Bull. Korean Math. Soc. **54** (2017), No. 1, pp. 17–28
https://doi.org/10.4134/BKMS.b150007
pISSN: 1015-8634 / eISSN: 2234-3016

MEAN-FIELD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS ON MARKOV CHAINS

Wen Lu and Yong Ren

Abstract. In this paper, we deal with a class of mean-field backward stochastic differential equations (BSDEs) related to finite state, continuous time Markov chains. We obtain the existence and uniqueness theorem and a comparison theorem for solutions of one-dimensional mean-field BSDEs under Lipschitz condition.

1. Introduction

The general (nonlinear) backward stochastic differential equations (BSDE in short) were firstly introduced by Pardoux and Peng [22] in 1990. Since then, BSDEs have been studied with great interest, and they have gradually become an important mathematical tool in many fields such as financial mathematics, stochastic games and optimal control, etc, see for example, Peng [23], Hamadène and Lepeltier [13] and El Karoui et al. [11].

McKean-Vlasov stochastic differential equation of the form

$$dX(t) = b(X(t), \mu(t))dt + dW(t), \quad t \in [0, T], \quad X(0) = x, \tag{1}$$

where

$$b(X(t), \mu(t)) = \int_{\Omega} b(X(t, \omega), X(t; \omega'))P(d\omega') = E[b(\xi, X(t)]|_{\xi = X(t)},$$

$b : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ being a (locally) bounded Borel measurable function and $\mu(t; \cdot)$ being the probability distribution of the unknown process $X(t)$, was suggested by Kac [14] as a stochastic toy model for the Vlasov kinetic equation of plasma. The study of which was initiated by Mckean [21]. Since then, many authors made contributions on McKean-Vlasov type SDEs and applications,

Received January 5, 2015; Revised July 8, 2016.
2010 *Mathematics Subject Classification.* Primary 60H20, 60H10.
Key words and phrases. mean-field backward stochastic differential equations, Markov chain, comparison theorem.
The work of Wen Lu is supported partially by the National Natural Science Foundation of China (61273128, 11371029) and a Project of Shandong Province Higher Educational Science and Technology Program (J13LI06). The work of Yong Ren is supported by the National Natural Science Foundation of China (11371029, 11501009) and the Natural Science Foundation of Anhui Province (1508085JGD10).

see for example, Ahmed [1], Ahmed and Ding [2], Borkar and Kumar [3], Chan [6], Crisan and Xiong [10], Kotelenez [15], Kotelenez and Kurtz [16], and so on.

Mathematical mean-field approaches have been used in many fields, not only in physics and chemistry, but also recently in economics, finance and game theory, see for example, Lasry and Lions [17], they have studied mean-field limits for problems in economics and finance, and also for the theory of stochastic differential games.

Inspired by Lasry and Lions [17], Buckdahn et al. [4] introduced a new kind of BSDEs of mean-field BSDEs. Furthermore, Buckdahn et al. [5] deepened the investigation of mean-field BSDEs in a rather general setting, they gave the existence and uniqueness of solutions for mean-field BSDEs with Lipschitz condition on coefficients, they also established the comparison principle for these mean-field BSDEs. On the other hand, since the works [4] and [5] on the mean-field BSDEs, there are some efforts devote to its generalization, Xu [24] obtained the existence and uniqueness of solutions for mean-field backward doubly stochastic differential equations; Li and Luo [19] studied reflected BSDEs of mean-field type, they proved the existence and the uniqueness for reflected mean-field BSDEs; Li [18] studied reflected mean-filed BSDEs in a purely probabilistic method, and gave a probabilistic interpretation of the nonlinear and nonlocal PDEs with the obstacles.

However, most previous contributions to BSDEs and mean-field BSDEs have been obtained in the framework of continuous time diffusion. Recently, Cohen and Elliott [7] introduced a new kind of BSDEs of the form, for $t \in [0, T]$

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s)ds - \int_t^T Z_s dM_s, \tag{2}$$

where M_t is a martingale related to a finite state continuous time Markov chain (the details of M_t will be given in Section 2). In Cohen and Elliott [7], the authors proved the existence and uniqueness of solutions for those equations under Lipschitz condition. Furthermore, Cohen and Elliott [8] gave a scalar and vector comparisons for solutions of the BSDEs on Markov chains. Furthermore, they discussed arbitrage and risk measure in scalar case. In Lu and Ren [20], we established the existence and uniqueness of the solutions of anticipated backward stochastic differential equations on finite state, continuous time Markov chains and a scalar comparison theorem. Very recently, Cohen and Elliott [9] established the existence and uniqueness as well as comparison theorem for BSDEs in general spaces.

Motivated by the above works, the present paper deal with a class of mean-field BSDEs on Markov Chains of the form

$$Y_t = \xi + \int_t^T E'[f(s, Y'_{s-}, Z'_s, Y_{s-}, Z_s)]ds - \int_t^T Z_s dM_s, \tag{3}$$

where (Y', Z') is a copy of (Y, Z). To the best of our knowledge, so far little is known about this new kind of BSDEs. Our aim is to find a pair of adapted processes (Y, Z) in an appropriate space such that (3) hold. We also present a comparison theorem for the solutions of BSDEs (3). We remark that our BSDE (3) includes BSDE (2) as a special case.

The paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to the proof of the existence and uniqueness of the solutions to mean-field BSDEs on Markov chains. In Section 4, we give a comparison theorem for the solutions of mean-field BSDEs.

2. Preliminaries

Let $T > 0$ be fixed throughout this paper. Let $X = \{X_t, t \in [0, T]\}$ be a continuous time finite state Markov chain. The state space of X can be identified with the set of unite column vectors $\{e_1, e_2, \ldots, e_N\}$ in $\mathbb{R}^N$, where $e_i = (0, \ldots, 1, \ldots, 0)^*$ with 1 in the i-th position, N is the number of states and $[\cdot]^*$ denotes vector/matrix transposition.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We denote by $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ the natural filtration generated by $X = \{X_t, t \in [0, T]\}$ and augmented by all P-null sets, i.e.,

$$\mathcal{F}_t = \sigma\{X_u, 0 \leq u \leq t\} \vee \mathcal{N}_P,$$

where $\mathcal{N}_P$ is the set of all P-null subsets.

Let A_t be the rate matrix for the chain X at time t, then this chain has the representation (see Appendix B of Elliott et al. [12])

$$X_t = X_0 + \int_0^t A_u X_{u-} du + M_t,$$

where M_t is a martingale related to the chain $X = \{X_t, t \in [0, T]\}$. The optional quadratic variation of M_t is given by the matrix process

$$[M, M]_t = \sum_{0<u\leq t} \Delta M_u \Delta M_u^*$$

and

$$\langle M, M\rangle_t = \int_{]0,\, t]} [\mathrm{diag}(A_u X_{u-}) - \mathrm{diag}(X_{u-}) A_u^* - A_u \mathrm{diag}(X_{u-})] du.$$

Let Φ_t be the nonnegative definite matrix

$$\Phi_t := \mathrm{diag}(A_t X_{t-}) - \mathrm{diag}(X_{t-}) A_t^* - A_t \mathrm{diag}(X_{t-})$$

and

$$\|Z\|_{X_{t-}} := \sqrt{\mathrm{Tr}(Z \Phi_t Z^*)}.$$

Then $\|\cdot\|_{X_{t-}}$ defines a (stochastic) seminorm, with the property that

$$\mathrm{Tr}(Z_t d\langle M, M\rangle_t Z_t^*) = \|Z\|_{X_{t-}}^2 dt.$$

Now, we provide some spaces and notations used in the sequel.

- $L^p(\Omega, \mathcal{F}_T, P) := \{\xi :$ real valued $\mathcal{F}_T$-measurable random variable $E|\xi|^p < +\infty,\ p \geq 1\}$;

- $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n) := \{\xi : \mathbb{R}^n\text{-valued } \mathcal{F}\text{-measurable random variable}\}$;
- $S^2_{\mathbb{F}}(\mathbb{R}) := \{Y : \Omega \times [0,T] \to \mathbb{R}$ càdlàg and $\mathbb{F}$-adapted, $E\big[\sup_{t\in[0,T]} |Y_t|^2\big] < +\infty\}$;
- $H^2_{X,\mathbb{F}}(\mathbb{R}^N) := \{Z : \Omega \times [0,T] \to \mathbb{R}^N$, left continuous and predictable, $E\int_0^T \|Z_t\|^2_{X_{t-}} dt < +\infty\}$.

Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P)$ be the (non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself. We denote the filtration of this product space by $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_t = \mathcal{F} \otimes \mathcal{F}_t, 0 \le t \le T\}$. A random variable $\xi \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ originally defined on Ω is extended canonically to $\overline{\Omega} : \xi'(\omega', \omega) = \xi(\omega'), (\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega$. For any $\theta \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ the variable $\theta(\cdot, \omega) : \Omega \to R$ belongs to $L^1(\Omega, \mathcal{F}, P)$, $P(d\omega)$-a.s.; we denote its expectation by

$$E'[\theta(\cdot, \omega)] = \int_\Omega \theta(\omega', \omega) P(d\omega').$$

Notice that $E'[\theta] = E'[\theta(\cdot, \omega)] \in L^1(\Omega, \mathcal{F}, P)$, and

$$\bar{E}[\theta] \Big(= \int_\Omega \theta d\bar{P} = \int_\Omega E'[\theta(\cdot, \omega)] P(d\omega) \Big) = E[E'[\theta]].$$

For convenience, we rewrite mean-field BSDEs (3) as below:

$$Y_t = \xi + \int_t^T E'[f(s, Y'_{s-}, Z'_s, Y_{s-}, Z_s)] ds - \int_t^T Z_s dM_s. \tag{4}$$

The coefficient of our mean-field BSDE is a function $f = f(\omega', \omega, t, y', z', y, z) : \bar{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which is $\bar{\mathbb{F}}$-progressively measurable for all (y', z', y, z). We make the following assumptions:

(A1) There exists a constant $C \ge 0$ such that, $dt \times \bar{P}$-a.s., $y_1, y_2, y'_1, y'_2 \in \mathbb{R}$,$z_1, z_2, z'_1, z'_2 \in \mathbb{R}^N$,

$$\begin{aligned} &|f(\omega', \omega, t, y'_1, z'_1, y_1, z_1) - f(\omega', \omega, t, y'_2, z'_2, y_2, z_2)| \\ &\le C\Big(|y'_1 - y'_2| + \|z'_1 - z'_2\|_{X_{t-}} + |y_1 - y_2| + \|z_1 - z_2\|_{X_{t-}}\Big); \end{aligned}$$

(A2) $\bar{E}\int_0^T |f(t, 0, 0, 0, 0)|^2 dt < +\infty$.

Remark 2.1. Since the integral in (4) is with respect to Lebesgue measure and our processes have at most countably many jumps, in this case the equation is unchanged whether the left limits are included or not.

Remark 2.2. We emphasize that, due to our notations, the driving coefficient f of (4) has to be interpreted as follows

$$\begin{aligned} E'[f(s, Y'_s, Z'_s, Y_s, Z_s)](\omega) &= E'[f(s, Y'_s, Z'_s, Y_s(\omega), Z_s(\omega))] \\ &= \int_\Omega f(s, Y'_s(\omega'), Z'_s(\omega'), Y_s(\omega), Z_s(\omega)) P(d\omega'). \end{aligned}$$

Definition 1. A solution to the mean-filed BSDE (4) is a couple $(Y, Z) = (Y_t, Z_t)_{0\le t\le T}$ satisfying (4) such that $(Y, Z) \in S^2_{\mathbb{F}}(\mathbb{R}) \times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$.

3. Existence and uniqueness of solutions

In this section, we aim to derive the existence and uniqueness result for the solutions of mean-field BSDEs on Markov chains.

Before stating our main theorem, we recall an existence and uniqueness result in Cohen and Elliott [7].

Lemma 3.1. *Given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Suppose assumptions* (A1) *and* (A2) *hold. Then BSDE* (2) *has a unique solution $(Y, Z) \in S^2_{\mathbb{F}}(\mathbb{R}) \times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$, and the solution is the unique such solution, up to indistinguishability for Y and equality $d\langle M, M\rangle_t \times P$-a.s. for Z.*

For the solutions of mean-field BSDE (4), we first establish the following unique result.

Lemma 3.2. *Given $\xi \in L^2(\Omega, \mathcal{F}_T, P)$. Suppose assumptions* (A1) *and* (A2) *hold. Then mean-field BSDE* (4) *has at most one solution $(Y, Z) \in S^2_{\mathbb{F}}(\mathbb{R}) \times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$.*

Proof. Let $(Y^i, Z^i) \in S^2_{\mathbb{F}}(\mathbb{R}) \times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$, $i = 1, 2$ be two solutions of mean-field BSDE (4). Define $\hat{Y} = Y^1 - Y^2$, $\hat{Z} = Z^1 - Z^2$, we then have

$$\hat{Y}(t) = \int_t^T E'[\hat{f}(s)]ds - \int_t^T \hat{Z}_s dM_s,$$

where $\hat{f}(s) = f(s, Y^{1\prime}_{s-}, Z^{1\prime}_s, Y^1_{s-}, Z^1_s) - f(s, Y^{2\prime}_{s-}, Z^{2\prime}_s, Y^2_{s-}, Z^2_s)$.

Using the Stieltjes chain rule for products, we get

$$|\hat{Y}_t|^2 = |\hat{Y}_0|^2 - 2\int_0^t \hat{Y}_{s-} E'[\hat{f}(s)]ds + 2\int_0^t \hat{Y}_{s-}\hat{Z}_s dM_s + \sum_{0<s\le t} |\Delta Y^1_s - \Delta Y^2_s|^2. \tag{5}$$

Taking expectation on both sides of (5) and evaluating at $t = T$, we obtain

$$\begin{aligned} E|\hat{Y}_t|^2 &= 2\int_t^T E[\hat{Y}_{s-}E'[\hat{f}(s)]]ds - E\sum_{t<s\le T} |\Delta Y^1_s - \Delta Y^2_s|^2 \\ &= 2\int_t^T E[\hat{Y}_{s-}E'[\hat{f}(s)]]ds - E\sum_{t<s\le T} |(Z^1_s - Z^2_s)\Delta M_s|^2 \\ &= 2\int_t^T E[\hat{Y}_{s-}E'[\hat{f}(s)]]ds - \int_t^T E\|\hat{Z}_s\|^2_{X_{s-}} ds. \end{aligned} \tag{6}$$

On the other hand, by (A1) and Young's inequality $2ab \le \frac{1}{\rho}a^2 + \rho b^2$, for any $\rho > 0$, it holds that

$$2\int_t^T E[\hat{Y}_{s-}E'[\hat{f}(s)]]ds$$

$$\leq 2C\int_t^T E\big[\hat{Y}_{s-}E'(|\hat{Y}'_{s-}|+\|\hat{Z}'_s\|_{X_{s-}}+|\hat{Y}_{s-}|+\|\hat{Z}_s\|_{X_{s-}})\big]ds$$
$$\leq 4C\int_t^T E|\hat{Y}_{s-}|^2ds+2C\int_t^T\big[\rho E|\hat{Y}_{s-}|^2+\frac{1}{\rho}E\|\hat{Z}_s\|^2_{X_{s-}}\big]ds.$$

Choosing $\rho=3C$, we obtain

$$2\int_t^T E\big\{\hat{Y}_{s-}E'[\hat{f}(s)]\big\}ds\leq(6C^2+4C)\int_t^T E|\hat{Y}_{s-}|^2ds+\frac{2}{3}\int_t^T E\|\hat{Z}_s\|^2_{X_{s-}}ds.$$

This together with (6) implies

$$E|\hat{Y}_t|^2+\frac{1}{3}\int_t^T E\|\hat{Z}_s\|^2_{X_{s-}}ds\leq(6C^2+4C)\int_t^T E|\hat{Y}_{s-}|^2ds.$$

An application of Grönwall's inequality gives

$$E|\hat{Y}_t|^2=0,\quad E\|\hat{Z}_t\|^2_{X_{t-}}=0,$$

i.e., $Y_t^1=Y_t^2$ and $Z_t^1=Z_t^2$ P-a.s. for each t. The proof is complete. □

Next, let's consider a simplified version of mean-field BSDEs (4) as follows

$$Y_t=\xi+\int_t^T E'[f(s,Y'_{s-},Y_{s-},Z_s)]ds-\int_t^T Z_s dM_s. \tag{7}$$

We have the following existence and uniqueness result.

Lemma 3.3. *Given $\xi\in L^2(\Omega,\mathcal{F}_T,P)$. Suppose assumptions* (A1) *and* (A2) *hold. Then mean-field BSDE* (7) *has a unique solution* $(Y,Z)\in S^2_{\mathbb{F}}(\mathbb{R})\times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$.

Proof. Let $Y_t^0=0$, $t\in[0,T]$, we consider the following mean-field BSDE:

$$Y_t^{n+1}=\xi+\int_t^T E'[f(s,Y_{s-}^{n\prime},Y_{s-}^n,Z_s^{n+1})]ds-\int_t^T Z_s^{n+1}dM_s. \tag{8}$$

According to Lemma 3.1, we can define recursively (Y^{n+1},Z^{n+1}) be the solution of BSDE (8). For $t\in[0,T]$, we have

$$\begin{aligned} &Y_t^{n+1}-Y_t^n \\ &=\int_t^T E'[f(s,Y_{s-}^{n\prime},Y_{s-}^n,Z_s^{n+1})-f(s,Y_{s-}^{n-1\prime},Y_{s-}^{n-1},Z_s^n)]ds \\ &\quad-\int_t^T(Z_s^{n+1}-Z_s^n)dM_s \\ &=Y_0^{n+1}-Y_0^n-\int_0^t E'[f(s,Y_{s-}^{n\prime},Y_{s-}^n,Z_s^{n+1})-f(s,Y_{s-}^{n-1\prime},Y_{s-}^{n-1},Z_s^n)]ds. \\ &\quad-\int_0^t(Z_s^{n+1}-Z_s^n)dM_s. \end{aligned} \tag{9}$$

Using the Stieltjes chain rule for products, we have

$$
\begin{aligned}
&|Y_t^{n+1} - Y_t^n|^2 \\
&= |Y_0^{n+1} - Y_0^n|^2 - 2\int_0^t (Y_{s-}^{n+1} - Y_{s-}^n)E'[f(s, Y_{s-}^{n}{}', Y_{s-}^n, Z_s^{n+1}) \\
&\quad - f(s, Y_{s-}^{n-1}{}', Y_{s-}^{n-1}, Z_s^n)]ds + 2\int_0^t (Y_{s-}^{n+1} - Y_{s-}^n)(Z_s^{n+1} - Z_s^n)dM_s \\
&\quad + \sum_{0<s\le t} |\Delta Y_s^{n+1} - \Delta Y_s^n|^2.
\end{aligned}
$$

Taking expectation and evaluating at $t = T$, we obtain

$$
\begin{aligned}
(10)\qquad &E|Y_t^{n+1} - Y_t^n|^2 \\
&= 2\int_t^T E\big\{(Y_{s-}^{n+1} - Y_{s-}^n)E'[f(s, Y_{s-}^{n}{}', Y_{s-}^n, Z_s^{n+1}) \\
&\quad - f(s, Y_{s-}^{n-1}{}', Y_{s-}^{n-1}, Z_s^n)]\big\}ds - \int_t^T E\|Z_s^{n+1} - Z_s^n\|_{X_{s-}}^2 ds,
\end{aligned}
$$

by (A1) and Young's inequality, for any $\rho > 0$, we have

$$
\begin{aligned}
(11)\qquad &2\int_t^T E\big\{(Y_{s-}^{n+1} - Y_{s-}^n)E'[f(s, Y_{s-}^{n}{}', Y_{s-}^n, Z_s^{n+1}) \\
&\quad - f(s, Y_{s-}^{n-1}{}', Y_{s-}^{n-1}, Z_s^n)]\big\}ds \\
&\le 2C\int_t^T E\big\{(Y_{s-}^{n+1} - Y_{s-}^n)E'[|Y_{s-}^{n}{}' - Y_{s-}^{n-1}{}'| \\
&\quad + |Y_{s-}^n - Y_{s-}^{n-1}| + \|Z_s^{n+1} - Z_s^n\|_{X_{s-}}]\big\}ds \\
&\le \frac{3C}{\rho}\int_t^T E|Y_{s-}^{n+1} - Y_{s-}^n|^2 ds + 2\rho C\int_t^T E|Y_{s-}^n - Y_{s-}^{n-1}|^2 ds \\
&\quad + \rho C\int_t^T E\|Z_s^{n+1} - Z_s^n\|_{X_{s-}}^2 ds.
\end{aligned}
$$

Choosing $\rho = \frac{1}{2C}$, combining (10) and (11), we then have

$$
\begin{aligned}
(12)\qquad &E|Y_t^{n+1} - Y_t^n|^2 \\
&\le c\Big[\int_t^T E|Y_s^{n+1} - Y_s^n|^2 ds + \int_t^T E|Y_s^n - Y_s^{n-1}|^2 ds\Big],
\end{aligned}
$$

where $c = \max\{6C^2, 1\}$. Let $u^n(t) = \int_t^T E|Y_s^n - Y_s^{n-1}|^2 ds$, it follows from (12)

$$
-\frac{du^{n+1}}{dt}(t) - cu^{n+1}(t) \le cu^n(t), \quad u^{n+1}(T) = 0.
$$

Integration gives

$$
u^{n+1}(t) \le c\int_t^T e^{c(s-t)}u^n(s)ds.
$$

Iterating above inequality, we obtain

$$u^{n+1}(0) \le \frac{(ce^c)^n}{n!} u^1(0).$$

This implies that $\{Y^n\}$ is a Cauchy sequence in $S^2_{\mathbb{F}}(\mathbb{R})$. Then by (12), $\{Z^n\}$ is a Cauchy sequence in $H^2_{X,\mathbb{F}}(\mathbb{R}^N)$.

Passing to the limit on both sides of (8), by (A2) and the dominated convergence theorem, it follows that

$$Y := \lim_{n\to\infty} Y^n, \quad Z := \lim_{n\to\infty} Z^n$$

solves BSDE (7). The uniqueness is a direct consequence of Lemma 3.2. The proof is complete. □

The main result of this section is the following theorem.

Theorem 3.4. *Assume that* (A1) *and* (A2) *hold true. Then for any given terminal conditions $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the mean-field BSDE* (4) *has a unique solution $(Y, Z) \in S^2_{\mathbb{F}}(\mathbb{R}) \times H^2_{X,\mathbb{F}}(\mathbb{R}^N)$.*

Proof. According to Lemma 3.2, all we need to prove is the existence of solution for mean-field BSDE (4).

let $Z^0_t = \mathbf{0}$, $t \in [0, T]$, in virtue of Lemma 3.3, we can define recursively the pair of processes (Y^{n+1}, Z^{n+1}) be the unique solution of the following mean-field BSDE:

$$(13) \quad Y^{n+1}_t = \xi + \int_t^T E'[f(s, Y^{n+1}_{s-}{}', Z^n_s{}', Y^{n+1}_{s-}, Z^{n+1}_s)]ds - \int_t^T Z^{n+1}_s dM_s.$$

Using the same procedure as above, we get

$$\begin{aligned}
&E|Y^{n+1}_t - Y^n_t|^2 \\
&= 2\int_t^T E\big\{(Y^{n+1}_{s-} - Y^n_{s-})E'[f(s, Y^{n+1}_{s-}{}', Z^n_s{}', Y^{n+1}_{s-}, Z^{n+1}_s) \\
&\quad - f(s, Y^n_{s-}{}', Z^{n-1}_s{}', Y^n_{s-}, Z^n_s)]\big\}ds - \int_t^T E\|Z^{n+1}_s - Z^n_s\|^2_{X_{s-}} ds \\
&\le 2C\int_t^T E\big\{(Y^{n+1}_{s-} - Y^n_{s-})E'[|Y^{n+1}_{s-}{}' - Y^n_{s-}{}'| + |Y^{n+1}_{s-} - Y^n_{s-}| \\
&\quad + \|Z^n_{s-}{}' - Z^{n-1}_{s-}{}'\|_{X_{s-}} + \|Z^{n+1}_{s-} - Z^n_{s-}\|_{X_{s-}}]\big\}ds \\
&\quad - \int_t^T E\|Z^{n+1}_s - Z^n_s\|^2_{X_{s-}} ds.
\end{aligned}$$

With the help of (A1) and Young's inequality, for any $\rho > 0$, we have

$$\begin{aligned}
&E|Y^{n+1}_t - Y^n_t|^2 \\
&\le 2C\int_t^T E\big\{(Y^{n+1}_{s-} - Y^n_{s-})E'[|Y^{n+1}_{s-}{}' - Y^n_{s-}{}'| + |Y^{n+1}_{s-} - Y^n_{s-}|
\end{aligned}$$

$$
\begin{aligned}
&+\|Z^{n\prime}_{s-}-Z^{n-1\prime}_{s-}\|_{X_{s-}}+\|Z^{n+1}_{s-}-Z^n_{s-}\|_{X_{s-}}]\big\}ds\\
&-\int_t^T E\|Z^{n+1}_s-Z^n_s\|^2_{X_{s-}}ds\\
\le\ &(4C+\frac{2C}{\rho})\int_t^T E\big[|Y^{n+1}_{s-}-Y^n_{s-}|^2ds+\rho C\int_t^T E\|Z^n_{s-}-Z^{n-1}_{s-}\|^2_{X_{s-}}ds\\
&+(\rho C-1)\int_t^T E\|Z^{n+1}_s-Z^n_s\|^2_{X_{s-}}ds.
\end{aligned}
$$

Define $k=4C+\frac{2C}{\rho}$, by the backward Grönwall's inequality, we obtain

$$
\begin{aligned}
&E|Y^{n+1}_t-Y^n_t|^2\\
\le\ &\rho C\int_t^T E\|Z^n_s-Z^{n-1}_s\|^2_{X_{s-}}ds+(\rho C-1)\int_t^T E\|Z^{n+1}_s-Z^n_s\|^2_{X_{s-}}ds\\
&+ke^{-kt}\int_t^T e^{-ks}\big[\int_s^T \rho C E\|Z^n_u-Z^{n-1}_u\|^2_{X_{u-}}du\\
&+(\rho C-1)\int_s^T E\|Z^{n+1}_u-Z^n_u\|^2_{X_{u-}}du\big]ds.
\end{aligned}
\tag{14}
$$

Choosing $\rho=\frac{1}{3C}$, we get

$$
\begin{aligned}
&\int_t^T E\|Z^{n+1}_s-Z^n_s\|^2_{X_{s-}}ds+ke^{-kt}\int_t^T e^{ks}\int_s^T E\|Z^{n+1}_u-Z^n_u\|^2_{X_{u-}}duds\\
\le\ &\frac{1}{2}\big(\int_t^T E\|Z^n_s-Z^{n-1}_s\|^2_{X_{s-}}ds+ke^{-kt}\int_t^T e^{ks}\int_s^T E\|Z^n_u-Z^{n-1}_u\|^2_{X_{u-}}duds\big).
\end{aligned}
$$

Iterating above inequality implies that $\{Z^n\}$ is a Cauchy sequence in $H^2_{X,\mathbb{F}}(\mathbb{R}^N)$ under the equivalent norm.

By (14), we know that $\{Y^n\}$ is a Cauchy sequence in $H^2_{\mathbb{F}}(\mathbb{R})$. We denote their limits by Y and Z respectively. By (A2) and the dominated convergence theorem, for any $t\in[0,T]$, we have

$$
\int_t^T E\big|E'[f(s,Y^{n+1\prime}_{s-},Z^{n\prime}_s,Y^{n+1}_{s-},Z^{n+1}_s)-f(s,Y'_{s-},Z'_s,Y_{s-},Z_s)]\big|ds\to 0, n\to\infty.
$$

We now pass to the limit on both sides of (13), it follows that (Y,Z) is the unique solution of mean-filed BSDE (4). □

4. A comparison theorem

In this section, we discuss a comparison theorem for the solutions of one-dimensional mean-field BSDEs on Markov chains.

Let (Y^1,Z^1) and (Y^2,Z^2) be respectively the solutions for the following two mean-field BSDEs

$$
Y^i_t=\xi^i+\int_t^T E'[f_i(s,Y^{i\prime}_s,Y^i_s,Z^{i\prime}_s,Z^i_s)]ds-\int_t^T Z^i_s dM_s,
\tag{15}
$$

where $i=1,2$.

Theorem 4.1. *Assume that* f_1, f_2 *satisfy* (A1) *and* (A2), $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P)$. *Moreover, we suppose*:
(i) $\xi^1 \geq \xi^2$, *P-a.s.*;
(ii) *for any* $t \in [0,T]$,

$$f_1(\omega', \omega, t, Y_t^{2\prime}, Z_t^{2\prime}, Y_t^2, Z_t^2) \geq f_2(\omega', \omega, t, Y_t^{2\prime}, Z_t^{2\prime}, Y_t^2, Z_t^2), \bar{P}\text{-a.s.}$$

It is then true that $Y^1 \geq Y^2$ *on* $[0,T]$, *P-a.s.*

Proof. We omit the ω', ω and s for clarity. By assumption (i), $(\xi^2-\xi^1)^+ = 0$, a.s.. Since for $t\in[0,T]$, $(Y_t^2 - Y_t^1)^+ = \frac{1}{2}[|Y_t^2 - Y_t^1| + (Y_t^2 - Y_t^1)]$, then by the Stieltjes chain rule for products, we have

$$\begin{aligned}
&((Y_t^2 - Y_t^1)^+)^2 \\
=&\ -2\int_t^T (Y_s^2 - Y_s^1)^+ d(Y_s^2 - Y_s^1)^+ - \sum_{t<s\leq T} \Delta(Y_s^2 - Y_s^1)^+ \Delta(Y_s^2 - Y_s^1)^+ \\
=&\ -\int_t^T (Y_s^2 - Y_s^1)^+ d[|Y_s^2 - Y_s^1| + (Y_s^2 - Y_s^1)] \\
&- \sum_{t<s\leq T} \Delta(Y_s^2 - Y_s^1)^+ \Delta(Y_s^2 - Y_s^1)^+ \\
=&\ -\int_t^T (Y_s^2 - Y_s^1)^+ d|Y_s^2 - Y_s^1| - \int_t^T (Y_s^2 - Y_s^1)^+ d(Y_s^2 - Y_s^1) \\
&- \sum_{t<s\leq T} \Delta(Y_s^2 - Y_s^1)^+ \Delta(Y_s^2 - Y_s^1)^+ \\
=&\ -2\int_t^T I_{\{Y_s^2 > Y_s^1\}} (Y_s^2 - Y_s^1) d(Y_s^2 - Y_s^1) \\
&- \sum_{t<s\leq T} I_{\{Y_s^2 > Y_s^1\}} \Delta(Y_s^2 - Y_s^1) \Delta(Y_s^2 - Y_s^1) \\
=&\ -2\int_t^T I_{\{Y_s^2 > Y_s^1\}} (Y_s^2 - Y_s^1) d(Y_s^2 - Y_s^1) \\
&- \sum_{t<s\leq T} I_{\{Y_s^2 > Y_s^1\}} |(Z_s^2 - Z_s^1)\Delta M_s|^2.
\end{aligned}$$

For $t \in [0,T]$, by assumption (ii), (A1) and Young's inequality, for any $\rho > 0$, we have

$$\begin{aligned}
&E((Y_t^2 - Y_t^1)^+)^2 + E\int_t^T I_{\{Y_s^2 > Y_s^1\}} \|(Z_s^2 - Z_s^1)\|_{X_{s-}}^2 ds \\
=&\ 2\int_t^T E I_{\{Y_s^2 > Y_s^1\}} (Y_s^2 - Y_s^1) E'[f_2(Y_s^{2\prime}, Z_s^{2\prime}, Y_s^2, Z_s^2) - f_1(Y_s^{1\prime}, Z_s^{1\prime}, Y_s^1, Z_s^1)] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_t^T E\big\{I_{\{Y_s^2>Y_s^1\}}(Y_s^2-Y_s^1)E'[f_1(Y_s^{2\prime},Z_s^{2\prime},Y_s^2,Z_s^2)-f_1(Y_s^{1\prime},Z_s^{1\prime},Y_s^1,Z_s^1)]\big\}ds\\
&\leq 2C\int_t^T E\big\{I_{\{Y_s^2>Y_s^1\}}(Y_s^2-Y_s^1)[|Y_s^2-Y_s^1|+\|(Z_s^2-Z_s^1)\|_{X_{s-}}\\
&\quad +E'|Y_s^{2\prime}-Y_s^{1\prime}|+E'\|(Z_s^{2\prime}-Z_s^{1\prime})\|_{X_{s-}}]\big\}ds\\
&\leq 2C\int_t^T E[(Y_s^2-Y_s^1)^+]^2ds\\
&\quad +2C\int_t^T E\big\{I_{\{Y_s^2>Y_s^1\}}(Y_s^2-Y_s^1)E[I_{\{Y_s^2>Y_s^1\}}|Y_s^2-Y_s^1|]\big\}ds\\
&\quad +\frac{2C}{\rho}\int_t^T E[(Y_s^2-Y_s^1)^+]^2ds+2\rho CE\int_t^T I_{\{Y_s^2>Y_s^1\}}\|(Z_s^2-Z_s^1)\|_{X_{s-}}^2ds\\
&\leq (4C+\frac{2C}{\rho})\int_t^T E[(Y_s^2-Y_s^1)^+]^2ds\\
&\quad +2\rho CE\int_t^T I_{\{Y_s^2>Y_s^1\}}\|(Z_s^2-Z_s^1)\|_{X_{s-}}^2ds.
\end{aligned}
$$

Choosing $\rho=\frac{1}{2C}$, it follows from Gronwall's inequality that $E((Y_t^2-Y_t^1)^+)^2=0, t\in[0,T]$. It is then rue that $Y^1\geq Y^2$ on $[0,T]$, P-a.s. The proof is complete. □

Remark 4.2. Compared to the comparison results in Cohen and Elliott [8], our assumptions on coefficients f_1 and f_2 are natural. Moreover, we don't make restrictions on the two solutions, hence it's easier to use.

Acknowledgements. We thank the editors Yongnam Lee and Hyun Sun Kim and the referees for their valuable comments and help. The first author acknowledges the partial support from the National Natural Science Foundation of China (71672166).

References

[1] N. Ahmed, *Nonlinear diffusion governed by McKean-Vlasov equation on Hilbert space and optimal control*, SIAM J. Control Optim. **46** (2007), no. 1, 356–378.

[2] N. Ahmed and X. Ding, *A semilinear McKean-Vlasov stochastic evolution equation in Hilbert space*, Stochastic Process. Appl. **60** (1995), no. 1, 65–85.

[3] V. Borkar and K. Kumar, *McKean-Vlasov limit in portfolio optimization*, Stoch. Anal. Appl. **28** (2010), no. 5, 884–906.

[4] R. Buckdahn, B. Djehiche, J. Li, and S. Peng, *Mean-field backward stochastic differential equations: a limit approach*, Ann. Probab. **37** (2009), no. 4, 1524–1565.

[5] R. Buckdahn, J. Li, and S. Peng,*Mean-field backward stochastic differential equations and related partial differential equations*, Stoch. Process. Appl. **119** (2009), no. 10, 3133–3154.

[6] T. Chan, *Dynamics of the McKean-Vlasov equation*, Ann. Probab. **22** (1994), no. 1, 431–441.

[7] S. N. Cohen and R. J. Elliott, *Solutions of backward stochastic differential equations on Markov chains*, Commun. Stoch. Anal. **2** (2008), no. 2, 251–262.

[8] ______, *Comparisons for backward stochastic differential equations on Markov chains and related no-arbitrage conditions*, Ann. Appl. Probab. **20** (2010), no. 1, 267–311.

[9] ______, *Existence, uniqueness and comparisons for BSDEs in general spaces*, Ann. Probab. **40** (2012), no. 5, 2264–2297.

[10] D. Crisan and J. Xiong, *Approximate McKean-Vlasov representations for a class of SPDEs*, Stochastics **82** (2010), no. 1-3, 53–68.

[11] N. El Karoui, S. Peng, and M. Quenez, *Backward stochastic differential equations in finance*, Math. Finance **7** (1997), no. 1, 1–71.

[12] R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models*: *Estimation and Control*, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[13] S. Hamadène and J. Lepeltier, *Zero-sum stochastic differential games and backward equations*, Systems Control Lett. **24** (1995), no. 4, 259–263.

[14] M. Kac, *Foundations of kinetic theory*, In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954955, vol. III, pp. 171–197. University of California Press, Berkeley and Los Angeles, 1956.

[15] P. Kotelenez, *A class of quasilinear stochastic partial differential equations of McKean-Vlasov type with mass conservation*, Probab. Theory Related Fields **102** (1995), no. 2, 159–188.

[16] P. Kotelenez and T. Kurtz,*Macroscopic limit for stochastic partial differential equations of McKean-Vlasov type*, Probab. Theory Related Fields **146** (2010), no. 1-2, 189–222.

[17] J. Lasry and P. Lions, *Mean field games*, Jpn J. Math. **2** (2007), no. 1, 229–260.

[18] J. Li, *Reflected Mean-Field Backward Stochastic Differential Equations, Approximation and Associated Nonlinear PDEs*, J. Math. Anal. Appl. **413** (2014), no. 1, 47–68.

[19] Z. Li and J. Luo, *Mean-field reflected backward stochastic differential equations*, Statist. Probab. Lett. **82** (2012), no. 11, 1961–1968.

[20] W. Lu and Y. Ren, *Anticipated backward stochastic differential equations on Markov chains*, Statist. Probab. Lett. **83** (2013), no. 7, 1711–1719.

[21] H. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 1907–1911.

[22] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett. **14** (1990), no. 1, 55–61.

[23] S. Peng, *Backward stochastic differential equations and applications to optimal control*, Appl. Math. Optim. **27** (1993), no. 2, 125–144.

[24] R. Xu, *Mean-field backward doubly stochastic differential equations and related SPDEs*, Bound. Value Probl. **2012** (2012), 114, 20 pp.

Wen Lu
School of Mathematics and Informational Science
Yantai University
Yantai 264005, P. R. China
E-mail address: `llcxw@163.com`

Yong Ren
Department of Mathematics
Anhui Normal University
Wuhu 241000, P. R. China
E-mail address: `renyong@126.com`