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A NOTE ON ALMOST CONTACT RIEMANNIAN 3-MANIFOLDS II

Jun-ichi Inoguchi

Abstract. We classify Kenmotsu 3-manifolds and cosymplectic 3-manifolds with η-parallel Ricci operator.

Introduction

It is well known that semi-symmetric Sasakian manifolds are of constant curvature 1. On the other hand, semi-symmetric Kenmotsu manifolds are of constant curvature -1. These facts mean that semi symmetry is a strong restriction for Sasakian and Kenmotsu manifolds.

In 3-dimensional geometry, local symmetry, *i.e.*, the parallelism of the Riemannian curvature R is equivalent to the parallelism of the Ricci operator S.

Cho and Kimura showed that Kenmotsu 3-manifolds whose Ricci operator is parallel along the characteristic flow are of constant curvature -1 [6].

In this paper we study more mild condition on the Ricci operator. More precisely we study Kenmotsu 3-manifolds and cosymplectic 3-manifolds satisfying the following *η-parallel condition*:

$$g((\nabla_X S)Y, Z) = 0$$

for all vector fields X, Y and Z orthogonal to the structure vector field ξ.

We classify Kenmotsu 3-manifolds satisfying this condition. Moreover we show that there exist Kenmotsu 3-manifolds of non-constant curvature which have η-parallel Ricci operator. In addition we also study cosymplectic 3-manifolds and Sasakian 3-manifolds with η-parallel Ricci operator.

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1. Preliminaries

1.1. Let (M, g) be a Riemannian m-manifold with its Levi-Civita connection ∇. Denote by R the Riemannian curvature of M:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M.

For an endomorphism field F on M, its *divergence* $\mathrm{div}F$ is a vector field defined by

$$\mathrm{div}F = \mathrm{tr}_g(\nabla F) = \sum_{i=1}^{m} (\nabla_{e_i} F)e_i.$$

Here $\{e_i\}_{i=1}^m$ is a local orthonormal frame field of (M, g).

One can see that the differential dr of the scalar curvature r is related to the divergence of the Ricci operator S by ([15]):

$$dr = 2g(\mathrm{div}S, \cdot). \tag{1.1}$$

A Riemannian manifold (M, g) is said to be *locally symmetric* if R is parallel, *i.e.*, $\nabla R = 0$. Clearly every Riemannian manifolds of constant curvature is locally symmetric. More generally (M, g) is said to be *semi-symmetric* if R is semi-parallel, *i.e.*, $R \cdot R = 0$.

1.2. In case $m = \dim M = 3$, the Riemannian curvature R is determined by the Ricci tensor ρ. In fact, R is expressed as

$$\begin{aligned} R(X, Y)Z = {} & \rho(Y, Z)X - \rho(Z, X)Y \\ & + g(Y, Z)SX - g(Z, X)SY - \frac{r}{2}(X \wedge Y)Z, \end{aligned} \tag{1.2}$$

where $(X \wedge Y)Z$ is a curvature-like tensor field defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y, Z \in \mathfrak{X}(M).$$

The formula (1.2) implies that a Riemannian 3-manifold (M, g) is locally symmetric if and only if R is *semi-parallel*, that is, $R \cdot S = 0$. More generally (M, g) is semi-symmetric if and only if S is semi-parallel.

2. Almost contact Riemannian manifolds

2.1. Let M be a $(2n+1)$-dimensional manifold. An *almost contact structure* on M is a quadruple of tensor fields (φ, ξ, η, g), where φ is an endomorphism field, ξ is a vector field, η is a one-form and g is a Riemannian metric, respectively, such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

An $(2n+1)$-dimensional manifold together with an almost contact structure is called an *almost contact Riemannian manifold* (or *almost contact metric manifold*) [2]. The *fundamental* 2-*form* Φ of M is defined by

$$\Phi(X,Y) = g(X,\varphi Y), \quad X,Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M;\varphi,\xi,\eta,g)$ satisfies the condition:

$$\rho = ag + b\eta \otimes \eta \tag{2.3}$$

for some functions a and b, then M is said to be *η-Einstein*.

An almost contact Riemannian manifold M is said to be *normal* if it satisfies $[\varphi,\varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ.

Definition 2.1. An almost contact Riemannian manifold M is said to be an *almost Kenmotsu manifold* if it satisfies $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu manifold is called a *Kenmotsu manifold*.

Definition 2.2. An almost contact Riemannian manifold M is said to be an *almost cosymplectic manifold* if it satisfies $d\eta = 0$ and $d\Phi = 0$. A normal almost cosymplectic manifold is called a *cosymplectic manifold*.

Definition 2.3. An almost contact Riemannian manifold M is said to be a *contact Riemannian manifold* if it satisfies $d\eta = \Phi$. A normal contact Riemannian manifolds is called a *Sasakian manifold*.

A tangent plane Π_p at a point p of an almost contact Riemannian manifold M is said to be *holomorphic* (or φ-section) if it is invariant under φ_p. It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_p. The sectional curvature $K(\Pi_p)$ of a holomorphic plane Π_p is called the *holomorphic sectional curvature* (or φ-sectional curvature) of M.

2.2. For an arbitrary almost contact Riemannian 3-manifold M, we have ([14]):

$$(\nabla_X \varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi. \tag{2.4}$$

Moreover, we have

$$\mathrm{d}\eta = \eta \wedge \nabla_\xi \eta + \alpha\Phi, \quad \mathrm{d}\Phi = 2\beta\eta \wedge \Phi,$$

where α and β are the functions defined by

$$\alpha = \frac{1}{2}\mathrm{tr}_g(\varphi\nabla\xi), \quad \beta = \frac{1}{2}\mathrm{tr}_g(\nabla\xi) = \frac{1}{2}\mathrm{div}\xi. \tag{2.5}$$

Olszak [14] showed that an almost contact Riemannian 3-manifold M is normal if and only if $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$ or, equivalently,

$$\nabla_X\xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M). \tag{2.6}$$

We call the pair (α,β) the *type* of a normal almost contact Riemannian 3-manifold M.

Using (2.4) and (2.6) we note that the covariant derivative $\nabla\varphi$ of φ on a 3-dimensional normal almost contact Riemannian manifold is given by

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X,Y)\xi - \eta(Y)\varphi X). \tag{2.7}$$

Moreover M satisfies (see [3]):

$$2\alpha\beta + \xi(\alpha) = 0.$$

Thus if α is a nonzero constant, then $\beta = 0$. In particular a Kenmotsu 3-manifold is a normal almost contact Riemannian 3-manifold of type $(0,1)$. Cosymplectc 3-manifolds are characterised as almost contact Riemannian 3-manifolds of type $(0,0)$. A Sasakian manifold is a normal almost contact Riemannian manifold of type $(1,0)$.

Next, we consider η-Einstein normal almost contact Riemannian 3-manifolds.

Proposition 2.1. *Let M be a normal almost contact Riemannian 3-manifold of type (α,β). Then M is η-Einstein if and only if*

$$g(\mathrm{grad}\beta - \varphi\mathrm{grad}\alpha, X) = 0$$

for all $X \in \mathfrak{X}(M)$ orthogonal to ξ. In this case, the Ricci operator $S = a\mathrm{I}+b\eta\otimes\xi$ has coefficients:

$$a = \frac{r}{2} + \mathrm{d}\beta(\xi) - (\alpha^2 - \beta^2), \quad b = -\frac{r}{2} - 3\mathrm{d}\beta(\xi) + 3(\alpha^2 - \beta^2).$$

In particular, cosymplectic 3-manifolds, Kenmotsu 3-manifolds and Sasakian 3-manifolds are η-Einstein.

2.3. *Kenmotsu 3-manifolds.* Let $(M;\varphi,\xi,\eta,g)$ be a Kenmotsu 3-manifold. Then we have

$$(\nabla_X\varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X, \tag{2.8}$$

$$\nabla_X\xi = X - \eta(X)\xi \tag{2.9}$$

for all X, $Y \in \mathfrak{X}(M)$.

In particular we have $\nabla_\xi\xi = 0$. Hence on Kenmotsu 3-manifolds, integral curves (trajectories) of ξ are geodesics.

Every Kenmotsu 3-manifold is η-Einstein with Ricci operator

$$S = \frac{1}{2}(r+2)\mathrm{I} - \frac{1}{2}(r+6)\eta\otimes\xi.$$

The scalar curvature r is related to the holomorphic sectional curvature function H by $H = r/2 + 2$.

Corollary 2.1. *The Riemannian curvature of a Kenmotsu 3-manifold is given by*

$$R(X,Y)Z = \frac{r+4}{2}(X\wedge Y)Z + \frac{r+6}{2}[\xi\wedge\{(X\wedge Y)\xi\}]Z.$$

This curvature formula implies that a Kenmotsu 3-manifold M has constant scalar curvature $r = -6$ if and only if it is of constant curvature -1.

More generally we have:

Proposition 2.2 (*cf.* [9])**.** *A Kenmotsu* 3*-manifold M has constant scalar curvature if and only if M is of constant curvature* -1*.*

Proof. The divergence $\text{div}S$ is computed as

$$\text{div}S = \frac{1}{2}\text{grad}r - \frac{1}{2}dr(\xi)\xi - (r+6)\xi.$$

Thus if r is constant, then $r = -6$ and hence M is of constant curvature -1. Conversely if M is of constant curvature -1, then $r = -6$. □

From the divergence formula for S, we have

$$dr(\xi) = 2g(\text{div}S, \xi) = \xi(r) - \xi(r) - (r+6) = -(r+6).$$

Hence we obtain the following result.

Proposition 2.3. *Let M be a Kenmotsu* 3*-manifold. Then M satisfies $dr(\xi) =$ 0 if and only if r is constant* -6*.*

Corollary 2.2. *A Kenmotsu* 3*-manifold satisfies the condition*

$$\varphi^2\{(\nabla_W R)(X,Y)Z\} = 0 \tag{2.10}$$

for all X, Y, Z, $W \in \mathfrak{X}(M)$ orthogonal to ξ if and only if M is of constant curvature -1*.*

Proof. De and Pathak [7, 8] showed that M satisfies (2.10) for all X, Y, Z, $W \in \mathfrak{X}(M)$ orthogonal to ξ if and only if M is of constant scalar curvature. As we have seen above, M is of constant scalar curvature if and only if M is of constant curvature -1. □

Note that all the examples of Kenmotsu 3-manifold exhibited in [7, Examples 5.1, 5.2, 5.3] are of constant curvature -1.

3. η-parallelism

3.1. Kenmotsu [11] showed that locally symmetric Kenmotsu manifolds are of constant curvature -1. Thus for Kenmotsu manifolds, local symmetry is a very strong restriction. Instead of local symmetry, we study η-parallelism for the Ricci operator.

First we recall the notion of η-parallelism in the sense of Kimura and Maeda.

Definition 3.1 (*cf.* [12])**.** An endomorphism field P of an almost contact Riemannian manifold M is said to be *η-parallel* if

$$g((\nabla_X P)Y, Z) = 0$$

for all vector fields X, Y and Z orthogonal to ξ.

On the other hand Kon introduced the notion of η-parallelism as follows:

Definition 3.2 ([13])**.** The Ricci tensor field ρ of an almost contact Riemannian manifold M is said to be *η-parallel* if

$$(\nabla_X \rho)(\varphi Y, \varphi Z) = 0$$

for all vector fields X, Y and Z on M.

Now we apply these η-parallelisms on Kenmotsu 3-manifolds. By definition we have

$$(\nabla_X \rho)(\varphi Y, \varphi Z) = g((\nabla_X S)\varphi Y, \varphi Z)$$

for all $X \in \mathfrak{X}(M)$ and Y and Z orthogonal to ξ. Hence the η-parallelism of the Ricci tensor field ρ on an almost contact Riemannian 3-manifold M in the sense of Kon is equivalent to

$$g((\nabla_X S)Y, Z) = 0$$

for all $X \in \mathfrak{X}(M)$ and Y and Z orthogonal to ξ. Thus the η-parallelism of ρ in the sense of Kon is stronger than that of S in the sense of Kimura-Maeda.

To distinguish these two η-parallelisms, we call the η-parallelism in the sense of Kon by the name, "strong η-parallelism".

4. Kenmotsu 3-manifolds with strongly η-parallel S

4.1. We start our discussions with η-Einstein almost contact Riemannian 3-manifolds with η-parallel Ricci operator.

Express the Ricci operator S of an η-Einstein almost contact Riemannian 3-manifold M as $S = a\mathrm{I} + b\eta \otimes \xi$, then we have

$$(\nabla_X S)Y = da(X)Y + db(X)\eta(Y)\xi + b\{(\nabla_X \eta)Y\}\xi \tag{4.1}$$

for all X, Y, $Z \in \mathfrak{X}(M)$.

4.2. Let us assume that M is a Kenmotsu 3-manifold. Then we have

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all X, $Y \in \mathfrak{X}(M)$. Hence from (4.1),

$$\begin{aligned} g((\nabla_X S)Y, Z) = {} & da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) \\ & + b\{g(X, Y) - \eta(X)\eta(Y)\}\eta(Z) \end{aligned} \tag{4.2}$$

for all X, Y, $Z \in \mathfrak{X}(M)$. Next, since

$$a = \frac{1}{2}(r + 2), \quad b = -\frac{1}{2}(r + 6)$$

on Kenmotsu 3-manifolds, we get

$$g((\nabla_X S)Y, Z) = \frac{1}{2}dr(X)g(Y, Z)$$

for all X, Y and $Z \in \mathfrak{X}(M)$ with $\eta(Y) = \eta(Z) = 0$.

Now we take a local orthonormal frame field $\{e_1, e_2, e_3\}$ of M of the form $e_2 = \varphi e_1$, $\eta(e_1) = 0$ and $e_3 = \xi$.

If we choose $X = Y = Z = e_i$ $(i = 1, 2)$, then we get $dr(e_i) = 0$ for $i = 1, 2$. Thus S is η-parallel if and only if $dr(X) = 0$ for any X orthogonal to ξ.

Proposition 4.1. *A Kenmotsu* 3*-manifold M has η-parallel Ricci operator if and only if its scalar curvature satisfies $dr(X) = 0$ for any tangent vector X orthogonal to ξ.*

Next we assume that M has *strongly* η-parallel Ricci operator, then we have

$$0 = g((\nabla_\xi S)e_i, e_i) = \frac{1}{2}dr(\xi), \quad i = 1, 2.$$

This implies that $r = -6$. Thus we obtain an alternative proof to the following result due to De and Pathak.

Proposition 4.2 ([8])**.** *A Kenmotsu* 3*-manifold M has strongly η-parallel Ricci operator if and only if M is of constant curvature -1.*

Summing up our results, we get:

Theorem 4.1. *Let M be a Kenmotsu* 3*-manifolds. Then the following properties are mutually equivalent*:

- *The scalar curvature r is constant along the trajectories of ξ, i.e., $\xi(r) = 0$.*
- *The scalar r is constant.*
- *The scalar curvature is -6.*
- *The holomorphic sectional curvature function H is constant.*
- *The Ricci operator is strongly η-parallel.*
- *M is locally symmetric.*
- *M is of constant curvature -1.*

Remark 1. Jun, De and Pathak showed that Kenmotsu manifolds of arbitrary odd dimension with strongly η-parallel Ricci operator has constant scalar curvature [10, Theorem 5].

In the next section we classify Kenmotsu 3-manifolds with η-parallel Ricci operator.

5. Kenmotsu 3-manifolds with η-parallel S

5.1. *Warped products.* We start with the standard examples of Kenmotsu 3-manifold.

Let (N, h, J) be an oriented Riemannian 2-manifold together with the compatible orthogonal complex structure J. Take a direct product $M = \mathbb{E}^1(t) \times N$ of real line and N. We denote π and σ the natural projections onto the first and second factors,

$$\pi : M \to \mathbb{E}^1, \quad \sigma : M \to N,$$

respectively. On the direct product M, we equip a Riemannian metric g defined by

$$g = \mathrm{d}t^2 + f(t)^2 \pi^* h.$$

Here f is a positive function on $\mathbb{E}^1(t)$. The resulting Riemannian manifold (M,g) is denoted by $\mathbb{E}^1 \times_f N$ and called the *warped product* with base $\mathbb{E}^1$ and fibre N. The function f is called the *warping function*.

On the warped product $M = \mathbb{E}^1 \times_f N$, we define the vector field ξ by $\xi = \frac{\partial}{\partial t}$. Then the Levi-Civita connection ∇ of M is given by (*cf.* [15]):

$$\begin{aligned}\nabla_{\overline{X}^{\mathrm{v}}}\overline{Y}^{\mathrm{v}} &= (\overline{\nabla}_{\overline{X}}\overline{Y})^{\mathrm{v}} - \frac{1}{f}g(\overline{X}^{\mathrm{v}},\overline{Y}^{\mathrm{v}})f'\xi,\\ \nabla_\xi \overline{X}^{\mathrm{v}} &= \nabla_{\overline{X}^{\mathrm{v}}}\xi = \frac{f'}{f}\overline{X}^{\mathrm{v}},\\ \nabla_\xi\xi &= 0.\end{aligned}$$

Here the superscript v means the vertical lift operation of vector fields from N to M. Define an endomorphim field φ on M by $\varphi X = \{J(\sigma_* X)\}^{\mathrm{v}}$. Then we get

$$\begin{aligned}\nabla_X\xi &= \beta(X - \eta(X)\xi),\\ (\nabla_X\varphi)Y &= \beta\{g(\varphi X, Y) - \eta(Y)\varphi X\}, \quad \beta = f'/f.\end{aligned}$$

Hence $M = \mathbb{E}^1 \times_f N$ is a normal almost contact Riemannian 3-manifold of type $(0,\beta)$. In particular, $\mathbb{E}^1 \times_f N$ is a Kenmotsu manifold if and only if $f(t) = ce^t$ for some positive constant c. Take a local orthonormal frame field $\{\bar{e}_1, \bar{e}_2\}$ of (N,h) such that $\bar{e}_2 = J\bar{e}_1$. Then we obtain a local orthonormal frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = \frac{1}{f}\bar{e}_1^{\mathrm{v}}, \quad e_2 = \frac{1}{f}\bar{e}_2^{\mathrm{v}} = \varphi e_1, \quad e_3 = \xi.$$

Then sectional curvatures of M are given by

$$K(e_1 \wedge e_2) = \frac{1}{f^2}\{\kappa - (f')^2\}, \quad K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f},$$

where κ is the Gaussian curvature of N. The components $\rho_{ij} = \rho(e_i, e_j)$ of Ricci tensor field are given by

$$\rho_{11} = \rho_{22} = \frac{\kappa}{f^2} - \frac{f''}{f} - \left(\frac{f'}{f}\right)^2, \quad \rho_{33} = -\frac{2f''}{f}.$$

Now we assume that M is a Kenmotsu manifold, that is, we choose $f(t) = ce^t$, then we have

$$\rho_{11} = \rho_{22} = \frac{\kappa}{c^2e^{2t}} - 2, \quad \rho_{33} = -2.$$

Thus we have

$$r = \frac{2\kappa}{c^2e^{2t}} - 6.$$

5.2. The local structure of Kenmotsu manifolds is described as follows.

Lemma 5.1 ([11])**.** *A Kenmotsu* 3*-manifold* M *is locally isomorphic to a warped product* $I \times_f N$ *whose base* $I \subset \mathbb{E}^1(t)$ *is an open interval,* N *is a surface and warping function* $f(t) = ce^t$, $c > 0$*. The structure vector field is* $\xi = \partial/\partial t$*.*

Now let M be a Kenmotsu 3-manifold and take a local warped product representation $I \times_{ce^t} N$.

Take a local isothermal coordinates (x, y) on N and represent h as $h = e^{\omega}(dx^2 + dy^2)$. Then

$$\bar{e}_1 = e^{-\omega/2}\frac{\partial}{\partial x}, \quad \bar{e}_2 = e^{-\omega/2}\frac{\partial}{\partial y}.$$

Thus we have that S is η-parallel if and only if $\kappa_x = \kappa_y = 0$, that is, κ is constant. Under the constancy of κ, $dr(\xi) = 0$ holds if and only if $\kappa = 0$. In this case M is of constant curvature -1.

Theorem 5.1. *A Kenmotsu* 3*-manifold has* η*-parallel Ricci operator if and only if it is locally isomorphic to the warped product* $\mathbb{E}^1 \times_{ce^t} N$*, where* N *is of constant curvature.*

Thus the global warped products

$$\mathbb{E}^1 \times_{ce^t} \mathbb{S}^2(\kappa), \quad \mathbb{E}^1 \times_{ce^t} \mathbb{H}^2(\kappa)$$

are Kenmotsu 3-manifolds whose Ricci operator is η-parallel but not strongly η-parallel.

Remark 2. In [6], Cho and Kimura showed that a Kenmotsu 3-manifold M satisfies $\pounds_{\xi} S = 0$ if and only if M is of constant curvature -1. They also showed that Kenmotsu 3-manifolds whose Ricci operator is parallel along the characteristic flow (*i.e.,* $\nabla_{\xi} S = 0$) are of constant curvature -1. Recently Cho classified locally symmetric almost Kenmotsu 3-manifolds [5].

Problem 5.1.
(1) Classify almost Kenmotsu 3-manifolds with η-parallel Ricci operator.
(2) Classify Kenmotsu 3-manifolds with *semi* η*-parallel* Ricci operator, *i.e.*,

$$g((R(X, Y)S)Z, W) = 0$$

for all vector fields X, Y, Z and W orthogonal to ξ.

6. Cosymplectic 3-manifolds

In this section we study cosymplectic 3-manifolds with η-parallel Ricci operator. On a cosymplectic 3-manifold M, we have

$$\nabla \varphi = 0, \quad \nabla \xi = 0.$$

In particular we have $\nabla_{\xi} \xi = 0$. Hence on cosymplectic 3-manifolds, integral curves (trajectories) of ξ are geodesics.

Example 6.1. Let (N, h, J) be an oriented Riemannian 2-manifold with the compatible complex structure J. On the direct product manifold $M = N \times \mathbb{E}^1$ of N with the real line $\mathbb{E}^1(t)$, we equip the product metric $g = \pi^* h + dt^2$. Here $\pi : M \to N$ is the natural projection. Define the endomorphism field φ on M by

$$\varphi X = \{J\pi_* X\}^{\mathtt{h}},$$

where $\mathtt{h}$ is the horizontal lift operation. Define the vector field ξ and the 1-form η by $\xi = \partial/\partial t$ and $\eta = dt$. Then the resulting almost contact Riemannain 3-manifold $(M, \varphi, \xi, \eta, g)$ is cosymplectic.

The local structure of cosymplectic 3-manifolds is described as follows.

Lemma 6.1 ([4, Lemma 2])**.** *A cosymplectic 3-manifold M is locally isomorphic to the Riemannian product $N \times I$ whose base $N = (N, h)$ is a Riemannin 2-mnanifold. The standard fibre I is an open interval with coordinate t. The metric is $g = \pi^* h + dt^2$, where $\pi : N \times I \to N$ is the natural projection. The structure vector field is $\xi = \partial/\partial t$.*

Every cosymplectic 3-manifold is η-Einstein with Ricci operator

$$S = \frac{r}{2}\mathrm{I} - \frac{r}{2}\eta \otimes \xi.$$

The holomorphic sectional curvature function H is given by $H = r/2$.

Corollary 6.1. *The Riemannian curvature of a cosymplectic 3-manifold is given by*

$$R(X,Y)Z = \frac{r}{2}[\xi \wedge \{(X \wedge Y)\xi\}]Z.$$

Using this formula, the covariant derivative of S is computed as

$$(\nabla_X S)Y = \frac{1}{2} dr(X)(Y - \eta(Y)\xi). \tag{6.1}$$

The divergence div$S$ is computed as

$$\mathrm{div}S = \frac{1}{2}(\mathrm{grad}r - \eta(\mathrm{grad}r)\xi).$$

This implies the formula

$$dr(X) = g(\mathrm{grad}r, X) - \eta(\mathrm{grad}r)\eta(X).$$

Equivalently,

$$dr = dr - \eta(\mathrm{grad}r)\eta$$

From this formula we have $\xi(r) = 0$. This implies that div$S = \mathrm{grad}r/2$ and $\nabla_\xi S = 0$.

Now let us consider cosymplectic 3-manifolds with η-parallel Ricci operator.

If we assume that $\eta(Y) = \eta(Z) = 0$, in (6.1), we obtain

$$g((\nabla_X S)Y, Z) = \frac{1}{2} dr(X) g(Y, Z) = 0$$

for all $X \in \mathfrak{X}(M)$.

Now we take a local orthonormal frame field $\{e_1, e_2, e_3\}$ of the form $e_2 = \varphi e_1$, $\eta(e_1) = 0$ and $e_3 = \xi$. If we choose $X = Y = Z = e_i$ in (6.1) for $i = 1,2$, then we get $dr(e_i) = 0$ for $i = 1, 2$. Thus S is η-parallel if and only if $dr(X) = 0$ for any vector field X orthogonal to ξ. Since we know that $dr(\xi) = 0$, S is η-parallel if and only if S is strongly η-parallel.

Proposition 6.1. *Let M be a cosymplectic 3-manifold with Ricci operator S. Then S is η-parallel if and only if r is constant. In such a case S is strongly η-parallel.*

Since the holomorphic sectional curvature function H is related to r by $H = r/2$, the η-parallelism of S is equivalent to the constancy of H.

Corollary 6.2. *Let M be a cosymplectic 3-manifold with Ricci operator S. Then S is η-parallel if and only if M has constant holomorphic sectional curvature.*

Let (N, h) be an oriented Riemannian 2-manifold and $M = N \times \mathbb{E}^1$ the direct product with product metric. We equip the natural cosymplectic structure on M. Then the scalar curvature r of M is $r = 2\kappa^{\mathbf{h}}$. Here κ is the Gaussian curvature. Hence S is η-parallel if and only if κ is constant.

Thus $\mathbb{S}^2(\kappa) \times \mathbb{E}^1$ and $\mathbb{H}^2(\kappa) \times \mathbb{E}^1$ are non-constant curvature cosymplectic manifolds with η-parallel Ricci operator.

Theorem 6.1. *Let M be a cosymplectic 3-manifold. Then the following properties are mutually equivalent*:

- *The scalar curvature is constant.*
- *The holomorphic sectional curvature function H is constant.*
- *The Ricci operator is η-parallel.*
- *The Ricci operator is strongly η-parallel.*
- *M is locally symmetric.*

7. Sasakian 3-manifolds

For a Sasakian 3-manifold M, we have

$$S = a\mathrm{I} + b\eta \otimes \xi, \quad a = \frac{1}{2}(r-2),\ b = \frac{1}{2}(6-r).$$

The holomorphic sectional curvature is $H = r/2 - 2$. Hence we get

$$\mathrm{div}S = \frac{1}{2}(\mathrm{grad}r - dr(\xi)\xi).$$

This formula implies $dr = dr - dr(\xi)\eta$. Hence we have $dr(\xi) = 0$.

Moreover we have

$$g((\nabla_X S)Y, Z) = \frac{1}{2}dr(X)g(Y, Z)$$

for all $X \in \mathfrak{X}(M)$ and Y, Z orthogonal to ξ. The Ricci operator S on a Sasakian 3-manifold M is η-parallel if and only if $dr(X) = 0$ for all X orthogonal to ξ. Since $\xi(r) = 0$ holds on every Sasakian 3-manifold, we have the following result.

Proposition 7.1. *The following properties are mutually equivalent for Sasakian* 3*-manifolds.*

- *The scalar curvature r is constant.*
- *The holomorphic sectional curvature H is constant.*
- *The Ricci operator is η-parallel.*
- *The Ricci operator is strongly η-parallel.*

Thus 3-dimensional Sasakian space forms are examples of Sasakian 3-manifolds with *strongly* η-parallel Ricci operator. As is well known Sasakian 3-manifold is locally symmetric if and only if it is of constant curvature 1.

Among the three classes (cosymplectic, Kenmotsu, Sasakian), only for the class of Kenmotsu 3-manifolds, η-parallelism of S is weaker than the strong η-parallelism of S.

Remark 3. Sasakian manifolds $\mathrm{Nil}_3 = \mathbb{R}^3(-3)$ and $\widetilde{\mathrm{SL}}_2\mathbb{R}$, cosymplectic 3-manifolds $\mathbb{S}^2(\kappa)\times\mathbb{E}^1$ and $\mathbb{H}^2(\kappa)\times\mathbb{E}^1$ are model spaces of *Thurston geometries*. Moreover these space are included in the 2-parameter family of homogeneous Riemannian spaces referred as to the *Bianchi-Cartan-Vranceanu family*, see [1].

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Jun-ichi Inoguchi
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan
E-mail address: inoguchi@math.tsukuba.ac.jp