Bull. Korean Math. Soc. **54** (2017), No. 1, pp. 51–69
https://doi.org/10.4134/BKMS.b150623
pISSN: 1015-8634 / eISSN: 2234-3016

ON COMMUTATIVITY OF SKEW POLYNOMIALS AT ZERO

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Abstract. We, in this paper, study the commutativity of skew polynomials at zero as a generalization of an α-rigid ring, introducing the concept of strongly skew reversibility. A ring R is be said to be *strongly α-skew reversible* if the skew polynomial ring $R[x;\alpha]$ is reversible. We examine some characterizations and extensions of strongly α-skew reversible rings in relation with several ring theoretic properties which have roles in ring theory.

1. Introduction

Throughout this paper, all rings are associative with identity, unless specified otherwise. We denote by $R[x]$ the polynomial ring with an indeterminate x over a ring R. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively.

Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Following Cohn [4], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A ring is usually called *abelian* if every idempotent is central. Commutative rings are clearly reversible. A simple computation gives that reduced rings are reversible, and reversible rings are abelian, but the converses do not hold in general.

Note that a ring R is reduced if and only if $R[x]$ is a reduced ring obviously, and a ring R is abelian if and only if $R[x]$ is abelian by [14, Lemma 8]. But polynomial rings over reversible rings need not be reversible by [15, Example 2.1]. However, the property of such rings with the Armendariz condition can be extended to their polynomial rings. Rege and Chhawchharia [20] called a ring R *Armendariz* if whenever any polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0+b_1x+\cdots+b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. This nomenclature was used by them since it was Armendariz [1, Lemma 1] who initially showed that a reduced ring always satisfies this condition. By [15, Proposition 2.4], if R is an Armendariz ring, then R is reversible if and only if $R[x]$ is reversible. Based on this result, Yang and Liu [22] called a

Received August 4, 2015.
2010 *Mathematics Subject Classification.* Primary 16W20, 16U80; Secondary 16S36.
Key words and phrases. strongly α-skew reversible ring, reversible ring, α-rigid ring, skew polynomial ring, Dorroh extension.

ring R *strongly reversible* if $R[x]$ is reversible, and studied its properties and extensions.

As generalizations of the concept of strongly reversible rings, Singh et al. [21] and Peng et al. [19] studied the reversibility in monoid rings and skew monoid rings, respectively.

For an endomorphism α of a ring R, a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) $R[x;\alpha]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r)x$ for all $r \in R$.

According to Krempa [16], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and R is called an α*-rigid* ring [8] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced rings by [8, Proposition 5], and $R[x;\alpha]$ is a reduced ring if and only if R is α-rigid by [9, Proposition 3]. In [2, Definition 2.1], an endomorphism α of a ring R is called *right* (resp., *left*) *skew reversible* if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called *right* (resp., *left*) α*-skew reversible* if there exists a right (resp., left) skew reversible endomorphism α of R, and R is α*-skew reversible* if it is both right and left α-skew reversible. Note that R is a α-rigid ring if and only if R is a semiprime right α-skew reversible ring and α is a monomorphism of R by [2, Proposition 2.5]. (We change over from "α-reversible" in [2] to "α-skew reversible", so as to cohere with other related definitions.)

Motivated by the above, in this paper, we extend the reversibility of polynomials to one of skew polynomials. We introduce the concept of a *strongly* α*-skew reversible* ring for an endomorphism α of given a ring which strictly contains the class of α-skew reversible as well as is a generalization of α-rigid rings and an extension of strongly reversible rings, and then study the structure of strongly α-skew reversible rings and their related properties. The relationship between strongly α-skew reversible rings and generalized reduced rings is also investigated. Consequently, several known results are obtained as corollaries of our results.

2. Basic properties of strongly α-skew reversible rings

We begin our study by examining an example which illuminates the need to investigate the reversibility of skew polynomial rings.

Example 2.1. Consider the reduced ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $\alpha : R \to R$ be an automorphism defined by $\alpha((a, b)) = (b, a)$. For $p(x) = (1, 0)$ and $q(x) = (0, 1)x$ in $R[x;\alpha]$, $p(x)q(x) = 0$ but $q(x)p(x) = (0, 1)x(1, 0) = (0, 1)x \neq 0$. Thus $R[x;\alpha]$ is not reversible (and hence not reduced).

We introduce the following definition from Example 2.1.

Definition 2.2. A ring R is called *strongly α-skew reversible* if $R[x;\alpha]$ is reversible.

Any α-rigid rings (i.e., $R[x;\alpha]$ is reduced) are clearly strongly α-skew reversible. However, there exist many strongly α-skew reversible rings which are not α-rigid as in Example 2.7 to follow. But any domain R with a monomorphism α is strongly α-skew reversible since R is α-rigid. Note that every subring S with $\alpha(S) \subseteq S$ of a strongly α-skew reversible ring is also strongly α-skew reversible. We will freely use these facts without reference.

Note that $\alpha(1) = 1$ for any skew polynomial ring $R[x;\alpha]$, since $1x = x1 = \alpha(1)x$ where 1 is the identity of R. Thus we always have $\alpha(1) = 1$, whenever we assume that R is a strongly α-skew reversible ring.

Lemma 2.3. *Let R be a strongly α-skew reversible ring. Then we have the following results.*

(1) *R is reversible.*

(2) *α is a monomorphism of R.*

(3) *For any $a, b \in R$ and nonnegative integer m and n,*

$$a\alpha^m(b) = 0 \Leftrightarrow ab = 0 \Leftrightarrow ba = 0 \Leftrightarrow b\alpha^n(a) = 0 \Leftrightarrow \alpha^m(b)\alpha^n(a) = 0$$
$$\Leftrightarrow \alpha^n(a)\alpha^m(b) = 0.$$

(4) *$\alpha(e) = e$ for any $e^2 = e \in R$.*

(5) *$p(x)q(x) = 0$ implies $p(x)R[x;\alpha]q(x) = 0$ and $q(x)R[x;\alpha]p(x) = 0$ for each $p(x), q(x) \in R[x;\alpha]$.*

Proof. (1) This is clear by the definition of a strongly α-skew reversible ring.

(2) Suppose that $\alpha(a) = 0$ for $a \in R$. Then $p(x)q(x) = 0$ for $p(x) = x$, $q(x) = a \in R[x;\alpha]$, and so $q(x)p(x) = ax = 0$ and therefore $a = 0$. This entails that α is a monomorphism.

(3) Let m and n be any nonnegative integers. Then

$$a\alpha^m(b) = 0 \Leftrightarrow ax^m b = 0 \Leftrightarrow bax^m = 0 \Leftrightarrow ba = 0 \Leftrightarrow ab = 0 \Leftrightarrow abx^n = 0$$
$$\Leftrightarrow bx^n a = 0 \Leftrightarrow b\alpha^n(a) = 0.$$

These immediately yield

$$ab = 0 \Leftrightarrow \alpha^m(b)\alpha^n(a) = 0 \Leftrightarrow \alpha^n(a)\alpha^m(b) = 0.$$

(4) Note that $\alpha(1) = 1$ as noted earlier. Let $e^2 = e \in R$. Then $e(1-e) = 0$ implies $0 = e\alpha(1-e) = e(\alpha(1) - \alpha(e)) = e - e\alpha(e)$ by (3), and hence $e\alpha(e) = e$. Similarly, $(1-e)e = 0$ implies $0 = (1-e)\alpha(e)$ and so $\alpha(e) = e\alpha(e)$ by (2). Consequently, $\alpha(e) = e$.

(5) This follows from the definition of a strongly α-skew reversible ring. □

The converse of Lemma 2.3(1) does not hold by Example 2.1.

There exists another generalization of α-rigid rings. For an endomorphism α of a ring R, the ring R is called an *α-compatible* ring [7] if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. In this case, clearly the endomorphism α is a monomorphism.

Corollary 2.4. (1) *If R is a strongly α-skew reversible ring, then R is α-compatible.*

(2) [2, Theorem 2.9(ii)] *If R is a strongly α-skew reversible ring, then R is α-skew reversible.*

Proof. These come directly from Lemma 2.3(3). □

The next example shows that the converses of Corollary 2.4 does not hold in general.

Example 2.5. (1) We refer to the argument and ring construction in [13, Example 2.8]. Consider the free algebra $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Define an automorphism δ of A by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively. Let B be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$\begin{aligned}
&a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,\\
&b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,\\
&(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2),\\
&a_0a_0, a_2a_2, a_0ra_0, a_2ra_2, b_0b_0, b_2b_2, b_0rb_0, b_2rb_2, r_1r_2r_3r_4,\\
&a_0a_1 + a_1a_0, a_0a_2 + a_1a_1 + a_2a_0, a_1a_2 + a_2a_1, b_0b_1 + b_1b_0,\\
&b_0b_2 + b_1b_1 + b_2b_0, b_1b_2 + b_2b_1,\\
&(a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2),
\end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \subseteq I$. Set $R = A/I$. Since $\alpha(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. Note that $\alpha^2 = 1_R$, where 1_R denotes the identity endomorphism of R. We identify every element of A with its image in R for simplicity.

We show that R is α-compatible. Notice that a monomial usually means a product of the indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$, and a monomial of degree n means a product of exactly n number of indeterminates. Let H_n be the set of all linear combinations of monomials of degree n over $\mathbb{Z}_2$. Notice that H_n is finite for any n and that the ideal I of R is homogeneous (i.e., if $\sum_{i=1}^{s} r_i \in I$ with $r_i \in H_i$, then every r_i is in I).

Claim 1. If $f_1\alpha(g_1) \in I$ for $f_1, g_1 \in H_1$, then $f_1g_1 \in I$.

Proof. By definition of I, we have only the following cases when $f_1\alpha(g_1) \in I$ for $f_1, g_1 \in H_1$:

$$\begin{aligned}
&(f_1 = a_0, \alpha(g_1) = b_0), (f_1 = a_0, \alpha(g_1) = a_0);\\
&(f_1 = b_0, \alpha(g_1) = a_0), (f_1 = b_0, \alpha(g_1) = b_0);\\
&(f_1 = a_2, \alpha(g_1) = b_2), (f_1 = a_2, \alpha(g_1) = a_2);
\end{aligned}$$

$$(f_1 = b_2, \alpha(g_1) = a_2), (f_1 = b_2, \alpha(g_1) = b_2);$$

$$(f_1 = a_0+a_1+a_2, \alpha(g_1) = b_0+b_1+b_2), (f_1 = a_0+a_1+a_2, \alpha(g_1) = a_0+a_1+a_2);$$

$$(f_1 = b_0+b_1+b_2, \alpha(g_1) = a_0+a_1+a_2), (f_1 = b_0+b_1+b_2, \alpha(g_1) = b_0+b_1+b_2).$$

Thus $f_1g_1 \in I$ by the construction of I and the definition of α, since $\alpha(b_i+I) = a_i + I$ for each i. □

Claim 2. If $f\alpha(g) \in I$ for $f, g \in B$, then $fg \in I$.

Proof. Let $f = f_1 + f_2 + f_3 + f_4$ and $g = g_1 + g_2 + g_3 + g_4$, where $f_1, g_1 \in H_1$, $f_2, g_2 \in H_2$, $f_3, g_3 \in H_3$, and $f_4, g_4 \in I$. Note that $H_i \subseteq I$ for $i \geq 4$. Let $\alpha(g) = k$ where $\alpha(g_i) = k_i$ for $i = 1, 2, 3, 4$. So $f\alpha(g) = fk = f_1k_1+f_1k_2+f_2k_1+h \in I$, with $h \in I$, implies $f_1k_1 \in I$ and $f_1k_2 + f_2k_1 \in I$ since I is homogeneous. By Claim 1, we have $f_1g_1 \in I$. We will show $f_1g_2 + f_2g_1 \in I$. From $f_1k_1 \in I$ and $f_1k_2 + f_2k_1 \in I$, we have the following cases:

(i) $f_1 = a_0, k_1 = b_0$;
(i)′ $f_1 = a_0, k_1 = a_0$;
(ii) $f_1 = b_0, k_1 = a_0$;
(ii)′ $f_1 = b_0, k_1 = b_0$;
(iii) $f_1 = a_2, k_1 = b_2$;
(iii)′ $f_1 = a_2, k_1 = a_2$;
(iv) $f_1 = b_2, k_1 = a_2$;
(iv)′ $f_1 = b_2, k_1 = b_2$;
(v) $f_1 = a_0 + a_1 + a_2, k_1 = b_0 + b_1 + b_2$;
(v)′ $f_1 = a_0 + a_1 + a_2, k_1 = a_0 + a_1 + a_2$;
(vi) $f_1 = b_0 + b_1 + b_2, k_1 = a_0 + a_1 + a_2$;
(vi)′ $f_1 = b_0 + b_1 + b_2, k_1 = b_0 + b_1 + b_2$.

If f_2 and k_2 are in I, then $g_2 \in I$ since $g_2 = \alpha(k_2) \in \alpha(I) \subseteq I$, and so $f_1g_2 + f_2g_1 \in I$. Hence, we consider the other cases of f_2 and k_2.

When we get the case (i), we can obtain the following cases:

$$(f_2 \in I, k_2 = b_0t), (f_2 \in I, k_2 = tb_0), (f_2 \in I, k_2 = a_0t), (f_2 \in I, k_2 = ta_0);$$

$$(f_2 = a_0s, k_2 \in I), (f_2 = sa_0, k_2 \in I), (f_2 = sb_0, k_2 \in I), (f_2 = b_0s, k_2 \in I);$$

$$(f_2=a_0s, k_2=b_0t), (f_2=a_0s, k_2=tb_0), (f_2=a_0s, k_2=a_0t), (f_2=a_0s, k_2=ta_0);$$

$$(f_2=sa_0, k_2=b_0t), (f_2=sa_0, k_2=tb_0), (f_2=sa_0, k_2=a_0t), (f_2=sa_0, k_2=ta_0);$$

$$(f_2=sb_0, k_2=b_0t), (f_2=sb_0, k_2=tb_0), (f_2=sb_0, k_2=a_0t), (f_2=sb_0, k_2=ta_0);$$

$$(f_2=b_0s, k_2=b_0t), (f_2=b_0s, k_2=tb_0), (f_2=b_0s, k_2=a_0t), (f_2=b_0s, k_2=ta_0),$$

where $s, t \in H_1$. Then $f_1g_2 + f_2g_1 = a_0g_2 + f_2b_0 \in I$ because $k_1 = b_0$, $\alpha(a_i + I) = b_i + I$, $\alpha(b_j + I) = a_j + I$ for each i, j and $\alpha(c + I) = c + I$.

The computations for (iii) and (v) are similar, and the ones for the rest are obtained by symmetry. Consequently, $f_1g_2 + f_2g_1 \in I$ and thus $fg \in I$. □

Now assume that $y\alpha(z) \in I$, where $y = u + y'$ and $z = v + z' \in A$ for some $u, v \in \mathbb{Z}_2$ and some $y', z' \in B$ to see that $yz \in I$. Then $uv + u\alpha(z') +$

$y'v + y'\alpha(z') = y\alpha(z) \in I$ implies that $u = 0$ or $v = 0$. Let $u = 0$. Then $y'v + y'\alpha(z') \in I$. Here $y'v, y'\alpha(z') \in I$ since I is homogeneous. This yields $y'z' \in I$ by Claim 2, entailing that $yz = y'v + y'z' \in I$. The computation for the case of $v = 0$ is similar.

Therefore it proved that $y\alpha(z) \in I$ implies $yz \in I$. In conclusion, R is α-compatible.

Note that R is not strongly α-skew reversible. For $a_0 + a_1x^2 + a_2x^4$, $b_0 + b_1x^2 + b_2x^4 \in R[x;\sigma]$, we have $(a_0 + a_1x^2 + a_2x^4)(b_0c + b_1cx^2 + b_2cx^4) = 0$, but $(b_0c + b_1cx^2 + b_2cx^4)(a_0 + a_1x^2 + a_2x^4) \neq 0$ since $b_0ca_1 + b_1ca_0 \neq 0$ by the construction of I, showing that R is not strongly α-skew reversible.

(2) Consider the polynomial ring $R = \mathbb{Z}[t]$ over $\mathbb{Z}$. Define $\alpha : R \to R$ by $\alpha(f(t)) = f(0)$ where $f(t) \in R$. Then R is a domain and so α-skew reversible clearly. But α is not a monomorphism, and hence R is not strongly α-skew reversible by Lemma 2.3(2).

Remark 2.6. (1) The ring R of Example 2.5(1) is also reversible by similar computation to above. Thus the converses of Lemma 2.3(1,2) need not hold, respectively.

(2) Consider the non strongly α-skew reversible R in Example 2.5(2). Then R obviously satisfies $\alpha(e) = e$ for any $e^2 = e \in R$, showing that the converse of Lemma 2.3(4) does not hold.

The Armendariz property with respect to polynomials was extended to one of skew polynomials. A ring R is called *α-skew Armendariz* [9, Definition] if for $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$, $p(x)q(x) = 0$ implies $a_i\alpha^i(b_j) = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. Any α-rigid ring is α-skew Armendariz, but the concepts of α-skew Armendariz rings and strongly α-skew reversible rings are independent of each other, by help of [11, Example 14] and [20, Example 3.2].

The following example illuminates that there exists a non α-rigid ring, even if it is a reversible α-compatible and α-skew Armendariz ring.

Example 2.7. We apply [9, Example 1]. Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z},\ b \in \mathbb{Q} \right\},$$

where $\mathbb{Q}$ is the set of all rational numbers. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & b/2 \\ 0 & a \end{smallmatrix}\right)$. Then R is commutative (and hence reversible) α-skew Armendariz, but not α-rigid by [9, Example 1]. Moreover, it can be easily checked that R is α-compatible.

However, we have next proposition which extends the result of [15, Proposition 2.4].

Theorem 2.8. *Let R be an α-skew Armendariz ring. Then R is reversible and α-compatible if and only if R is strongly α-skew reversible.*

Proof. It suffices to show the necessity by Lemma 2.3(1) and Corollary 2.4(1). Suppose that R is reversible and α-compatible. Let $p(x)q(x) = 0$ for $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x; \alpha]$. Since R is α-skew Armendariz, $a_i\alpha^i(b_j) = 0$ and so we have

$$a_ib_j = 0 \Rightarrow b_ja_i = 0 \Rightarrow b_j\alpha^j(a_i) = 0$$

for all i and j by hypothesis. Therefore $q(x)p(x) = 0$ and hence R is strongly α-skew reversible. □

Corollary 2.9. *If R is an Armendariz ring, then R is reversible if and only if $R[x]$ is.*

The following example shows that each condition "R is an α-skew Armendariz ring" and "R is an α-compatible ring" in Theorem 2.8 cannot be dropped.

Example 2.10. (1) Consider the ring R in Example 2.5(1) (and Remark 2.6(1)). Then R is a reversible and α-compatible ring but not strongly α-skew reversible. However R is not α-skew Armendariz. In fact, for $a_0 + a_1x^2 + a_2x^4, b_0 + b_1x^2 + b_2x^4 \in R[x; \alpha]$ we have $(a_0 + a_1x^2 + a_2x^4)(b_0 + b_1x^2 + b_2x^4) = 0$ but $a_0b_1 \neq 0$ by the construction of I, entailing that R is not α-skew Armendariz.

(2) Consider the non strongly α-skew reversible ring $R = \mathbb{Z}[t]$, in Example 2.5(2). Then R is clearly reversible. Moreover R is α-skew Armendariz by [9, Example 5]. But R is not α-compatible, since α is not a monomorphism.

As a parallel example to Example 2.10(2), there exists an α-compatible ring which is neither reversible nor α-skew reversible as follows.

Example 2.11. We refer to [10, Example 1]. Let R be the semigroup ring without identity of S over $\mathbb{Z}_2$, where $S = \{a, b\}$ is the semigroup with multiplication $a^2 = ab = a, b^2 = ba = b$, i.e., $R = \mathbb{Z}_2S = \{0, a, b, a + b\}$. Then we have only the following cases when $uv = 0$ for nonzero $u, v \in R$:

$$(\dagger) \qquad (u = a, v = a + b), (u = b, v = a + b), (u = a + b, v = a + b).$$

We see that R is not reversible, since $vu = a + b \neq 0$ for $u = a$ and $v = a + b$.

Now we define an automorphism α of R by

$$a \mapsto b, b \mapsto a \text{ and } a + b \mapsto a + b.$$

Then R is clearly α-compatible, noting that $\alpha(v) = v$ in (†). But R is not α-skew reversible since $a(a + b) = 0$ and $(a + b)\alpha(a) = a + b \neq 0$.

However, we can easily check that the concept of α-skew reversibility coincides with the concept of α-compatibility for a reversible ring with a monomorphism α. Moreover, we have the following.

Proposition 2.12. *Let α be a monomorphism of a reduced ring R. Then the following statements are equivalent*:

(1) *R is α-rigid.*

(2) *R is strongly α-skew reversible.*
(3) *R is α-skew reversible.*
(4) *R is right α-skew reversible.*
(5) *R is α-skew Armendariz.*
(6) *R is α-compatible.*

Proof. (1)⇒(2) is clear, (2)⇒(3) follows from Corollary 2.4(2) and (3)⇔(4) comes from [2, Proposition 2.4]. (3)⇒(5) and (5)⇒(1) are by [2, Theorem 2.9] and [3, Theorem 1], respectively. (4)⇔(6) comes from the above noted fact, since reduced rings are reversible. □

It is seen that the class of semiprime rings and the class of strongly α-skew reversible rings are independent of each other by Example 2.5(2) and Example 2.7. Indeed, the ring R in Example 2.7 is strongly α-skew reversible by Theorem 2.8 but not semiprime.

Theorem 2.13. *A ring R is α-rigid if and only if R is strongly α-skew reversible and semiprime.*

Proof. It is enough to show that R is α-rigid when R is strongly α-skew reversible and semiprime. If R is strongly α-skew reversible, then it is right α-skew reversible and α is a monomorphism by Lemma 2.3(2) and Corollary 2.4(2). Thus R is α-rigid by [2, Proposition 2.5(iii)]. □

Theorem 2.13 leads to the following result which includes the result of [15, Lemma 2.7].

Corollary 2.14. *Let R be a semiprime ring. Then the following statements are equivalent*:
(1) *R is reduced.*
(2) *R is strongly reversible.*
(3) *R is reversible.*

Let α_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. Then the map $\alpha : \prod_{\gamma\in\Gamma} R_\gamma \to \prod_{\gamma\in\Gamma} R_\gamma$ defined by $\alpha((a_\gamma)_\gamma) = (\alpha_\gamma(a_\gamma))_\gamma$ is an endomorphism of the product $\prod_{\gamma\in\Gamma} R_\gamma$ of R_γ.

Lemma 2.15. *Let R_γ be a ring with an endomorphism α_γ for each $\gamma \in \Gamma$. The following statements are equivalent*:
(1) *R_γ is strongly α_γ-skew reversible for each $\gamma \in \Gamma$.*
(2) *The direct product of R_γ for $\gamma \in \Gamma$ is strongly α-skew reversible.*
(3) *The direct sum of R_γ for $\gamma \in \Gamma$ is strongly α-skew reversible.*

Proof. The proof is trivial, noting that the class of strongly α-skew reversible is closed under subrings. □

Recall that R is called *local* if $R/J(R)$ is a division ring where $J(R)$ denotes the Jacobson radical of R, R is called *semilocal* if $R/J(R)$ is semisimple Artinian, and R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are abelian and semilocal.

Proposition 2.16. *Let α be an endomorphism of a ring R. Then we have the following.*

(1) *R is strongly α-skew reversible and semiperfect if and only if R is a finite direct sum of local strongly α_γ-skew reversible rings for $\gamma \in \Gamma$.*

(2) *For a central idempotent e of R, eR and $(1-e)R$ are strongly α-skew reversible if and only if R is strongly α-skew reversible.*

Proof. (1) Suppose that R is strongly α-skew reversible and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [17, Proposition 3.7.2], say $R = \sum_{i=1}^{n} e_i R$ such that each $e_i R e_i$ is a local ring. Since strongly α-skew reversible is abelian by Lemma 2.3(1), each $e_i R$ is an ideal of R with $e_i R = e_i R e_i$. But each $e_i R$ is also a strongly α-skew reversible ring as a subring, noting that $\alpha(e_i R) \subseteq e_i R$ for all i, by Lemma 2.3(4).

Conversely assume that R is a finite direct sum of strongly α_γ-skew reversible local rings. Then R is semiperfect since local rings are semiperfect, and moreover R is strongly α-skew reversible by Lemma 2.15.

(2) It comes from Lemma 2.15 with $R \cong eR \oplus (1-e)R$, since the fact that the class of strongly α-skew reversible is closed under subrings, noting that $\alpha(eR) \subseteq eR$ and $\alpha((1-e)R) \subseteq (1-e)R$ by Lemma 2.3(4). □

3. Extensions of strongly α-skew reversible rings

Let R be a ring with a monomorphism α. We consider the Jordan's construction of an over-ring of R by α (see [12] for more details). Let $A(R,\alpha)$ be the subset $\left\{x^{-i} r x^i \mid r \in R \text{ and } i \geq 0\right\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Note that $x^j r = \alpha^j(r) x^j$ implies $r x^{-j} = x^{-j} \alpha^j(r)$ for $j \geq 0$ and $r \in R$. Thus $x^{-i} r x^i = x^{-(i+j)} \alpha^j(r) x^{i+j}$ for each $j \geq 0$. It follows that $A(R,\alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following natural operations: $x^{-i} r x^i + x^{-j} s x^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s)) x^{i+j}$ and $(x^{-i} r x^i)(x^{-j} s x^j) = x^{-(i+j)} \alpha^j(r) \alpha^i(s) x^{i+j}$ for $r, s \in R$ and $i, j \geq 0$. Note that $A(R,\alpha)$ is an over-ring of R, and α is actually an automorphism of $A(R,\alpha)$ with $\alpha(x^{-i} r x^i) = x^{-i} \alpha(r) x^i$. Jordan showed, with the use of left localization of the skew polynomial $R[x;\alpha]$ with respect to the set of powers of x, that for any pair (R,α), such an extension $A(R,\alpha)$ always exists in [12]. This ring $A(R,\alpha)$ is usually said to be *Jordan extension* of R by α.

Proposition 3.1. *Let α be an endomorphism of a ring R.*

(1) *Assume that $A(R,\alpha)$ is the corresponding Jordan extension of R, where α is a monomorphism. Then R is a strongly α-skew reversible ring if and only if $A(R,\alpha)$ is a strongly α-skew reversible ring.*

(2) *Let $\sigma : R \to S$ be a ring isomorphism. Then R is a strongly α-skew reversible if and only if S is a strongly $\sigma\alpha\sigma^{-1}$-skew reversible.*

Proof. (1) It is enough to show the necessity. Let

$$A = A(R,\alpha) = \left\{t^{-i} r t^i \mid r \in R \text{ and } i \geq 0\right\},$$

using t in place of x in the argument above. Suppose that R is strongly α-skew reversible and let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{n} b_j x^j \in A[x;\alpha]$, where $a_i = t^{-v(i)} a'_i t^{v(i)}$ and $b_j = t^{-u(j)} b'_j t^{u(j)}$ for $a'_i, b'_j \in R$ and $v(i), u(j) \geq 0$. We use the property that $t^{-i} r t^i = t^{-(i+j)} \alpha^j(r) t^{i+j}$ for each $j \geq 0$ and that α is an automorphism of A. Then we can express all a_i, b_j by

$$a_i = t^{-s} c_i t^s \text{ and } b_j = t^{-s} d_j t^s$$

for some $s \geq 0$ and $c_i, d_j \in R$. This yields that

$$\begin{aligned} 0 &= \sum_{k=0}^{m+n} \left(\sum_{k=i+j} a_i \alpha^i(b_j) \right) x^k = \sum_{k=0}^{m+n} \left(\sum_{k=i+j} t^{-s} c_i t^s \alpha^i(t^{-s} d_j t^s) \right) x^k \\ &= \sum_{k=0}^{m+n} \left(\sum_{k=i+j} t^{-s} c_i t^s t^{-s} \alpha^i(d_j) t^s \right) x^k = t^{-s} \left(\sum_{k=0}^{m+n} \left(\sum_{k=i+j} c_i \alpha^i(d_j) \right) x^k \right) t^s. \end{aligned}$$

This entails $\sum_{k=0}^{m+n} \left(\sum_{k=i+j} c_i \alpha^i(d_j) \right) x^k = 0$. Next let

$$p'(x) = \sum_{i=0}^{m} c_i x^i \text{ and } q'(x) = \sum_{j=0}^{n} d_j x^j \in R[x;\alpha].$$

Then we get $p'(x)q'(x) = 0$ by the result above, noting that $p(x) = t^{-s} p'(x) t^s$ and $q(x) = t^{-s} q'(x) t^s$. Since R is strongly α-skew reversible, $q'(x)p'(x) = 0$ and thus $q(x)p(x) = (t^{-s} q'(x) t^s)(t^{-s} p'(x) t^s) = t^{-s}(q'(x)p'(x)) t^s = 0$. Therefore A is strongly α-skew reversible.

(2) It follows immediately from the basic fact that an isomorphism $\sigma : R \to S$ induces an isomorphism $R[x;\alpha] \cong S[x;\sigma\alpha\sigma^{-1}]$. □

As a corollary, we have the following result.

Corollary 3.2. *For an automorphism α of a ring R, the following statements are equivalent*:

(1) *R is strongly α-skew reversible and semiprime.*
(2) *R is α-rigid.*
(3) *$R[x, x^{-1};\alpha]$ is a reduced ring.*
(4) *$A(R,\alpha)[x, x^{-1};\alpha]$ is a reduced ring.*
(5) *$A(R,\alpha)$ is α-rigid.*
(6) *$A(R,\alpha)$ is strongly α-skew reversible and semiprime.*

Proof. (1)⇔(2) and (5)⇔(6) come from Theorem 2.13 and (1)⇔(6) is obtained by Proposition 3.1(1).

(2)⇔(3) and (4)⇔(5) follow from [18, Theorem 3]. □

For $n \geq 2$, denote the n by n full matrix (resp., upper triangular matrix) ring over a ring R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$). Consider the rings

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\},$$

which are subrings of $\mathrm{Mat}_n(R)$. Note that $V_n(R) \cong R[x]/(x^n)$, where (x^n) is the ideal of $R[x]$ generated by x^n.

For a ring R with an endomorphism α, the corresponding $(a_{ij}) \mapsto (\alpha(a_{ij}))$ induces endomorphisms of $\mathrm{Mat}_n(R)$, $D_n(R)$ and $V_n(R)$, respectively. We denote all of them by $\bar{\alpha}$.

If R is a reduced ring, then $V_n(R)$ $(n \geq 2)$ is strongly reversible by [22, Proposition 3.5], but $D_n(R)$ $(n \geq 3)$ is not reversible by [15, Example 1.3 and Example 1.5] and hence it is not strongly $\bar{\alpha}$-skew reversible for any endomorphism α of R by Lemma 2.3(1). Moreover, $U_2(A)$ over any ring A is not abelian obviously, and hence $U_n(A)$ and $\mathrm{Mat}_n(A)$ for $n \geq 2$ are not strongly $\bar{\alpha}$-skew reversible for any endomorphism α of A by Lemma 2.3(1) and the fact that the class of strongly $\bar{\alpha}$-skew reversible rings is closed under subrings. We will use this fact freely.

In the following theorem, we use $(a_1, a_2, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

Theorem 3.3. *If R is an α-rigid ring, then $V_n(R)$ is strongly $\bar{\alpha}$-skew reversible for any $n \geq 2$.*

Proof. Note that $V_n(R)[x; \bar{\alpha}] \cong V_n(R[x; \alpha])$ for $n \geq 2$. Then every polynomial $p(x) = \sum_{u=0}^{m} A_u x^u \in V_n(R)[x; \bar{\alpha}]$ can be expressed by the form of

$$(p_1, p_2, \ldots, p_n),$$

where $A_u = (a_i^u) \in V_n(R)$ for any $0 \leq u \leq m$ and $p_i = \sum_{u=0}^{m} a_i^u x^u \in R[x; \alpha]$ for any $1 \leq i \leq n$. Assume $p(x)q(x) = 0$ where $p(x) = \sum_{u=0}^{s} A_u x^u = (p_1, p_2, \ldots, p_n)$ and $q(x) = \sum_{v=0}^{t} B_v x^v = (q_1, q_2, \ldots, q_n) \in V_n(R)[x; \bar{\alpha}]$, $A_u =$

$(a_i^u), B_v = (b_j^v) \in V_n(R)$ for any $0 \le u \le s$, $0 \le v \le t$ and $p_i, q_j \in R[x;\alpha]$ for any $1 \le i, j \le n$. From $p(x)q(x) = 0$, we have the following equalities:

$$p_1q_1 = 0, \tag{1}$$

$$p_1q_2 + p_2q_1 = 0, \tag{2}$$

$$p_1q_3 + p_2q_2 + p_3q_1 = 0, \tag{3}$$

$$\vdots$$

$$p_1q_{n-1} + p_2q_{n-2} + \cdots + p_{n-1}q_1 = 0, \tag{4}$$

$$p_1q_n + p_2q_{n-1} + \cdots + p_{n-1}q_2 + p_nq_1 = 0 \tag{5}$$

in $R[x;\alpha]$. Note that $R[x;\alpha]$ is reduced. Recall that $ab = 0$ implies $aSb = 0$ and $ba = 0$, and that $ab^2 = 0$ implies that $ab = 0$ for any reduced ring S and any $a, b \in S$. We will freely use these facts in the following procedure. From Eq. (1), we see

$$p_1R[x;\alpha]q_1 = 0 \text{ and } q_1p_1 = 0. \tag{6}$$

If we multiply Eq. (2) on the right-hand side by q_1, then $p_1q_2q_1 + p_2q_1^2 = 0$ and hence $p_2q_1 = 0$ and $p_1q_2 = 0$. Thus

$$p_2R[x;\alpha]q_1 = 0,\ p_1R[x;\alpha]q_2 = 0,\ q_1p_2 = 0 \text{ and } q_2p_1 = 0. \tag{7}$$

If we multiply Eq. (3) on the right-hand side by q_1, then $p_1q_3q_1+p_2q_2q_1+p_3q_1^2 = 0$ and so $p_3q_1 = 0$, using Eqs. (6,7). Then Eq. (3) becomes

$$p_1q_3 + p_2q_2 = 0. \tag{8}$$

If we multiply Eq. (8) on the right-hand side by q_2, then $p_2q_2 = 0$ and $p_1q_3 = 0$ by the similar argument to above. Thus, we have

$$p_iR[x;\alpha]q_j = 0 \text{ and } q_jp_i = 0 \text{ for all } 2 \le i + j \le 4.$$

Inductively, we assume that

$$p_iR[x;\alpha]q_j = 0 \text{ and } q_jp_i = 0 \text{ for all } i + j \le n.$$

If we multiply Eq. (5) on the right-hand side by $q_1, q_2, \ldots, q_{n-1}$ in turn, then

$$p_nq_1 = 0,\ p_{n-1}q_2 = 0, \ldots,\ p_2q_{n-1} = 0 \text{ and } p_1q_n = 0$$

by the similar computation to above, and so

$$q_ip_j = 0 \text{ for all } i + j = n + 1.$$

Consequently, we get $q(x)p(x) = 0$ and therefore $V_n(R)$ is strongly $\bar{\alpha}$-skew reversible. □

Corollary 3.4. (1) [15, Proposition 1.6] *If R is a reduced ring, then $V_2(R)$ is reversible.*

(2) [22, Proposition 3.5] *Let R be a reduced ring and any $n \ge 2$. Then $R[x]/(x^n)$ is strongly reversible.*

Observe that for an ideal I and an endomorphism α of a ring R, if I is an α-ideal (i.e., $\alpha(I) \subseteq I$) of R, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of a factor ring R/I. Recall that $\mathrm{Mat}_n(A)$ ($n \geq 2$) over any ring A is not strongly $\bar{\alpha}$-skew reversible for any endomorphism α of A. This fact implies that the class of strongly α-skew reversible rings is not closed under the homomorphic images. In fact, if R is the ring of quaternions with integer coefficients with a monomorphism α, then R is a domain, and so strongly α-skew reversible; while for any odd prime integer q, we have $R/qR \cong \mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [6, Exercise 2A]. Since $\mathrm{Mat}_2(\mathbb{Z}_q)$ cannot be strongly $\bar{\alpha}$-skew reversible (as noted before), the factor ring R/qR is not strongly $\bar{\alpha}$-skew reversible.

On the other hand, we obtain the following which extends the result of [22, Proposition 3.8] and which can be obtained by [19, Proposition 2.15], but we provide another simple proof.

Proposition 3.5. *Let R be a ring with an endomorphism α and I a proper α-ideal of R. Suppose that R/I is a strongly $\bar{\alpha}$-skew reversible ring. If I is α-rigid as a ring without identity, then R is strongly α-skew reversible.*

Proof. Assume that R/I is strongly $\bar{\alpha}$-skew reversible and I is an α-rigid ring. Let $p(x)q(x) = 0$ for $p(x), q(x) \in R[x;\alpha]$. Then $q(x)p(x) \in I[x;\alpha]$ by assumption. Note $(q(x)p(x))^2 = q(x)p(x)q(x)p(x) = 0$. Since I is an α-rigid ring, $I[x;\alpha]$ is reduced by [9, Proposition 3]. This implies $q(x)p(x) = 0$ and therefore R is strongly α-skew reversible. □

The following example shows that the condition "I is α-rigid as a ring without identity" of Proposition 3.5 is not superfluous (see also [19, Example 2.14]).

Example 3.6. Consider a ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

and an automorphism α of R defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

where F is a division ring. Then R is not α-skew reversible as noted earlier. Note that the ideal $I = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ of R is not α-rigid as a ring without identity. However, the factor ring

$$R/I = \left\{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + I \mid a, c \in F\right\}$$

is reduced and $\bar{\alpha}$ is an identity map on R/I. That is, R/I is $\bar{\alpha}$-rigid and hence strongly $\bar{\alpha}$-skew reversible.

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha} : R[x] \to R[x]$ defined by

$$\bar{\alpha}\left(\sum_{i=0}^{m} a_i x^i\right) = \sum_{i=0}^{m} \alpha(a_i) x^i$$

is an endomorphism of $R[x]$. This map extends α clearly.

Let $\deg f(x)$ denote the degree of a polynomial $f(x)$.

Theorem 3.7. *Let R be a ring with an endomorphism α such that $\alpha^t = 1_R$ for some positive integer t, where 1_R denotes the identity endomorphism of R. Then R is strongly α-skew reversible if and only if $R[x]$ is strongly $\bar{\alpha}$-skew reversible.*

Proof. Assume that R is strongly α-skew reversible, and let $p(y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m$, $q(y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n$ in $R[x][y; \bar{\alpha}]$ such that $p(y)q(y) = 0$. Let

$$f_i(x) = a_{i_0} + a_{i_1}x + \cdots + a_{i_{w_i}} x^{w_i} \text{ and } g_j(x) = b_{j_0} + b_{j_1}x + \cdots + b_{j_{v_j}} x^{v_j}$$

for each $0 \le i \le m$, and $0 \le j \le n$, where $a_{i_0}, \ldots, a_{w_i}, b_{j_0}, \ldots, b_{v_j} \in R$. Set

$$C = \{a_{i_0}, \ldots, a_{i_{w_i}}, b_{j_0}, \ldots, b_{j_{v_j}} \mid 0 \le i \le m \text{ and } 0 \le j \le n\}.$$

Let $R[y; \alpha]$ be the skew polynomial ring with an indeterminate y over R, subject to $yr = \alpha(r)y$ for $r \in R$. We can rewrite $p(y)$ and $q(y)$ by

$$h_0(y) + h_1(y)x + \cdots + h_u(y)x^u \text{ and } l_0(y) + l_1(y)x + \cdots + l_s(y)x^s \in R[y; \alpha][x],$$

where $h_i(y), l_j(y) \in R[y; \alpha]$, and $R[y; \alpha][x]$ is the polynomial ring with an indeterminate x over $R[y; \alpha]$. Set $h(x) = p(y)$ and $l(x) = q(y)$, then $h(x)l(x) = 0$.

Take a positive integer k such that k is larger than the maximal integer in

$$\{\deg h_i(y), \deg l_j(y) \mid 0 \le i \le u \text{ and } 0 \le j \le s\},$$

where the degree of zero polynomial is taken to be 0. Set

$$h(y^{tk}) = h_0(y) + h_1(y)y^{tk} + \cdots + h_u(y)y^{mtk} \text{ and}$$
$$l(y^{tk}) = l_0(y) + l_1(y)y^{tk} + \cdots + l_s(y)y^{ntk}.$$

Then $h(y^{tk}), l(y^{tk}) \in R[y; \alpha]$. Note that the set of coefficients of the $h_i(y)$'s (resp., $l_j(y)$'s) equals the set of coefficients of $h(y^{tk})$ (resp., $l(y^{tk})$). Moreover this set is equal to the set C. Therefore, since $p(y)q(y) = 0 = h(x)l(x)$ and $\alpha^{tk} = 1_R$, we get $h(y^{tk})l(y^{tk}) = 0$ in $R[y; \alpha]$, noting that $\bar{\alpha}(r) = \alpha(r)$ for $r \in R$.

Now since R is strongly α-skew reversible by assumption, we have

$$l(y^{tk})h(y^{tk}) = 0.$$

This also yields $l(x)h(x) = 0$ also since $\alpha^{tk} = 1_R$, entailing $q(y)p(y) = 0$. Thus $R[x]$ is strongly $\bar{\alpha}$-skew reversible.

The converse comes from the fact that the class of strongly α-skew reversible rings is closed under subrings, recalling that $\bar{\alpha}(r) = \alpha(r)$ for $r \in R$. □

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* is defined, and *regular* means if it is both left and right regular (and hence not a zero divisor). Assume that M is a multiplicatively closed subset of R consisting of central regular elements.

Let α be an automorphism of R and assume $\alpha(m) = m$ for every $m \in M$. Then $\alpha(m^{-1}) = m^{-1}$ in $M^{-1}R$ and the induced map $\bar{\alpha} : M^{-1}R \to M^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = u^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.8. *Let R be a ring with an automorphism α and assume that there exists a multiplicatively closed subset M of R consisting of central regular elements and $\alpha(m) = m$ for every $m \in M$. Then R is a strongly α-skew reversible ring if and only if $M^{-1}R$ is a strongly $\bar{\alpha}$-skew reversible ring.*

Proof. Let R be strongly α-skew reversible, and suppose that $P(x)Q(x) = 0$ for $P(x) = u^{-1}(a_0 + a_1x + \cdots + a_kx^k), Q(x) = v^{-1}(b_0 + b_1x + \cdots + b_nx^n) \in M^{-1}R[x; \bar{\alpha}]$. Then

$$\begin{aligned}0 &= P(x)Q(x) \\ &= u^{-1}(a_0v^{-1} + a_1\alpha(v)^{-1}x + \cdots + a_k\alpha^k(v)^{-1}x^k)(b_0 + b_1x + \cdots + b_nx^n).\end{aligned}$$

Since $\alpha^i(v)^{-1} = v^{-1}$ for $0 \leq i \leq k$ by hypothesis, we have

$$0 = P(x)Q(x) = (uv)^{-1}(a_0 + a_1x + \cdots + a_kx^k)(b_0 + b_1x + \cdots + b_nx^n).$$

Let $p(x) = a_0 + a_1x + \cdots + a_kx^k$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$. Then we get $p(x)q(x) = 0$.

Now since R is strongly α-skew reversible, we have $q(x)p(x) = 0$. Then

$$\begin{aligned}Q(x)P(x) &= v^{-1}(b_0 + b_1x + \cdots + b_nx^n)u^{-1}(a_0 + a_1x + \cdots + a_kx^k) \\ &= (uv)^{-1}q(x)p(x) = 0\end{aligned}$$

by using $\alpha^i(v)^{-1} = v^{-1}$ for $0 \leq i \leq k$. Thus $M^{-1}R$ is strongly $\bar{\alpha}$-skew reversible.

The converse can be obtained from that the class of strongly α-skew reversible rings is closed under subrings. □

For the ring of Laurent polynomials $R[x, x^{-1}]$, the map $\bar{\alpha} : R[x, x^{-1}] \to R[x; x^{-1}]$ defined by $\bar{\alpha}(\sum_{i=k}^{n} a_ix^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ extends α and is also an endomorphism of $R[x; x^{-1}]$.

Corollary 3.9. *For a ring R, $R[x]$ is strongly $\bar{\alpha}$-skew reversible if and only if $R[x, x^{-1}]$ is strongly $\bar{\alpha}$-skew reversible.*

Proof. It directly follows from Proposition 3.8. For, letting $M = \{1, x, x^2, \ldots\}$, M is a multiplicatively closed subset of $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$ and $\alpha(x) = x$. □

By Proposition 2.16(2), Theorem 3.7 and Corollary 3.9, we have the following result which includes [22, Proposition 3.1 and Theorem 3.3].

Corollary 3.10. *For a ring R, the following statements are equivalent*:

(1) *R is strongly reversible.*

(2) *eR and $(1-e)R$ are strongly reversible for an central idempotent e of R.*

(3) *$R[x]$ is strongly reversible.*

(4) *$R[x, x^{-1}]$ is strongly reversible.*

Next consider the case of $\alpha(M) \subseteq M$ in Proposition 3.8, substituting the condition "$\alpha(m) = m$ for every $m \in M$". Let M be a multiplicatively closed subset of R consisting of central regular elements. For an automorphism α of R satisfying $\alpha(M) \subseteq M$, the induced map $\bar{\alpha} : M^{-1}R \to M^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = \alpha(u)^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.11. *Let R be an α-skew Armendariz ring and α an automorphism of R. Assume that there exists a multiplicatively closed subset M of R consisting of central regular elements and $\alpha(M) \subseteq M$. Then R is a strongly α-skew reversible ring if and only if $M^{-1}R$ is a strongly $\bar{\alpha}$-skew reversible ring.*

Proof. It is sufficient to show the necessity. Let R be strongly α-skew reversible, and suppose that $P(x)Q(x) = 0$ for $P(x) = u^{-1}(a_0 + a_1x + \cdots + a_mx^m), Q(x) = v^{-1}(b_0 + b_1x + \cdots + b_nx^n) \in M^{-1}R[x; \bar{\alpha}]$. Note

$$\begin{aligned} 0 &= P(x)Q(x) \\ &= u^{-1}(a_0v^{-1} + a_1\alpha(v)^{-1}x + \cdots + a_m\alpha^m(v)^{-1}x^m)(b_0 + b_1x + \cdots + b_nx^n). \end{aligned}$$

Since $\alpha^i(v)^{-1} \in M$ for $0 \le i \le m$ by hypothesis, there exist $w \in M$ and $a'_i \in R$ such that $a_i\alpha^i(v)^{-1} = w^{-1}a'_i$. Then we have

$$\begin{aligned} P(x)v^{-1} &= u^{-1}(a_0 + a_1x + \cdots + a_mx^m)v^{-1} = u^{-1}w^{-1}p'(x) \text{ and} \\ 0 &= P(x)Q(x) = u^{-1}w^{-1}p'(x)q(x), \end{aligned}$$

where $p'(x) = a'_0 + a'_1x + \cdots + a'_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$. This entails $p'(x)q(x) = 0$.

Now since R is strongly α-skew reversible, we have $q(x)p'(x) = 0$. Here we have $b_j\alpha^j(a'_i) = 0$ for all $0 \le i \le m$ and $0 \le j \le n$ since R is α-skew Armendariz. Then $b_j\alpha^j(a'_i)\alpha^j(uw)^{-1} = 0$. But since $\alpha^j(uw)^{-1}$ is central in $M^{-1}R$, we have

$$b_j\alpha^j(uw)^{-1}\alpha^j(a'_i) = b_j\bar{\alpha}^j((uw)^{-1}a'_i) = 0 \text{ for all } i, j.$$

This allows us to have

$$\begin{aligned} &q(x)P(x)v^{-1} \\ &= q(x)((uw)^{-1}p'(x)) \\ &= (b_0 + b_1x + \cdots + b_nx^n)((uw)^{-1}a'_0 + (uw)^{-1}a'_1x + \cdots + (uw)^{-1}a'_mx^m) = 0, \end{aligned}$$

entailing that $q(x)P(x) = 0$. This also yields that $Q(x)P(x) = v^{-1}q(x)P(x) = 0$. Thus $M^{-1}R$ is strongly $\bar{\alpha}$-skew reversible. □

Remark 3.12. Regarding Proposition 3.8 and Proposition 3.11, Z. Peng et al. [19, Theorem 2.16] showed the strongly monoid reversibility between a skew monoid ring and its classical right quotient ring, but there exists a gap in the proof of [19, Theorem 2.16], for taking elements in the classical right quotient ring.

Let R be an algebra over a commutative ring S. Following [5], the *Dorroh extension* of R by S is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. For an endomorphism α of R and the Dorroh extension D of R by S, $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S-algebra homomorphism.

Theorem 3.13. *Let R be an algebra over a commutative domain S and α an endomorphism of R. Then R is strongly α-skew reversible if and only if the Dorroh extension D of R by S is strongly $\bar{\alpha}$-skew reversible.*

Proof. Recall that $\alpha(1) = 1$ and note that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$. It is sufficient to show that the Dorroh extension D is strongly $\bar{\alpha}$-skew reversible when R is strongly α-skew reversible. Assume that R is strongly α-skew reversible. Let $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{m}(a_i, b_i)x^i$, $q(x) = \sum_{j=0}^{n}(c_j, d_j)x^j \in D[x; \bar{\alpha}]$. Then we have

(9)
$$\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i, b_i)\bar{\alpha}^i(c_j, d_j)\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i, b_i)(\alpha^i(c_j), d_j)\right)x^k = 0.$$

Put $f(x) = \sum_{i=0}^{m} b_ix^i$, $g(x) = \sum_{j=0}^{n} d_jx^j \in S[x]$, then we get $f(x)g(x) = 0$ by Eq. (9). Since $S[x]$ is a domain, $f(x) = 0$ or $g(x) = 0$. Note that $\alpha(s) = s$ for all $s \in S$, since $\alpha(1) = 1$. We will freely use this fact without reference in the following procedure.

If $f(x) = 0$, then Eq. (9) becomes $\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(a_i(\alpha^i(c_j) + d_j), 0)\right)x^k$, and so we have

$$(10) \quad 0 = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i(\alpha^i(c_j) + d_j)\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i\alpha^i(c_j + d_j)\right)x^k.$$

We let $p'(x) = \sum_{i=0}^{m} a_ix^i$, $q'(x) = \sum_{j=0}^{n}(c_j + d_j)x^j \in R[x; \alpha]$. Then we get $p'(x)q'(x) = 0$ by Eq. (10). Since R is strongly α-skew reversible, $q'(x)p'(x) = 0$ and so we obtain

$$0 = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(c_j + d_j)\alpha^j(a_i)\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(c_j\alpha^j(a_i) + d_j\alpha^j(a_i))\right)x^k.$$

Hence,

$$\sum_{k=0}^{m+n}\left(\sum_{i+j=k}(c_j,d_j)\bar{\alpha}^j((a_i,0))\right)x^k = \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(c_j,d_j)(\alpha^j(a_i),0)\right)x^k$$
$$= \sum_{k=0}^{m+n}\left(\sum_{i+j=k}(c_j\alpha^j(a_i)+d_j\alpha^j(a_i),0)\right)x^k$$
$$= 0,$$

showing that $q(x)p(x) = 0$.

In case of $g(x) = 0$, we can show that $q(x)p(x) = 0$ by the similar argument to above.

Consequently, the Dorroh extension D of R by S is strongly $\bar{\alpha}$-skew reversible if R is strongly α-skew reversible. □

Corollary 3.14. *Let R be an algebra over a commutative domain S, and D be the Dorroh extension of R by S. Then R is strongly reversible if and only if D is strongly reversible.*

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