$ of rationals with indeterminates x, y, z, t and I the ideal generated by seven elements

$$\begin{aligned}
&y^4 - x^2yz - xyz^2 - y^2z^2 + x^2yt + xy^2t + y^2zt - yt^3,\\
&xy^2 - xyz + yz^2 + x^2t + xyt - xzt - yzt - zt^2,\\
&x^3 - xy^2 + xyz + z^3 - y^2t - xzt + yt^2 - zt^2,\\
&-x^3y + xy^3 + xyz^2 - 2y^2zt + xyt^2,\\
&x^2y^2 - y^4 + y^2z^2 - 2xyzt + y^2t^2,\\
&x^2yz + y^3z - yz^3 - 2xy^2t + yzt^2,\\
&-2xy^2z + x^2yt + y^3t + yz^2t - yt^3.
\end{aligned}$$

Let A and Y be 4×3 and 4×4 matrices given by

$$A = \begin{bmatrix} t & z & y \\ z & t & x \\ y & x & t \\ x & y & z \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & x & 0 & t \\ -x & 0 & y & 0 \\ 0 & -y & 0 & z \\ -t & 0 & -z & 0 \end{bmatrix}.$$

For $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, let c_i, e_j and w be the elements in (2.2), (2.3), (2.6), respectively. Let F be the matrix given in (2.7). Then we can rewrite I

in the form

$$I = (yc_1, c_2, c_3, ye_1, ye_2, ye_3, ye_4).$$

By Algebra system, CoCoA 4.7.4, we can check that $\mathbf{x} = yc_1, c_2, c_3$ is a regular sequence. Then $J = (\mathbf{x}) : I = (\mathbf{x}, w)$ is an almost complete intersection of grade 3 with type 4. Thus it follows from Theorem 2.10 that I is a perfect ideal of grade 3 with type 2. Let K and G be 3×4 and 7×7 matrices given by

$$K = \begin{bmatrix} t & z & y & x \\ -yz & -yt & -xy & -y^2 \\ y^2 & xy & yt & yz \end{bmatrix} \text{ and } G = \left[\begin{array}{c|c} \mathbf{0} & K \\ \hline -F^t & Y \end{array}\right]. \tag{2.13}$$

The minimal free resolution $\mathbb{F}$ of R/I is

$$\mathbb{F} : 0 \longrightarrow R^2 \xrightarrow{f_3} R^8 \xrightarrow{f_2} R^7 \xrightarrow{f_1} R\ ,$$

where

$$f_1 = \begin{bmatrix} yc_1 & c_2 & c_3 & ye_1 & ye_2 & ye_3 & ye_4 \end{bmatrix},$$

$$f_2 = \begin{bmatrix} G & U \end{bmatrix},\ U = \begin{bmatrix} U_1 & \mathbf{0} \end{bmatrix}^t,\ U_1 = \begin{bmatrix} 0 & c_3 & -c_2 \end{bmatrix},$$

$$f_3 = \begin{bmatrix} Q & C^t \\ S_1 & E^t \\ -y & 0 \end{bmatrix},\ Q = \begin{bmatrix} -w \\ 0 \\ 0 \end{bmatrix},\ S_1 = \begin{bmatrix} s_{11} \\ s_{12} \\ s_{13} \\ s_{14} \end{bmatrix}.$$

3. Resolutions of two classes of Gorenstein ideals of grade 4

We construct the minimal free resolutions of two classes of Gorenstein ideals of grade 4 mentioned in the introduction.

For a positive integer r with $r > 1$, let A and Y be the $r \times 3$ matrix and the $r \times r$ alternating matrix, respectively, defined in Section 2. Let K and G be the $3 \times r$ matrix and the $(r+3) \times (r+3)$ alternating matrix, respectively, given by

$$K = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ -a_{12} & -a_{22} & \cdots & -a_{r2} \\ a_{13} & a_{23} & \cdots & a_{r3} \end{bmatrix} \quad \text{and} \quad G = \left[\begin{array}{c|c} \mathbf{0} & K \\ \hline -K^t & Y \end{array}\right]. \tag{3.1}$$

We define H to be an ideal associated with the pfaffians of some alternating submatrices of the alternating matrix G as follows. If r is even, then H is an ideal minimally generated by the maximal order pfaffians of G and the pfaffian of the alternating submatrix Y of G, that is,

$$H = (G_1, G_2, \ldots, G_{r+3}, G_{123}) \tag{3.2}$$

or an ideal minimally generated by the pfaffians of some $r \times r$ alternating submatrices of G and the pfaffian of a certain $(r+2) \times (r+2)$ alternating submatrix of G, that is,

$$H = (G_{ijk}, G_{ij4}, G_{ij5}, \ldots, G_{ijr+3}, G_k), \text{ where } \{i, j, k\} = \{1, 2, 3\}. \tag{3.3}$$

If r is odd, then H is an ideal minimally generated by the maximal order pfaffians of the $r \times r$ alternating submatrix Y of G and the pfaffian of G, that is,

$$H = (G_{1234}, G_{1235}, \ldots, G_{123r+3}, \mathrm{Pf}(G)) \tag{3.4}$$

or an ideal minimally generated by the pfaffians of some $(r+1) \times (r+1)$ alternating submatrices of G, that is,

$$H = (G_{ki}, G_{kj}, G_{k4}, G_{k5}, \ldots, G_{kr+3}, G_{ij}), \text{ where } \{i, j, k\} = \{1, 2, 3\}. \tag{3.5}$$

Yong Sung Cho proved that if H is of grade 4, then H is Gorenstein by constructing the minimal free resolution $\mathbb{F}$ of R/H although H is slightly a different form from [7].

Theorem 3.1 ([7]). *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. With the above notation, let A and Y be matrices with entries in $\mathfrak{m}$. If H is of grade 4, then H is Gorenstein.*

Proof. For the case that r is even, the proof is referred to [7]. For the case that r is odd, the proof is similar to that of the case that r is even. □

We construct the minimal free resolutions of R/H, where H is of the form in class (E) or (O). Let t_i be an element given by, for $i = 1, 2, 3$,

$$t_i = \begin{cases} c_i & \text{if } r \text{ is even} \\ Z_i & \text{if } r \text{ is odd.} \end{cases}$$

Let H be an ideal generated by $(r+4)$ elements $t_1, t_2, t_3, w, p_{11}, p_{21}, \ldots, p_{r1}$ defined in Section 2. We construct the minimal free resolution $\mathbb{H}$ of R/H

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{h_4} R^{r+4} \xrightarrow{h_3} R^{2(r+3)} \xrightarrow{h_2} R^{r+4} \xrightarrow{h_1} R\,. \tag{3.6}$$

Define h_1 to be a map from R^{4+r} to R given by

$$h_1 = \begin{bmatrix} t_1 & t_2 & t_3 & w & p_{11} & p_{21} & \cdots & p_{r1} \end{bmatrix} = \begin{bmatrix} f_1 & P \end{bmatrix},$$

where f_1 is the 1×4 matrix defined in Theorem 2.9 and $P = (p_{k1})$ is the $1 \times r$ matrix. First in order to define h_2, we introduce following four matrices $M(i)$ for $i = 1, 2, 3, 4$. Let $M(1)$ be a 3×3 matrix defined by if r is even, then

$$M(1) = \begin{bmatrix} 0 & \bar{D}_{124} & \bar{D}_{134} \\ -\bar{D}_{124} & 0 & \bar{D}_{234} \\ -\bar{D}_{134} & -\bar{D}_{234} & 0 \end{bmatrix}$$

and if r is odd, then

$$M(1) = \begin{bmatrix} -\bar{D}_{123} & 0 & 0 \\ 0 & -\bar{D}_{123} & 0 \\ 0 & 0 & -\bar{D}_{123} \end{bmatrix}.$$

Define $M(2)$ to be a $3 \times r$ matrix given by

$$M(2) = \begin{cases} \bar{D}_{123}F & \text{if } r \text{ is even} \\ \bar{A} = (\bar{a}_{ki}) & \text{if } r \text{ is odd,} \end{cases}$$

where $\bar{a}_{ki}$ is an element defined by

$$\bar{a}_{1i} = \bar{D}_{124}a_{i2} + \bar{D}_{134}a_{i3},\ \bar{a}_{2i} = -\bar{D}_{124}a_{i1} + \bar{D}_{234}a_{i3},\ \bar{a}_{3i} = -\bar{D}_{134}a_{i1} - \bar{D}_{234}a_{i2}$$

for each i. Let $M(3)$ be a 1×3 matrix given by

$$M(3) = \mathbf{0} \quad \text{if } r \text{ is even} \quad \text{and} \quad M(3) = \begin{bmatrix} -\bar{D}_{234} & \bar{D}_{134} & -\bar{D}_{124} \end{bmatrix} \quad \text{if } r \text{ is odd.}$$

Finally we define $M(4)$ to be a $1 \times r$ matrix given by if r is even, then

$$M(4) = \begin{bmatrix} m_1 & m_2 & \cdots & m_r \end{bmatrix}, \ \text{where } m_i = a_{i1}\bar{D}_{234} + a_{i2}\bar{D}_{134} + a_{i3}\bar{D}_{124}$$

for each i, and if r is odd, then

$$M(4) = \mathbf{0}.$$

Now we can define M to be a $4 \times (r+3)$ matrix given by

$$M = \left[\begin{array}{c|c} M(1) & M(2) \\ \hline M(3) & M(4) \end{array}\right],$$

and N to be an $r \times (r+3)$ matrix given by

$$N = \begin{bmatrix} -F^t & Y \end{bmatrix} \text{ if } r \text{ is even} \quad \text{and} \quad N = \begin{bmatrix} A & Y \end{bmatrix} \text{ if } r \text{ is odd.}$$

Define h_2 and h_3 to be maps from $R^{2(r+3)}$ to R^{r+3} and from R^{r+3} to $R^{2(r+3)}$, respectively, given by

$$h_2 = \left[\begin{array}{c|c} M & f_2 \\ \hline N & \mathbf{0} \end{array}\right] \quad \text{and} \quad h_3 = \begin{bmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{bmatrix} h_2^t,$$

where I is an $(r+3) \times (r+3)$ identity matrix. More concretely, if r is even, then

$$h_2 = \begin{bmatrix} M(1) & \bar{D}_{123}F & Z & S \\ \mathbf{0} & M(4) & C & E \\ -F^t & Y & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad h_3 = \begin{bmatrix} Z & C^t & \mathbf{0} \\ S^t & E^t & \mathbf{0} \\ -M(1) & \mathbf{0} & -F \\ \bar{D}_{123}F^t & M(4)^t & -Y \end{bmatrix},$$

and if r is odd, then

$$h_2 = \begin{bmatrix} M(1) & \bar{A} & Z & S \\ M(3) & \mathbf{0} & \mathbf{0} & \mathbf{y} \\ A & Y & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad h_3 = \begin{bmatrix} -Z & \mathbf{0} & \mathbf{0} \\ S^t & \mathbf{y}^t & \mathbf{0} \\ M(1) & M(3)^t & A^t \\ \bar{A}^t & \mathbf{0} & -Y \end{bmatrix}.$$

Finally we define h_4 to be a map from R to R^{r+3} given by

$$h_4 = h_1^t.$$

Lemma 3.2. *With the above notation, $h_ih_{i+1} = 0$ for $i = 1, 2, 3$.*

Proof. We have two cases: r is even and r is odd. We prove Lemma 3.2 for the even case. The proof for the odd case is similar to that of the even case.

(i) $h_1h_2 = 0$.

Let f_1 and f_2 be maps defined in Theorem 2.9. Since $f_1f_2 = 0$, it is sufficient to show that $f_1M + PN = 0$. First we show that $CM(1) - PF^t = 0$.

$$\begin{aligned}
(CM(1))_{11} = &\ - c_2\bar{D}_{124} - c_3\bar{D}_{134},\\
(-PF^t)_{11} = &\ - (a_{11}p_{11} + \cdots + a_{r1}p_{r1})\\
= &\ - (e_1a_{11} + \cdots + e_ra_{r1})\bar{D}_{123}\\
&+ (s_{11}a_{11} + s_{12}a_{21} + \cdots + s_{1r}a_{r1})\bar{D}_{234}\\
&- (s_{21}a_{11} + s_{22}a_{21} + \cdots + s_{2r}a_{r1})\bar{D}_{134}\\
&+ (s_{31}a_{11} + s_{32}a_{21} + \cdots + s_{3r}a_{r1})\bar{D}_{124}\\
= &\ c_3\bar{D}_{134} + c_2\bar{D}_{124}.
\end{aligned}$$

The last identity follows from parts (1) and (2) of Proposition 3.1 in [15], that is,

$$EA = 0 \quad \text{and} \quad SA = \begin{bmatrix} 0 & -c_3 & -c_2\\ -c_3 & 0 & c_1\\ c_2 & c_1 & 0 \end{bmatrix}.$$

Hence

$$(CM(1))_{11} - (PF^t)_{11} = 0.$$

In a similar way, we get

$$(CM(1))_{1i} - (PF^t)_{1i} = 0 \quad \text{for} \ \ i = 2, 3.$$

Finally we show that $CM(2) + wM(4) + PY = 0$. For each i, we have

$$\begin{aligned}
(CM(2))_{1i} &= c_1a_{i1}\bar{D}_{123} - c_2a_{i2}\bar{D}_{123} + c_3a_{i3}\bar{D}_{123},\\
(wM(4))_{1i} &= wm_i = wa_{i1}\bar{D}_{234} + wa_{i2}\bar{D}_{134} + wa_{i3}\bar{D}_{124},\\
(PY)_{1i} &= (e_1y_{1i} + e_2y_{2i} + \cdots + e_ry_{ri})\bar{D}_{123}\\
&\quad - (s_{11}y_{1i} + s_{12}y_{2i} + \cdots + s_{1r}y_{ri})\bar{D}_{234}\\
&\quad + (s_{21}y_{1i} + s_{22}y_{2i} + \cdots + s_{2r}y_{ri})\bar{D}_{134}\\
&\quad - (s_{31}y_{1i} + s_{32}y_{2i} + \cdots + s_{3r}y_{ri})\bar{D}_{124}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&(CM(2) + wM(4) + PY)_{1i}\\
= &\ \bar{D}_{123}(c_1a_{i1} - c_2a_{i2} + c_3a_{13} + e_1y_{1i} + e_2y_{2i} + \cdots + e_ry_{ri})\\
&- \bar{D}_{234}(-wa_{i1} + s_{11}y_{1i} + s_{12}y_{2i} + \cdots + s_{1r}y_{ri})\\
&+ \bar{D}_{134}(-w(-a_{i2}) + s_{21}y_{1i} + s_{22}y_{2i} + \cdots + s_{2r}y_{ri})\\
&- \bar{D}_{124}(-wa_{i3} + s_{31}y_{1i} + s_{32}y_{2i} + \cdots + s_{3r}y_{ri}) = 0.
\end{aligned}$$

The last identity follows from parts (1)(b) and (1)(c) of Lemma 2.11.

(ii) $h_2h_3 = 0$.

We note that

$$h_2h_3 = \begin{bmatrix} M(1) & \bar{D}_{123}F & Z & S \\ \mathbf{0} & M(4) & C & E \\ -F^t & Y & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z & C^t & \mathbf{0} \\ S^t & E^t & \mathbf{0} \\ -M(1) & \mathbf{0} & -F \\ \bar{D}_{123}F^t & M(4)^t & -Y \end{bmatrix}.$$

We complete the proof of this part by showing the following eight identities.

(a) $M(1)Z + \bar{D}_{123}FS^t - ZM(1) + S\bar{D}_{123}F^t = 0$.

(i) $M(1)Z - ZM(1) = 0$. This follows from a direct computation.

(ii) $\bar{D}_{123}FS^t + S\bar{D}_{123}F^t = \bar{D}_{123}(FS^t + SF^t) = 0$.

$$SF^t = \begin{bmatrix} 0 & c_3 & -c_2 \\ -c_3 & 0 & c_1 \\ c_2 & -c_1 & 0 \end{bmatrix} \Rightarrow FS^t + SF^t = (SF^t)^t + SF^t = 0.$$

(b) $M(1)C^t + \bar{D}_{123}FE^t + SM(4)^t = 0$.

Since $EA = 0$, $A^tE^t = 0$. So we have $FE^t = 0$. For $i = 1$, we have

$$\begin{aligned} (M(1)C^t)_{i1} &= c_2\bar{D}_{124} + c_3\bar{D}_{134}, \\ (SM(4)^t)_{i1} &= \sum_{k=1}^{n} s_{1k}m_k = \sum_{k=1}^{n} s_{1k}(a_{k1}\bar{D}_{234} + a_{k2}\bar{D}_{134} + a_{k3}\bar{D}_{124}) \\ &= \sum_{k=1}^{n}(a_{k1}s_{1k}\bar{D}_{234} + a_{k2}s_{1k}\bar{D}_{134} + a_{k3}s_{1k}\bar{D}_{124}) \\ &= -c_3\bar{D}_{134} - c_2\bar{D}_{124}. \end{aligned}$$

So we have

$$(M(1)C^t)_{11} + (SM(4)^t)_{11} = 0.$$

Similarly, we have

$$(M(1)C^t)_{i1} + (SM(4)^t)_{i1} = 0 \quad \text{for } i = 2, 3.$$

Hence

$$M(1)C^t + \bar{D}_{123}FE^t + SM(4)^t = 0.$$

(c) $ZF + SY = 0$. This follows from part (1)(c) of Lemma 2.11.

(d) $M(4)S^t - CM(1) + EM(2)^t = 0$. This follows from part (b).

$$M(4)S^t - CM(1) + E\bar{D}_{123}F^t = (M(1)C^t + \bar{D}_{123}FE^t + SM(4)^t)^t = 0.$$

(e) $M(4)E^t + EM(4)^t = 0$. Since $EA = 0$, we have

$$\begin{aligned} M(4)E^t &= \sum_{k=1}^{r} m_ie_i = \sum_{k=1}^{r}(a_{k1}\bar{D}_{234} + a_{k2}\bar{D}_{134} + a_{k3}\bar{D}_{124})e_k = 0, \\ EM(4)^t &= (M(4)E^t)^t = 0. \end{aligned}$$

(f) $CF + EY = 0$. This follows from part (1)(b) of Lemma 2.11.

(g) $-F^tZ + YS^t = 0$. This follows from part (1)(c) of Lemma 2.11:

$$-F^tZ + YS^t = -(Z^tF + SY)^t = -(ZF + SY)^t = 0.$$

(h) $-F^tC^t + YE^t = 0$. This follows from part (1)(b) of Lemma 2.11:

$$-F^tC^t + YE^t = -(CF + EY)^t = 0.$$

The second identity follows from the fact that $Y^t = -Y$.

(iii) $h_3h_4 = 0$. This follows from the definitions of h_3 and h_4:

$$h_3h_4 = \begin{bmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{bmatrix} h_2^th_1^t = \begin{bmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{bmatrix} (h_1h_2)^t = 0.$$

□

Now we show that if H is of grade 4, then H is Gorenstein by constructing the minimal free resolution $\mathbb{H}$ of R/H.

Theorem 3.3. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. With the above notation, let A and Y be the matrices with entries in $\mathfrak{m}$.*

(1) *The sequence $\mathbb{H}$ of free R-modules and R-maps defined in* (3.6) *is a complex of free R-modules and R-maps.*

(2) *Let $d_i = \bar{D}_{abc}$ where $\{i, a, b, c\} = \{1, 2, 3, 4\}$. Assume that d_i is contained in $\mathfrak{m}$ for $i = 1, 2, 3$ and d_4 is not contained in $\mathfrak{m}$ if r is even, and that every d_i is contained in $\mathfrak{m}$ if r is odd. If $H = I_1(h_1)$ is of grade* 4*, then the complex $\mathbb{H}$ defined in* (3.6) *is exact and hence H is Gorenstein.*

Proof. (1) The proof for this part follows from Lemma 3.2.

(2) The exactness of the complex $\mathbb{H}$ defined in (3.6) follows from the Buchsbaum and Eisenbud acyclicity criterion [5]. The proof of this part is similar to that of Theorem 3.1 [8]. □

4. Structure theorems for two classes of Gorenstein ideals of grade 4

We give structure theorems for two classes of Gorenstein ideals of grade 4 mentioned in the introduction.

As shown by Golod [11], linkage can be used in the set of perfect ideals in a noetherian ring. Hence in Lemma 1.4 [19], Gorenstein local ring can be replaced with a noetherian local ring. If I and J are linked perfect ideals of grade g such that I is Gorenstein, then $\mu(J) \leq g+1$ where $\mu(J)$ is the minimal number of the generators for I. It will sometimes happen that J is a complete intersection. The following lemma determines precisely when this occurs.

Lemma 4.1 ([19])**.** *Let R be a noetherian local ring and I a perfect ideal of grade g. Assume that K is a complete intersection of grade g which is properly contained in I. Then $K : I$ is a complete intersection if and only if I is a complete intersection and $\mu(I/K) = 1$.*

The following corollary gives us a characterization of complete intersections of grade $g+1$ that every complete intersection of grade $g+1$ is expressed as the sum of two complete intersections of grade g geometrically linked by a regular sequence.

Corollary 4.2. *Let R be a noetherian local ring and H a complete intersection of grade $g+1$. Then there exist two complete intersections I and J of grade g such that*

(1) *they are geometrically linked by a regular sequence $\mathbf{z} = z_1, z_2, \ldots, z_g$ and*
(2) *the sum of these two ideals is equal to H.*

Proof. Let $H = (y_1, y_2, \ldots, y_{g+1})$ be a complete intersection of grade $g+1$. Then $I = (y_1, y_2, \ldots, y_g)$ and $J = (y_1, y_2, \ldots, y_{g-1}, y_{g+1})$ are complete intersections of grade g. We set

$$z_1 = y_1, z_2 = y_2, \ldots, z_{g-1} = y_{g-1}, z_g = y_g y_{g+1}.$$

Then $\mathbf{z} = z_1, z_2, \ldots, z_g$ is a regular sequence. Let K be a complete intersection of grade g generated by $z_1, z_2, \ldots, z_g$. Then $\mu(I/K) = \mu(J/K) = 1$. By Lemma 4.1, $K : I$ and $K : J$ are complete intersections of grade g. By Theorem 2.6, we have $I = K : J$ and $J = K : I$. So I and J are linked by a regular sequence $\mathbf{z}$. Since $I \cap J = (\mathbf{z})$, they are geometrically linked. Clearly, $H = I + J$. □

Yong Sung Cho gave a structure theorem for a class of the Gorenstein ideals H of grade 4 expressed as the sum of a Gorenstein ideal of grade 3 and an almost complete intersection of grade 3 geometrically linked by a regular sequence.

Theorem 4.3 ([7]). *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $\widetilde{G}$ be the $n \times n$ alternating submatrix of G defined in* (3.1) *and t the pfaffian of an alternating submatrix of G. If $H = (Pf_{n-1}(\widetilde{G}), t)$ is an ideal of grade* 4 *defined in* (3.2) *or* (3.3) *or* (3,4) *or* (3.5), *then H is a Gorenstein ideal of grade* 4 *such that H is expressed as the sum of a Gorenstein ideal of grade* 3 *and an almost complete intersection of grade* 3 *geometrically linked by a regular sequence.*

(2) *Every Gorenstein ideal of grade* 4 *expressed as the sum of a Gorenstein ideal of grade* 3 *and an almost complete intersection of grade* 3 *geometrically linked by a regular sequence arises in the way of* (1).

Proof. (1) Let $H = (\mathrm{Pf}_{n-1}(\widetilde{G}), t)$ be an ideal of grade 4 for some $n \times n$ alternating submatrix $\widetilde{G}$ of G. Then we have proved in Theorem 3.1 that H is Gorenstein. Let $I = \mathrm{Pf}_{n-1}(\widetilde{G})$ be an ideal generated by the maximal order pfaffians of $\widetilde{G}$. Since H is of grade 4, I is of grade g $(3 \leq g \leq 4)$. It follows from Lemma 2.3 that I is of grade 3. Theorem 2.4 implies that I is Gorenstein. Let $\mathbf{z} = z_1, z_2, z_3$ be a regular sequence in I and $J = (\mathbf{z}) : I$. Since I is a perfect ideal of grade 3, by Theorem 2.6, J is a perfect ideal of grade 3. It is well known from the Bass' result that the type of I is equal to the minimal number of generators for the canonical module $\mathrm{Ext}_R^3(R/I,R)$ and

$$\mathrm{Ext}_R^3(R/I,R) \cong (\mathbf{z}) : I/(\mathbf{z}) \cong J/(\mathbf{z}). \tag{4.1}$$

Since I is of type 1, it follows from Proposition 2.7 and (4.1) that J is an almost complete intersection. Now we want to show that $H = I + J$ and I is geometrically linked to J. Let r be the type of J. We have two cases: r is even or r is odd. Assume that r is even. By Theorem 2.9, $J = (c_1, c_2, c_3, w)$, where c_i is an element defined in (2.2) for $i = 1, 2, 3$ and $w = \mathrm{Pf}(Y)$ is an element defined in (2.6). We have two cases: $r = 2$ or $r > 2$.

Case (a) $r = 2$. In this case, G has the form

$$G = \left[\begin{array}{c|c} \mathbf{0} & F \\ \hline -F^t & Y \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & a_{11} & a_{21} \\ 0 & 0 & 0 & -a_{12} & -a_{22} \\ 0 & 0 & 0 & a_{13} & a_{23} \\ -a_{11} & a_{12} & -a_{13} & 0 & y_{12} \\ -a_{21} & a_{22} & -a_{23} & -y_{12} & 0 \end{bmatrix}, \tag{4.2}$$

where A, F and Y are matrices in Section 2 with entries in $\mathfrak{m}$, respectively. Moreover $t = y_{12}$ and $c_i = \det A_i$ for $i = 1, 2, 3$, where A_i is the 2×2 submatrix of A obtained by deleting the ith column of A. Since r is even, H has the form defined in (3.2) or (3.3). Assume that H has the form defined in (3.2). Then $I = (c_1, c_2, c_3)$. Hence I is of grade less than or equal to 2. This is contrary to the fact that I is of grade 3. Hence we may assume that H has the form defined in (3.3). For $\{i, j, k\} = \{1, 2, 3\}$, we take $\widetilde{G} = G(i, j)$, the alternating submatrix of G obtained by deleting rows and columns i, j of G. Then $H = (y_{12}, a_{jk}, a_{ik}, c_k)$. We note that $I = (y_{12}, a_{jk}, a_{ik})$. Since H is of grade 4, c_k is regular on R/I. Hence c_k is not contained in I. Since J is of type 2, it follows from the Bass' result that $\mu(I/(\mathbf{z})) = 2$. Since $J = (c_1, c_2, c_3, y_{12})$ and c_k is not contained in I, we can choose $\mathbf{z} = y_{12}, c_i, c_j$. Since c_k is not contained in I, $I \cap J = (\mathbf{z})$. Hence I and J are geometrically linked by $\mathbf{z}$. Clearly,

$$H = (y_{12}, a_{jk}, a_{ik}, c_k) = I + J.$$

Case (b) $r > 2$. First we let H be an ideal of grade 4 defined in (3.2). In this case, we take $\widetilde{G} = G$ and $t = G_{123}$. So $H = (G_1, G_2, \ldots, G_{r+3}, G_{123})$. Since H is of grade 4, it follows from Lemma 2.3 and Theorem 2.4 that $I = \mathrm{Pf}_{r+2}(G)$ is a Gorenstein ideal of grade 3. Direct computations show that

$$G_i = c_i \ \text{ for } \ i = 1, 2, 3, \ \ G_i = e_i \ \text{ for } \ i = 4, 5, \ldots, r+3, \ \text{ and } \ t = G_{123} = -w.$$

Hence $I = (c_1, c_2, c_3, e_1, e_2, \ldots, e_r)$, where e_i is an element defined in (2.3). Since J is of type r, by the Bass' result, $\mu(I/(\mathbf{z})) = r$. Since H is of grade 4, $t = -w$ is regular on R/I. Hence w is not contained in I. Since $J = (c_1, c_2, c_3, w)$, we can choose $\mathbf{z} = c_1, c_2, c_3$. Since w is not contained in I, $I \cap J = (\mathbf{z})$. Hence I and J are geometrically linked by a regular sequence $\mathbf{z}$. Clearly,

$$H = (c_1, c_2, c_3, e_1, e_2, \ldots, e_r, w) = I + J.$$

Next we let H be an ideal of grade 4 defined in (3.3). The proof for this part is similar to that of the case mentioned above. In this case we take $\widetilde{G} = G(i,j)$ and $t = G_k$, where $\{i,j,k\} = \{1,2,3\}$. Direct computations show that

$$G_{ijk} = -w, G_{ij4} = \pm s_{k1}, G_{ij5} = \pm s_{k2}, \ldots, G_{ijr+3} = \pm s_{kr}, G_k = c_k.$$

We note that $I = \mathrm{Pf}_{n-1}(G(i,j)) = (w, s_{k1}, s_{k2}, \ldots, s_{kr})$ is Gorenstein. In this case since c_k is regular on R/I, we can choose $\mathbf{z} = c_i, c_j, w$. The proof for the case that r is odd is similar to that of the case (b).

(2) The proof is similar to that of part (2) in Theorem 3.4 [7]. □

The following example demonstrates Theorem 4.3.

Example 4.4. Let $R = \mathbb{C}[[x,y,z,t]]$ be the formal power series over the field $\mathbb{C}$ of complex numbers with indeterminates x, y, z, t. Let A and Y be the 4×3 matrix and the 4×4 alternating matrix given by

$$A = \begin{bmatrix} x & y & t \\ y & x & z \\ z & t & x \\ t & z & y \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & x & 0 & t \\ -x & 0 & y & 0 \\ 0 & -y & 0 & z \\ -t & 0 & -z & 0 \end{bmatrix}.$$

Define G to be a 7×7 alternating matrix given by

$$G = \begin{bmatrix} 0 & 0 & 0 & x & y & z & t \\ 0 & 0 & 0 & -y & -x & -t & -z \\ 0 & 0 & 0 & t & z & x & y \\ -x & y & -t & 0 & x & 0 & t \\ -y & x & -z & -x & 0 & y & 0 \\ -z & t & -x & 0 & -y & 0 & z \\ -t & z & -y & -t & 0 & -z & 0 \end{bmatrix}.$$

Let c_i and e_i be elements defined in (2.2) and (2.3). Then

$$c_1 = -y^3 - x^2z - yz^2 - x^2t + xyt + xzt + yzt + zt^2,$$
$$c_2 = -xy^2 + xyz - xz^2 - x^2t - xyt + yzt + z^2t + yt^2,$$
$$c_3 = -x^2z - xyz + y^2z + xz^2 + y^2t + xzt - xt^2 - yt^2,$$
$$e_1 = -2xyz + z^3 + x^2t + y^2t - zt^2, \;\; e_2 = x^2z + y^2z - 2xyt - z^2t + t^3,$$
$$e_3 = x^2y - y^3 - xz^2 + 2yzt - xt^2, \;\; e_4 = -x^3 + xy^2 - yz^2 + 2xzt - yt^2$$

and $w = \mathrm{Pf}(Y) = -(xz + yt)$. Using CoCoA 4.7.4, we can easily check that $\mathbf{c} = c_1, c_2, c_3$ is a regular sequence. $I = (c_1, c_2, c_3, e_1, e_2, e_3, e_4)$ is a Gorenstein ideal of grade 3 and the minimal free resolution $\mathbb{F}$ of R/I is

$$\mathbb{F} : 0 \longrightarrow R \xrightarrow{f_3} R^7 \xrightarrow{f_2} R^7 \xrightarrow{f_1} R,$$

where

$$f_1 = \begin{bmatrix} c_1 & c_2 & c_3 & e_1 & e_2 & e_3 & e_4 \end{bmatrix}, \;\; f_2 = \begin{bmatrix} \mathbf{0} & F \\ -F^T & Y \end{bmatrix}, \;\; f_3 = f_1^T,$$

and F is a matrix defined in (2.7). Theorem 2.6 and part (a) of Lemma 2.11 say that $J = (\mathbf{c}) : I = (c_1, c_2, c_3, w)$ is an almost complete intersection of grade 3. It follows from the Bass' result that I is of type 1. We have proved in the proof of Theorem 4.3 that I and J are geometrically linked by a regular sequence $\mathbf{c}$. We note that w is regular on R/I. Hence by Theorem 2.8, $H = I + J = (c_1, c_2, c_3, w, e_1, e_2, e_3, e_4)$ is a Gorenstein ideal of grade 4. Let D be a 7×7 diagonal matrix whose main diagonal entries are equal to $-w$ and $\mathbf{f}_2$ the row vector of the maximal order pfaffians of f_2. Then the minimal free resolution $\mathbb{H}$ of R/L is

$$\mathbb{H} = \mathbb{F} \otimes \mathbb{G} : 0 \longrightarrow R \xrightarrow{h_4} R^8 \xrightarrow{h_3} R^{14} \xrightarrow{h_2} R^8 \xrightarrow{h_1} R\ ,$$

where

$$h_1 = \begin{bmatrix} \mathbf{f}_2 & w \end{bmatrix},\ h_2 = \begin{bmatrix} f_2 & D \\ 0 & \mathbf{f}_2 \end{bmatrix},\ h_3 = \begin{bmatrix} \mathbf{f}_2^t & -D \\ 0 & f_2 \end{bmatrix},\ h_4 = \begin{bmatrix} -w \\ \mathbf{f}_2^t \end{bmatrix}$$

and

$$\mathbb{G} : 0 \longrightarrow R \xrightarrow{w} R$$

is a free complex.

Let r be a positive integer with $r > 1$. Let $I = \mathrm{Pf}_{r+2}(T)$ be a Gorenstein ideal of grade 3 for an $(r+3) \times (r+3)$ alternating matrix T which is not a complete intersection, and u a regular on R/I. Under what condition, does the Gorenstein ideal (I, u) of grade 4 has the form in Theorem 4.3?

Corollary 4.5. *Let R be a noetherian local ring. With the above notation, if (I, u) is of grade* 4, *then u is contained in $(\mathbf{x}) : I$ for a regular sequence $\mathbf{x}$ in I if and only if (I, u) has the form in Theorem* 4.3.

Proof. Let $J = (\mathbf{x}) : I$. Since I is a Gorenstein ideal of grade 3 which is not a complete intersection, by Proposition 2.7 and Lemma 4.1, J is an almost complete intersection of grade 3. By Theorem 4.3, it suffices to show that I and J are geometrically linked by a regular sequence $\mathbf{x}$. Since u is contained in $(\mathbf{x}) : I = J$, we have $J = (\mathbf{x}, u)$. Since u is regular on R/I, u is not contained in I. Thus $I \cap J = (\mathbf{x})$ and hence I and J are geometrically linked by a regular sequence $\mathbf{x}$. Clearly, $(I, u) = I + J$. Theorem 4.3 gives us the proof for this part.

To prove the converse, we assume that u is not contained in $(\mathbf{x}) : I$. Let $J = (\mathbf{x}) : I$. Since (I, u) has the form in Theorem 4.3, $(I, u) = I + J$. Since u is not contained in J, u is contained in I. Hence (I, u) is of grade 3. This is contrary to the assumption that (I, u) is of grade 4. □

The following example demonstrates Corollary 4.5.

Example 4.6. Let $\mathbb{C}$ be the field of complex numbers and $R = \mathbb{C}[[x, y, z, t]]$ the formal power series ring over $\mathbb{C}$ with indeterminates x, y, z, and t. Let H_5 be a 5×5 alternating matrix introduced by Buchsbaum and Eisenbud [6, Proposition 6.2]. By Theorem 2.4, $I = \mathrm{Pf}_4(H_5) = (y^2, -xz, xy + z^2, -yz, x^2)$ is a Gorenstein ideal of grade 3. Since t is regular on R/I, $(y^2, -xz, xy+z^2, -yz, x^2, t)$ is a Gorenstein ideal of grade 4. First we show that there is no regular sequence $\mathbf{z} = z_1, z_2, z_3$ in I such that t is contained in $(\mathbf{z}) : I$. If not, then t is contained in $(\mathbf{a}) : I$ for some regular sequence $\mathbf{a} = a_1, a_2, a_3$ in I. Let $J = (\mathbf{a}) : I$. Since I is Gorenstein, by Proposition 2.7, $J = (\mathbf{a}, s)$ is an almost complete intersection of grade 3 for some element s of R. Consider the ideal $K = (\mathbf{a}, t)$. Then K is contained in J. Since R/K is isomorphic to $\mathbb{C}[[x, y, z]]/(\mathbf{a})$ and $\dim \mathbb{C}[[x, y, z]]/(\mathbf{a}) = 0$, $\dim R/K = 0$. Since R is Cohen-Macaulay, we have

$$4 = \dim R = \dim(R/K) + \text{ht } K = 0 + \text{ht } K = 0 + \text{grade } K.$$

Hence K is of grade 4. However since $J = (\mathbf{a}) : I$ and J is a perfect ideal of grade 3, by Theorem 2.6, J is of grade 3. Since K is contained in J, $4 = \operatorname{grade} K \leq \operatorname{grade} J = 3$. This is contrary. Thus t is not contained in J. Hence we can see from the argument mentioned above that for any regular sequence $\mathbf{z}$ in I, $(y^2, -xz, xy+z^2, -yz, x^2, t) \neq I + J$ where I is geometrically linked to J by a regular sequence $\mathbf{z}$ in $I \cap J$.

We give a structure theorem for class (E). Kang and Ko introduced the skew-symmetrizable matrix in [16] to define a complete matrix of grade 4 which plays a key role in describing a structure theorem for complete intersections of grade 4.

Definition 4.7. Let R be a commutative ring with identity. An $n \times n$ matrix X over R is said to be *skew-symmetrizable* if there exist nonzero diagonal matrices $D' = \operatorname{diag}\{u_1, u_2, \ldots, u_n\}$ and $D = \operatorname{diag}\{w_1, w_2, \ldots, w_n\}$ with entries in R such that $D'XD$ is an alternating matrix.

We denote by GA_n the set of all $n \times n$ skew-symmetrizable matrices over R. Let X be an $n \times n$ skew-symmetrizable matrix. We define $\mathcal{A}(X)$ to be the alternating matrix induced by X as follows:

$$\mathcal{A}(X) = \begin{cases} X & \text{if } X \text{ is alternating} \\ D'XD & \text{if } X \text{ is not alternating.} \end{cases}$$

For example, if r is even, then

$$L = \left[\begin{array}{c|c} M(1) & M(2) \\ \hline -F^t & Y \end{array}\right] \tag{4.3}$$

is an $(r+3) \times (r+3)$ skew-symmetrizable submatrix of h_2 in the complex $\mathbb{H}$ defined in (3.6) which becomes an alternating matrix $\mathcal{A}(L)$ by multiplying the first three columns of L by $\bar{D}_{123}$. Hence $\mathcal{A}(L)$ has the following form

$$\mathcal{A}(L) = \left[\begin{array}{c|c} \bar{D}_{123}M(1) & M(2) \\ \hline -M(2)^t & Y \end{array}\right]. \tag{4.4}$$

The maximal order pfaffians of $\mathcal{A}(L)$ are expressed as R-linear combination of $c_1, c_2, c_3,\ w, p_{11}, p_{21}, \dots, p_{r1}$.

Lemma 4.8. *With the above notation,*

$$\mathcal{A}(L)_i = \begin{cases} (-1)^{i+1} d_4 d_i w + d_4^2 c_i & \text{for } i = 1,2,3, \\ (-1)^{i+1} d_4^2 p_{i'1} & \text{for } i = 4,5,\dots,r+3, \end{cases}$$

where

$$\{i,a,b,c\} = \{1,2,3,4\},\ \ d_i = \bar{D}_{abc} \text{ for } i = 1,2,3, \text{ and } i = i' + 3.$$

Proof. Let $\mathcal{A}(L) = (\widetilde{l}_{ij})$. From Lemma 2.2 we have

$$\begin{aligned} \mathcal{A}(L)_1 &= \sum_{i=1}^{r+3} \widetilde{l}_{i2} \mathcal{A}(L)_{i12} = \widetilde{l}_{32} \mathcal{A}(L)_{312} + \sum_{i=4}^{r+3} \widetilde{l}_{i2} \mathcal{A}(L)_{i12} \\ &= \bar{D}_{123} \bar{D}_{234} w + \sum_{i=4}^{r+3} \widetilde{l}_{i2} \mathcal{A}(L)_{i12} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=4}^{r+3} \widetilde{l}_{i2} \mathcal{A}(L)_{i12} &= \sum_{i=4}^{r+3} \widetilde{l}_{i2} \sum_{j=1}^{r+3} \widetilde{l}_{j3} \mathcal{A}(L)_{ji123} = \sum_{4 \le i < j \le r+3} \begin{vmatrix} \widetilde{l}_{i2} & \widetilde{l}_{i3} \\ \widetilde{l}_{j2} & \widetilde{l}_{j3} \end{vmatrix} \mathcal{A}(L)_{ji123} \\ &= \bar{D}_{123}^2 \sum_{1 \le u < v \le r} \begin{vmatrix} a_{u2} & a_{u3} \\ a_{v2} & a_{v3} \end{vmatrix} Y_{uv} = \bar{D}_{123}^2 c_1. \end{aligned}$$

Hence we have

$$\mathcal{A}(L)_1 = \bar{D}_{123} \bar{D}_{234} w + \bar{D}_{123}^2 c_1.$$

Similarly, we have the following for $i = 1, 2, 3$,

$\mathcal{A}(L)_i = (-1)^{i+1} d_4 d_i w + d_4^2 c_i$, where $\{i,a,b,c\} = \{1,2,3,4\}$ and $d_i = \bar{D}_{abc}$.

For $i = 4$, we have

$$\mathcal{A}(L)_4 = -\sum_{i=1}^{r+3} \widetilde{l}_{i3} \mathcal{A}(L)_{i34} = -\widetilde{l}_{13} \mathcal{A}(L)_{134} - \widetilde{l}_{23} \mathcal{A}(L)_{234} - \sum_{i=5}^{r+3} \widetilde{l}_{i3} \mathcal{A}(L)_{i34}.$$

Direct computations by Lemma 2.2 show that

$$\begin{aligned} -\widetilde{l}_{13} \mathcal{A}(L)_{134} &= -\bar{D}_{123} \bar{D}_{134} \sum_{k=1}^{r+3} \widetilde{l}_{k2} \mathcal{A}(L)_{k1234} = -\bar{D}_{123} \bar{D}_{134} \sum_{k=5}^{r+3} \widetilde{l}_{k2} \mathcal{A}(L)_{k1234} \\ &= -\bar{D}_{123}^2 \bar{D}_{134} \sum_{l=1}^{r} a_{l2} Y_{l1} = \bar{D}_{123}^2 \bar{D}_{134} \sum_{l=1}^{r} Y_{1l} a_{l2} = -\bar{D}_{123}^2 \bar{D}_{134} s_{21}, \end{aligned}$$

$$-\widetilde{l}_{23} \mathcal{A}(L)_{234} = \bar{D}_{123} \bar{D}_{234} \sum_{k=1}^{r+3} \widetilde{l}_{k1} \mathcal{A}(L)_{k1234} = \bar{D}_{123} \bar{D}_{234} \sum_{k=5}^{r+3} \widetilde{l}_{k1} \mathcal{A}(L)_{k1234}$$

$$= -\bar{D}_{123}^2\bar{D}_{234}\sum_{l=1}^{r} a_{l1}Y_{l1} = \bar{D}_{123}^2\bar{D}_{234}\sum_{l=1}^{r} Y_{1l}a_{l1} = \bar{D}_{123}^2\bar{D}_{234}s_{11},$$

$$\begin{aligned}
-\sum_{i=5}^{r+3}\widetilde{l}_{i3}\mathcal{A}(L)_{i34} &= \sum_{i=5}^{r+3}\widetilde{l}_{i3}\sum_{j=1}^{r+3}\widetilde{l}_{j2}\mathcal{A}(L)_{j2i34} \\
&= \sum_{i=5}^{r+3}\widetilde{l}_{i3}\left(\widetilde{l}_{12}\mathcal{A}(L)_{12i34} + \sum_{j=2}^{r+3}\widetilde{l}_{j2}\mathcal{A}(L)_{j2i34}\right) \\
&= \sum_{i=5}^{r+3}\widetilde{l}_{i3}\widetilde{l}_{12}\mathcal{A}(L)_{12i34} + \sum_{i=5}^{r+3}\widetilde{l}_{i3}\sum_{j=2}^{r+3}\widetilde{l}_{j2}\mathcal{A}(L)_{j2i34} \\
&= \sum_{i=5}^{r+3}\widetilde{l}_{12}\widetilde{l}_{i3}\mathcal{A}(L)_{i1234} + \sum_{i=5}^{r+3}\sum_{j=5}^{r+3}\widetilde{l}_{j2}\widetilde{l}_{i3}\mathcal{A}(L)_{j2i34} \\
&= -\bar{D}_{123}^2\bar{D}_{124}\sum_{i=1}^{r} Y_{i1}a_{i3} - \bar{D}_{123}^2\sum_{i=5}^{r+3}\sum_{j=5}^{r+3} a_{j2}a_{i3}\mathcal{A}(L)_{j2i34} \\
&= \bar{D}_{123}^2\bar{D}_{124}\sum_{i=1}^{r} Y_{1i}a_{i3} - \bar{D}_{123}^2\sum_{i=5}^{r+3}\sum_{j=5}^{r+3} a_{j2}a_{i3}\mathcal{A}(L)_{234ij} \\
&= \bar{D}_{123}^2\bar{D}_{124}s_{31} - \bar{D}_{123}^2\sum_{i=5}^{r+3}\sum_{j=5}^{r+3} a_{j2}a_{i3}\mathcal{A}(L)_{234ij}.
\end{aligned}$$

Since

$$\mathcal{A}(L)_{234ij} = -\sum_{k=1}^{r+3}\widetilde{l}_{k1}\mathcal{A}(L)_{k1234ij} = -\sum_{k=1}^{r+3}\widetilde{l}_{k1}\mathcal{A}(L)_{1234ijk} = \bar{D}_{123}\sum_{k'=1}^{r} a_{k'1}Y_{1i'j'k'},$$

where $i = i' + 3$ and $j = j' + 3$, we have

$$\begin{aligned}
&-\sum_{i=1}^{r+3}\sum_{j=1}^{r+3} a_{j2}a_{i3}\mathcal{A}(L)_{234ij} = -\bar{D}_{123}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} a_{k1}a_{j2}a_{i3}Y_{1ijk} \\
= &-\bar{D}_{123}\sum_{1\le i<j<k\le r}(a_{k1}a_{j2}a_{i3} - a_{j1}a_{k2}a_{i3} - a_{k1}a_{i2}a_{j3} + a_{i1}a_{k2}a_{j3} + a_{j1}a_{i2}a_{k3} \\
&- a_{i1}a_{j2}a_{k3})Y_{1ijk} = -\bar{D}_{123}\sum_{1\le i<j<k\le r} -D_{ijk}Y_{1ijk} = -\bar{D}_{123}e_1.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{A}(L)_4 &= -\sum_{i=1}^{r+3}\widetilde{l}_{i3}\mathcal{A}(L)_{i34} = -\widetilde{l}_{13}\mathcal{A}(L)_{134} - \widetilde{l}_{23}\mathcal{A}(L)_{234} - \sum_{i=5}^{r+3}\widetilde{l}_{i3}\mathcal{A}(L)_{i34} \\
&= \bar{D}_{123}^2\left(-\bar{D}_{134}s_{21} + \bar{D}_{234}s_{11} + D_{124}s_{31} - \bar{D}_{123}e_1\right)
\end{aligned}$$

$$= -\bar{D}_{123}^2 \left(\bar{D}_{123}e_1 - (\bar{D}_{234}s_{11} - \bar{D}_{134}s_{21} + D_{124}s_{31})\right)$$
$$= -\bar{D}_{123}^2 p_{11}.$$

For $i = 5, 6, \ldots, r+3$, in a similar way, we have the following

$$\mathcal{A}(L)_i = (-1)^{i+1}\bar{D}_{123}^2 p_{i'1} \quad \text{for } i = 4, 5, \ldots, r+3, \text{ and } i = i'+3. \qquad \square$$

Now we define $\overline{\mathrm{Pf}_{r+2}(\mathcal{A}(L))}$ to be the ideal obtained from the alternating matrix $\mathcal{A}(L)$ defined in (4.4) as follows.

Definition 4.9. Let R be a commutative ring with identity. With the above notation, let L be the skew-symmetrizable matrix defined in (4.3). We set

$$L_i = \begin{cases} (\mathcal{A}(L)_i + (-1)^i d_4 d_i w)/d_4^2 & \text{for } i = 1, 2, 3, \\ (-1)^{i+1}\mathcal{A}(L)_i/d_4^2 & \text{for } i = 4, 5, \ldots, r+3. \end{cases}$$

We define $\overline{\mathrm{Pf}_{r+2}(\mathcal{A}(L))}$ to be the ideal generated by $(r+3)$ elements $L_1, L_2, \ldots,$ L_{r+3}.

We know that there exist $n \times n$ skew-symmetrizable matrices characterizing structures of some classes of perfect ideals of grade 3 with types 2 and 3 [9].

Now we are in a good position to describe one of our main theorems, a structure theorem for class (E) of Gorenstein ideals of grade 4.

Theorem 4.10. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *With the above notation, we let d_i be an element defined in Theorem* 3.3. *We assume that $\mathbf{x}$ is a regular sequence in an almost complete intersection of grade* 3 *with even type r and $2 \le \mu(J/(\mathbf{x})) \le 4$. If $H = (\overline{\mathrm{Pf}_{r+2}(\mathcal{A}(L))}, \mathcal{A}(L)_{123})$ is an ideal of grade* 4, *then H is a Gorenstein ideal such that*

- (a) *$H = I + J$ where I is a perfect ideal of grade* 3 *which is not Gorenstein and J is a type r almost complete intersection of grade* 3.
- (b) *I and J are geometrically linked by the regular sequence $\mathbf{x} = x_1, x_2, x_3$ in $I \cap J$.*

(2) *Every Gorenstein ideal of grade* 4 *expressed as the sum of a perfect ideal I of grade* 3 *with type τ and $2 \le \tau \le 4$ and an almost complete intersection J of grade* 3 *with even type geometrically linked by a regular sequence $\mathbf{x} = x_1, x_2, x_3$ in $I \cap J$ arises in the way of* (1).

Proof. (1) Since d_i is contained in $\mathfrak{m}$ for $i = 1, 2, 3$ and H is of grade 4, it follows from Theorem 3.3 that H is Gorenstein. Let J be an almost complete intersection of grade 3 with even type r. Theorem 2.9 says that $J = (c_1, c_2, c_3, w)$. Let $\mathbf{x} = x_1, x_2, x_3$ be a regular sequence in J with $2 \le \mu(J/(\mathbf{x})) \le$ 4 and $I = (\mathbf{x}) : J$. Then it follows from Theorems 2.6 and 2.10 that $I = (x_1, x_2, x_3, p_{11}, p_{21}, \ldots, p_{r1})$ is a perfect ideal of grade 3 with type $\mu(J/(\mathbf{x}))$. Since I is of type $\mu(J/(\mathbf{x}))$ and $\mu(J/(\mathbf{x})) \neq 1$, I is not Gorenstein. Since $I + J = (c_1, c_2, c_3, w, p_{11}, p_{21}, \ldots, p_{r1})$, by Lemma 4.8, $H = I + J$. This prove part (a) of Theorem 4.10. Now we prove part (b) of it. We have three cases:

$\mu(J/(\mathbf{x})) = 2$, $\mu(J/(\mathbf{x})) = 3$ and $\mu(J/(\mathbf{x})) = 4$. We consider only the first case. For other cases, the proofs are similar to that of the first case. Assume that $\mu(J/(\mathbf{x})) = 2$. This implies that only two of the four generators for J are contained in the complete intersection $(\mathbf{x})$. We have two cases: (i) c_i and c_j are contained in $(\mathbf{x})$ or (ii) c_k and w are contained in $(\mathbf{x})$. We prove only the first case. The proof of other case is the same as that of the first case.

Case (i) : c_i and c_j are contained in $(\mathbf{x})$. In this case we may assume that $x_i = c_i$, $x_j = c_j$, $x_k = ac_k + bw$ for some elements a and b of R. Moreover, c_k and w are not contained in $(\mathbf{x})$ for $k \neq i, j$. We want to show that both c_k and w are not contained in I. If both c_k and w are contained in I, then $I = H$. This is contrary since I is of grade 3 and H is of grade 4. Assume that only one of the two is contained in I, say c_k. Then we have $I = (c_1, c_2, c_3, p_{11}, p_{21}, \dots, p_{r1})$. Hence $\mu(I/(\mathbf{x})) = r + 1$. However since J is of type r, the Bass' result says that $\mu(I/(\mathbf{x})) = r$. This is contrary. Thus c_k and w are not contained in I and $I \cap J = (\mathbf{x})$. So I and J are geometrically linked by the regular sequence $\mathbf{x}$.

Case (ii) : c_k and w are contained in $(\mathbf{x})$. The proof of this part is the same as that of the case (i).

(2) Let I and J be a perfect ideal of grade 3 with type τ and $2 \leq \tau \leq 4$ and an almost complete intersection of grade 3 with even type r, respectively. Since I is of type τ and $\tau \geq 2$, I is not Gorenstein. Since I is geometrically linked to J by the regular sequence $\mathbf{x}$, by Theorem 2.10, I is the following form: $I = (x_1, x_2, x_3, p_{11}, p_{21}, \dots, p_{r1})$, where every c_i and every p_{k1} are elements defined in (2.2) and (2.11). Since J is of type r and r is even, by Theorem 2.9, $J = (c_1, c_2, c_3, w)$. Thus

$$H = I + J = (c_1, c_2, c_3, w, p_{11}, p_{21}, \dots, p_{r1}).$$

Let L be the $(r+3) \times (r+3)$ generalized alternating matrix defined in (4.3). It follows from Lemma 4.8 that $H = (\overline{\mathrm{Pf}_{r+2}(\mathcal{A}(L))}, \mathcal{A}(L)_{123})$. □

The following example illustrates Theorem 4.10.

Example 4.11. Let R and I be the ring and ideal defined in Example 2.12, respectively. Let $\mathbf{x} = yc_1, c_2, c_3$ be a regular sequence mentioned in Example 2.12. Then $J = (\mathbf{x}) : I = (c_1, c_2, c_3, w)$ is an almost complete intersection of grade 3 with type 4. Then I and J are geometrically linked by the regular sequence. By Theorem 2.8, $H = I + J = (c_1, c_2, c_3, w, ye_1, ye_2, ye_3, ye_4)$ is a Gorenstein ideal of grade 4. The minimal free resolution $\mathbb{H}$ of R/H is

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{h_4} R^8 \xrightarrow{h_3} R^{14} \xrightarrow{h_2} R^8 \xrightarrow{h_1} R\,,$$

where

$$h_1 = \begin{bmatrix} c_1 & c_2 & c_3 & w & ye_1 & ye_2 & ye_3 & ye_4 \end{bmatrix},$$

$$h_2 = \begin{bmatrix} \mathbf{0} & yF & Z & S \\ \mathbf{0} & \mathbf{0} & C & E \\ -F^t & Y & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad h_3 = \begin{bmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{bmatrix} h_2^t, \quad h_4 = h_1^t$$

and I is a 7×7 identity matrix. We note that the following submatrix of h_2 is a 7×7 skew-symmetrizable matrix

$$\tilde{G} = \left[\begin{array}{c|c} \mathbf{0} & yF \\ \hline -F^t & Y \end{array}\right].$$

By multiplying the first three columns of $\tilde{G}$ by y, $\tilde{G}$ becomes an alternating matrix. Simple computation shows that the generators for H are

$$\begin{aligned} c_1 &= \mathcal{A}(\tilde{G})_1/y^2, c_2 = \mathcal{A}(\tilde{G})_2/y^2, c_3 = \mathcal{A}(\tilde{G})_3/y^2, \\ ye_1 &= \mathcal{A}(\tilde{G})_4/y^2, \dots, ye_4 = \mathcal{A}(\tilde{G})_7/y^2, \\ w &= \mathcal{A}(\tilde{G})_{123}. \end{aligned}$$

We give a structure theorem for class (O). We note that r is odd in this case. To describe a structure theorem for this class, we define an $(r+3) \times (r+3)$ alternating matrix $\tilde{L}$ as follows:

$$T = \tilde{L} = \left[\begin{array}{c|c} \mathbf{0} & A^t \\ \hline -A & Y \end{array}\right], \tag{4.5}$$

where $\mathbf{0}$ is a 3×3 zero matrix.

Lemma 4.12. *With the above notation,*

(1) $T_{ij} = (-1)^{k+1} Z_k$ *for* $\{i,j,k\} = \{1,2,3\}$,
(2) *Pf(T)=w defined in* (2.6),
(3) $T_{l\,t+3} = s_{lt}$ *for* $l = 1,2,3,$ *and for* $t = 1,2,\dots,r$,
(4) $T_{123t} = -Y_{t-3}$ *for* $t = 4,5,\dots,r+3$,
(5) $p_{t1} = d_4 T_{123t+3} + (T_{1\,t+3}d_1 - T_{2\,t+3}d_2 + T_{3\,t+3}d_3)$ *for* $t = 1,2,\dots,r$, *where* d_i *is an element defined in Theorem* 3.3 *for* $i = 1,2,3,4$.

Proof. The first four parts of Lemma 4.12 follow from Lemma 2.2 and part (5) does from (2.12). □

Now we are ready to describe a structure theorem for class (O).

Theorem 4.13. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *With the above notation, we let d_i be an element defined in Theorem* 3.3 *and* $\bar{h}_i = T_{123\,i+3}d_4 + T_{1\,i+3}d_1 - T_{2\,i+3}d_2 + T_{3\,i+3}d_3$ *an element defined in part* (5) *of Lemma* 4.12 *for* $i = 1,2,\dots,r$. *We assume that* $\mathbf{x}$ *is a regular sequence in an almost complete intersection of grade* 3 *with odd type r and* $2 \le \mu(J/(\mathbf{x})) \le 4$. *If* $H = (Z_1, Z_2, Z_3, w, \bar{h}_1, \bar{h}_2, \dots, \bar{h}_r)$ *has an ideal of grade* 4, *then H is a Gorenstein ideal such that*

(a) $H = I + J$ *where I is a perfect ideal of grade* 3 *which is not Gorenstein and J is a type r almost complete intersection of grade* 3.
(b) *I and J are geometrically linked by the regular sequence* $\mathbf{x} = x_1, x_2, x_3$ *in* $I \cap J$.

(2) *Every Gorenstein ideal of grade* 4 *expressed as the sum of a perfect ideal* I *of grade* 3 *with type* τ *and* $2 \le \tau \le 4$ *and an almost complete intersection* J *of grade* 3 *with odd type geometrically linked by a regular sequence* $\mathbf{x} = x_1, x_2, x_3$ *in* $I \cap J$ *arises in the way of* (1).

Proof. (1) Proof of this part is similar to that of part (1) of Theorem 4.10.
(2) The argument mentioned in part (2) of Theorem 4.10 gives us proof of this part. In this case Lemma 4.12 is used. □

The following example illustrates Theorem 4.13.

Example 4.14. Let R be the formal power series ring defined in Example 2.12. Let I be an ideal generated by eight elements

$$\begin{aligned}
&-x^4 + 2x^2y^2 - x^3z - xy^2z - x^2z^2 + xz^3 + x^2yt - xyzt + x^2t^2,\\
&-x^2y^2 + xy^3 + y^4 - 2x^2yz + xy^2z - 2y^2z^2 - x^2yt + 2yz^2t - y^2t^2,\\
&x^2y + y^3 - x^2z - 3xyz + y^2z - z^3 + x^2t + xyt - yzt + zt^2,\\
&-x^3y + xy^3 - xyz^2,\ \ x^2y^2 - xy^2z + xyzt,\ \ -x^3y + xyz^2 - xy^2t,\\
&x^2y^2 - xy^2z + x^2yt,\ \ xy^3 - 2x^2yz.
\end{aligned}$$

Let A and Y be a 5×3 matrix and a 5×5 alternating matrix, respectively, given by

$$A = \begin{bmatrix} x & y & z \\ y & z & t \\ z & t & 0 \\ t & 0 & x \\ 0 & x & y \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} 0 & x & y & z & t \\ -x & 0 & x & y & z \\ -y & -x & 0 & z & y \\ -z & -y & -z & 0 & x \\ -t & -z & -y & -x & 0 \end{bmatrix}.$$

For $i = 1, 2, 3$, and for $j = 1, 2, \ldots, 5$, let Z_i and w be the elements defined in (2.5) and (2.6), respectively. Then

$$\begin{aligned}
w = {} & x^4 - x^2y^2 - xy^3 + x^3z + x^2yz + xz^3 + yz^3 + 2xy^2t - x^2zt - 2xyzt\\
& - 2y^2zt - 2xz^2t - yz^2t + x^2t^2 + 3xyt^2 + y^2t^2 + z^2t^2 - t^4,
\end{aligned}$$

and we can rewrite I in the form

$$I = (xZ_1, yZ_2, Z_3, -xyY_1, -xyY_2, -xyY_3, -xyY_4, -xyY_5).$$

We can check by Algebra System, CoCoA 4.7.4, that $\mathbf{x} = xZ_1, yZ_2, Z_3$ is a regular sequence. Then $J = (\mathbf{x}) : I = (Z_1, Z_2, Z_3, w)$ is an almost complete intersection of grade 3 with type 5. Thus it follows from Theorem 2.10 that I is a perfect ideal of grade 3 with type 3. Since J is an odd type, by (2.9) $J = (Z_1, Z_2, Z_3, w)$. Since $I \cap J = (\mathbf{x})$, by Theorem 2.7, $H = I + J$ is a Gorenstein ideal of grade 4 which has the form

$$H = (Z_1, Z_2, Z_3, w, xyY_1, xyY_2, xyY_3, xyY_4, xyY_5).$$

On the other hand, we can get a 4×3 matrix B such that

$$B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} Z_1 & Z_2 & Z_3 \end{bmatrix} B.$$

Hence we have $\bar{D}_{123} = xy$ and $\bar{D}_{ijk} = 0$ for $\{i, j, k\} \neq \{1, 2, 3\}$. Then the matrices $M(1), M(2), M(3)$ defined for an odd type r in section 3 are given by

$$M(1) = \operatorname{diag}\{-xy, -xy, -xy\}, \;\; M(2) = \mathbf{0}, \;\; M(3) = \mathbf{0},$$

and the entries of a matrix S defined for the case are

$$s_{11} = x^3 + xyz - yz^2 + y^2t + xt^2, \;\; s_{21} = -xy^2 + xz^2 + xyt - xzt + yt^2,$$
$$s_{31} = xz^2 - x^2t - xyt - yzt + zt^2, \ldots, s_{35} = -xz^2 + z^3 + 2xyt - 2yzt + xt^2.$$

The minimal free resolution $\mathbb{H}$ of R/H is

$$\mathbb{H} : 0 \longrightarrow R \xrightarrow{h_4} R^9 \xrightarrow{h_3} R^{16} \xrightarrow{h_2} R^9 \xrightarrow{h_1} R\,,$$

where

$$h_1 = \begin{bmatrix} Z_1 & Z_2 & Z_3 & w & xyY_1 & xyY_2 & xyY_3 & xyY_4 & xyY_5 \end{bmatrix},$$

$$h_2 = \begin{bmatrix} M(1) & \mathbf{0} & Z & S \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{y} \\ A & Y & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad h_3 = \begin{bmatrix} -Z & \mathbf{0} & \mathbf{0} \\ S^t & \mathbf{y}^t & \mathbf{0} \\ M(1) & \mathbf{0} & A^t \\ \bar{A}^t & \mathbf{0} & -Y \end{bmatrix},$$

$$h_4 = h_1^t.$$

We note that the following submatrix of h_2 is a 8×8 alternating matrix

$$T = \left[\begin{array}{c|c} \mathbf{0} & A^t \\ \hline -A & Y \end{array}\right].$$

Simple computations show the identities in Lemma 4.12.

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