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A RECURSIVE FORMULA FOR THE KHOVANOV COHOMOLOGY OF KANENOBU KNOTS

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Abstract. Kanenobu has given infinite families of knots with the same HOMFLY polynomial invariant but distinct Alexander module structure. In this paper, we give a recursive formula for the Khovanov cohomology of all Kanenobu knots $K(p,q)$, where p and q are integers. The result implies that the rank of the Khovanov cohomology of $K(p,q)$ is an invariant of $p+q$. Our computation uses only the basic long exact sequence in knot homology and some results on homologically thin knots.

1. Introduction

In recent years, there has been tremendous interest in developing Khovanov cohomology theory [9]. One of the main advantages of the theory is that its definition is combinatorial and there is a straightforward algorithm for computing Khovanov cohomology groups of links. Consequently, there are various computer programs that efficiently calculate them with 50 crossings and more (See [3, 22]).

Since then, some results about the cohomology groups of larger classes of links have been obtained. E. S. Lee [12] pointed out that ranks of the cohomology groups of "alternating links" were determined by their Jones polynomial and signature. The notion of "quasi-alternating links" was introduced by P. Ozsváth and Z. Szabó in [18]. Furthermore, it was shown quasi-alternating links were homologically thin for both Khovanov cohomology and knot Floer homology in [15].

J. Greene [5] exhibited the first examples of links which were homologically thin but "not quasi-alternating", including the $11n50$ knot. L. Watson [25, 26] pointed out the knot $11n50$ occurred as $K(0,3)$ in Kanenobu knots whose HOMFLY polynomial depended only on $p+q$. Amongst them, the knot $K(1,2)=K(2,1)=11n132$ was quasi-alternating. However, J. Greene conjectured that this is the only knot in this family that is quasi-alternating

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(See Conjecture 3.1 of [5]). Later K. Qazaqzeh and N. Chbili [19] showed that there were only finite quasi-alternating links in the family of Kanenobu knots. In this sense, the Kanenobu knots can be seen as the "most non-quasi-alternating" class of knots (See [5, 6]). However, a formula for calculating cohomology groups of general Kanenobu knots remains unknown.

Let $K(p,q)$ be the Kanenobu knots of type (p,q). For simplicity, we denote the Khovanov cohomology groups, odd Khovanov cohomology groups [16] by $H(K(p,q))$ and $H^{odd}(K(p,q))$ respectively. In this paper, we will concentrate on the cohomology groups with coefficients in $\mathbb{F}_2$ (the field of two elements).

In [5], the following formula was obtained by J. Greene.

Lemma 1.1 ([5, Theorem 7]). *For all $p,q \in \mathbb{Z}$,*

$$H^{i,j}((K(p,q);\mathbb{Z}) = H^{i,j}(K(p+1,q-1);\mathbb{Z}).$$

In Proposition 3.6 of the present paper, we first generalize this formula to the case with coefficients $\mathbb{F}_2$. Then in Theorem 3.8, we can give a calculating formula of $H^{i,j}((K(p,0))$. Combining with Theorem 3.15, we give a recursive formula for the Khovanov cohomology of Kanenobu knots $K(p,q)$.

The organization of the coming sections is as follows. In Section 2, we review some fundamental facts on Kanenobu knots, and give a brief summary of the Khovanov cohomology. We show the main theorems about the calculation for $H^{i,j}(K(p,q))$ and give some corollaries in Section 3.

2. Preliminaries

2.1. The Kanenobu knots

The knot shown as in Fig. 1 is called Kanenobu knots $K(p,q)$, where p, q are the number of half twists.

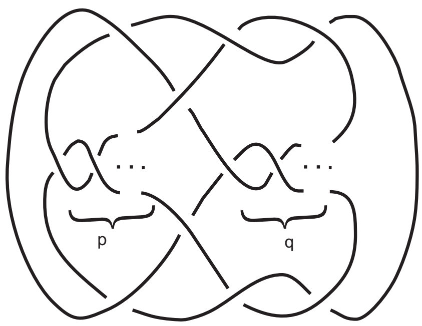

FIGURE 1. Kanenobu knots $K(p,q)$.

Kanenobu knots with small crossings are the following knots [8].

$$K(0,0) = 4_1\sharp 4_1,\ K(0,-1) = 8_8,\ K(1,-1) = 8_9,\ K(2,-1) = 10_{129},$$

$K(2,0) = 10_{137},\ K(1,1) = 10_{155},\ K(2,-3) = 13_{6714}.$

Lemma 2.1 ([7]). *There exist infinitely many examples in Kanenobu knots which are ribbon, hyperbolic, fibred, genus 2 and 3-bridge, with the same HOMFLY polynomial invariant and, therefore, the same Jones polynomial but distinct Alexander module structure.*

Remark 2.2. Two Kanenobu knots $K(p_1, q_1)$ and $K(p_2, q_2)$ are in the same subfamily of Kanenobu knots mentioned in Lemma 2.1 if and only if they satisfy the following conditions.

(1) $p_1 + q_1 = p_2 + q_2$,
(2) $|p_1 - q_1| \neq |p_2 - q_2|$,
(3) $(p, q) \neq (0, 0)$.

We summarize some main properties of Kanenobu knots as follows.

Lemma 2.3 ([7], [8]).

(1) $K(p_1, q_1) \cong K(p_2, q_2)$ *if and only if* $(p_1, q_1) = (p_2, q_2)$ *or* $(p_1, q_1) = (q_2, p_2)$,
(2) $K^!(p, q) \cong K(-p, -q)$ *where* $K^!$ *denotes the mirror image of* K,
(3) $K(p, q)$ *is prime except for* $K(0, 0)$.

2.2. Khovanov cohomology

Given an oriented link diagram D, Khovanov builds the "cube of resolutions" from it, where a cube is of all possible 0 and 1 resolutions of the crossing as shown in Fig. 2.

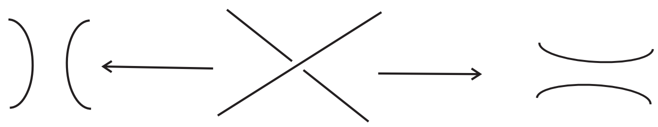

FIGURE 2. The resolution 0 and 1 of a crossing

Khovanov constructs a chain complex of bigraded modules $\overline{C}^{i,j}(D)$. We refer to the i-degree as homological degree and j-degree as quantum degree. This construction is dependent of the choice of a link diagram. Just as the Jones polynomial needs a writher correction term, this construction needs an overall normalization to be a link invariant.

Let $x(D)$ and $y(D)$ be the number of negative crossings and positive crossings of D respectively. $C^{i,j}(D) = \overline{C}^{i,j}(D)[x(D)]\{2x(D) - y(D)\}$, where $[\cdot]$ and $\{\cdot\}$ denote the shift of the cohomological degree and quantum degree respectively. We denote $\overline{H}^{i,j}(D)$ and $H^{i,j}(D)$ to be the cohomology group of the complex $\overline{C}^{i,j}(D)$ and $C^{i,j}(D)$ respectively. Then

$$H^{i,j}(D) = \overline{H}^{i+x(D),j+2x(D)-y(D)}(D).$$

Proposition 2.4 ([9, Proposition 9]). *Let $R = \mathbb{Z}[c]$ denote the ring of polynomials with integral coefficients. The bigraded R-modules $H(L) := H(D)$ will be an invariant of link L. In addition, the graded Euler characteristic of the complex $C(D)$ is equal to the Jones polynomial of the link L, that is,*

$$V_L(t) = (q + q^{-1})^{-1} \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}}(H^{i,j}(L) \otimes_{\mathbb{Z}} \mathbb{Q})|_{q=-t^{1/2}},$$

where $V_L(t)$ is the Jones polynomial of L.

We denote $D(*0)$ and $D(*1)$ by the diagram obtained by applying the 0-resolution and the 1-resolution at the crossing of D. It is clear that $C(D(*0))$ and $C(D(*1))[-1]\{-1\}$ are subcomplexes of $C(D)$ and form a short exact sequence.

$$0 \to \overline{C}(D(*1))[-1]\{-1\} \to \overline{C}(D) \to \overline{C}(D(*0)) \to 0.$$

It induces a long exact sequence on cohomology as follows.

$$\cdots \to \overline{H}^{i-1,j}(D(*0)) \to \overline{H}^{i-1,j-1}(D(*1)) \to \overline{H}^{i,j}(D) \to \overline{H}^{i,j}(D(*0)) \to \overline{H}^{i,j-1}(D(*1)) \to \cdots$$

3. Khovanov cohomology of Kanenobu knots

We denote $P(L)$ to be the graded Poincaré polynomial of Khovanov cohomology of the oriented link L, that is,

$$P(L)(t,q) := \sum_{i,j\in\mathbb{Z}} t^i q^j \dim H^{i,j}(L),$$

where the Poincaré polynomial of the cohomology use coefficients in $\mathbb{F}_2$.

The following lemmas will be used in our calculation.

Lemma 3.1 ([23, Theorem 5.2]). *For the homologically thin knot L, if its cohomology is concentrated on diagonals $j = s - 1 + 2i$ and $j = s + 1 + 2i$, then there exists a polynomial $P'(L)$ such that*

$$P(L) = q^{s-1}(1 + q^2)(1 + (1 + tq^2)P'(L),$$

where $P'(L)$ is a polynomial in tq^2.

Lemma 3.2 ([13, Theorem 1.2 and Theorem 4.3]). *For the homologically thin knot L, the integer s is equal to the signature of L and the polynomial $P'(L)$ contains only powers of tq^2.*

Lemma 3.3 ([13, Theorem 1.4]). *If L is the homologically thin knot with $s(L) = 0$ and $j - 2i = \sigma(k) \pm 1$, then*

$$\overline{H}^{x(D),2x(D)-y(D)-1}(D) = \overline{H}^{x(D)+1,2x(D)-y(D)+3}(D) \oplus \mathbb{F}_2,$$

$$\overline{H}^{x(D),2x(D)-y(D)+1}(D) = \overline{H}^{x(D)-1,2x(D)-y(D)-3}(D) \oplus \mathbb{F}_2,$$

$$H^{i,j}(D) = H^{i+1,j+4}(D).$$

Next, we shall give the calculating formula of $H^{i,j}((K(p,0))$ over $\mathbb{F}_2$.

For save further space, we've underlined negative numbers $\underline{1} := -1$ and used the notation a^r_m to denote the monomial at^rq^m. We've suppressed all " + " signs (See [2]).

Theorem 3.4. *Let 4_1 denote the figure eight knot. The Khovanov cohomology groups of 4_1 are given as follows.*

$$H^{i,j}(4_1) = \begin{cases} \mathbb{F}_2, & (i,j) \in \{(0,1),(1,1),(1,3),(2,3),(2,5),(0,-1), \\ & \quad (-1,-1),(-1,-3),(-2,-3),(-2,-5)\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. As $s(4_1) = 0$, according to Lemmas 3.1 and 3.2, it follows that

$$P'(4_1) = 1\frac{2}{\underline{4}}1\frac{1}{2} = \frac{1}{q^4t^2} + q^2t,$$

$$\begin{aligned} P(4_1) &= q^{-1}(1+q^2)[1+(1+tq^2)(q^{-4}t^{-2}+q^2t)] \\ &= (q^{-1}+q)(1+q^{-4}t^{-2}+q^2t+t^{-1}q^{-2}+t^2q^4) \\ &= q^{-1}+q^{-5}t^{-2}+qt+t^{-1}q^{-3}+t^2q^3+q+q^{-3}t^{-2}+q^3t+t^{-1}q^{-1} \\ &\quad + t^2q^5 \\ &= t^0(q+q^{-1})+t^{-1}(q^{-1}+q^{-3})+t(q+q^3)+t^2(q^3+q^5) \\ &\quad + t^{-2}(q^{-3}+q^{-5}) \\ &= t^0q^1+t^0q^{-1}+t^{-1}q^{-1}+t^{-1}q^{-3}+t^1q^1+t^1q^3+t^2q^3+t^2q^5 \\ &\quad + t^{-2}q^{-3}+t^{-2}q^{-5}. \end{aligned}$$

This completes the proof. □

Proposition 3.5. *$s(K(p,0)) = 0$, where $s(L)$ denotes the Rasmussen s-invariant of L, as is given in* [21].

Proof. The Kanenobu knot $K(p,0)$ is a ribbon knot, so it is a slice knot. The inequality $|s(K)| \leq 2g^*(K)$ ([21]), where $g^*(K)$ is the slice genus of the knot K, thus $s(K(p,0)) = 0$. □

In order to describe Khovanov cohomology over a ring R in terms of integral Khovanov cohomology, we use the universal coefficient theorem in homological algebra.

Since $C(D,R) = C(D,\mathbb{Z}) \otimes_{\mathbb{Z}} R$, the universal coefficient theorem tells us that there is a short exact sequence.

$$0 \to H^{i,j}(D;\mathbb{Z}) \otimes_{\mathbb{Z}} R \to H^{i,j}(D;R) \to Tor(H^{i-1,j}(D;\mathbb{Z}),R).$$

Therefore, $H^{i,j}(D;R) = H^{i,j}(D;\mathbb{Z}) \otimes_{\mathbb{Z}} R \oplus Tor(H^{i-1,j}(D;\mathbb{Z}),R)$.

Proposition 3.6. *For any $p,q \in \mathbb{Z}$, then*

$$H^{i,j}(K(p,q);\mathbb{F}_2) = H^{i,j}(K(p-1,q+1);\mathbb{F}_2).$$

Proof. Consider $R = \mathbb{F}_2$ and $D = K(p,q)$. According to Lemma 1.1 and the universal coefficient theorem, the proof is trivial. □

Theorem 3.7. *The khovanov cohomology groups of the Kanenobu knot $K(0,0)$ are given by the following.*

$$H^{i,j}(K(0,0)) = \begin{cases} {\mathbb{F}_2}^{\oplus 5}, & (i,j) = \{(0,1),(0,-1)\}; \\ {\mathbb{F}_2}^{\oplus 4}, & (i,j) = \{(1,1),(1,3),(-1,-1),(-1,-3)\}; \\ {\mathbb{F}_2}^{\oplus 3}, & (i,j) = \{(2,3),(2,5),(-2,-3),(-2,-5)\}; \\ {\mathbb{F}_2}^{\oplus 2}, & (i,j) = \{(3,5),(3,7),(-3,-5),(-3,-7)\}; \\ \mathbb{F}_2, & (i,j) = \{(4,7),(4,9),(-4,-7),(-4,-9)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $K(0,0) = 4_1\sharp 4_1$, it is a homologically thin knot and $K(1,-1) = 8_9$, by Proposition 3.6, it follows that

$$H^{i,j}(K(0,0)) = H^{i,j}(K(1,-1)) = H^{i,j}(8_9).$$

By Lemmas 3.1, 3.2 and $s(8_9) = 0$, we obtain

$$\begin{aligned} P'(8_9) &= 1\frac{4}{\underline{8}}1\frac{3}{\underline{6}}2\frac{2}{\underline{4}}2\frac{1}{\underline{2}}2^0_0 2^1_2 1^2_4 1^3_6 \\ &= \frac{1}{q^8t^4} + \frac{1}{q^6t^3} + \frac{2}{q^4t^2} + \frac{2}{q^2t^1} + 2 + 2q^2t + q^4t^2 + q^6t^3, \end{aligned}$$

$$\begin{aligned} P(K(0,0)) &= q^{-1}(1+q^2)[1+(1+tq^2)P'(8_9)] \\ &= tq^{-1} + q^{-9}t^{-4} + q^{-7}t^{-3} + 2q^{-5}t^{-2} + 2t^{-1}q^{-3} + 2q^{-1} + 2qt \\ &\quad + q^3t^2 + q^5t^3 + t^{-3}q^{-7} + t^{-2}q^{-5} + 2q^{-3}t^{-1} + 2q^{-1} + 2qt \\ &\quad + 2t^2q^3 + t^3q^5 + t^4q^7 + q^{-7}t^{-4} + q^{-5}t^{-3} + 2t^{-12}q^{-3} + 2t^{-1}q^{-1} \\ &\quad + 5q + 2q^3t + t^2q^5 + q^7t^3 + t^{-3}q^{-5}) + t^{-2}q^{-3} + 2q^{-1}t^{-1} + 2q \\ &\quad + 2tq^3 + 2t^2q^5 + t^3q^7 + t^4q^9 = 5tq^{-1} + q^{-9}t^{-4} + 2q^{-7}t^{-3} \\ &\quad + 2q^{-5}t^{-2} + 2t^{-3}q^{-5} + 4t^{-1}q^{-3} + 2q^{-1} + 4qt + 3q^3t^2 + 2q^5t^3 \\ &= 4t^3q^7 + t^4q^7 + 3t^{-2}q^{-3} + 4q^{-1}t^{-1} + 5q^1 + 4t^1q^3 + 3t^2q^5. \end{aligned}$$

We have thus proved the theorem. □

Theorem 3.8. *The Kanenobu knot $K(p,0)$ for a negative integer p is homologically thin over $\mathbb{F}_2$ and its Khovanov cohomology groups are given as follows.*

$$H^{i,j}(K(p,0)) = \begin{cases} H^{-p,-2p+1}(K(0,0)) \oplus F_2, & \{(i,j) = (0,1)\}; \\ H^{-p,-2p-1}(K(0,0)) \oplus F_2, & \{(i,j) = (0,-1)\}; \\ H^{-p-1,-2p-3}(K(0,0)) \oplus F_2, & \{(i,j) = (-1,-3),\ p \neq -1\}; \\ H^{-p-1,-2p-1}(K(0,0)), & \{(i,j) = (-1,-1),\ p \neq -1\}; \\ H^{i-p,j-2p}(K(0,0)), & \text{otherwise} \end{cases}$$

and

$$H^{i,j}(K(p,0)) \oplus \mathbb{F}_2 = \begin{cases} H^{0,1}(K(0,0)), & \{(i,j) = (p, 2p+1)\}; \\ H^{0,-1}(K(0,0)), & \{(i,j) = (p, 2p-1)\}. \end{cases}$$

To prove the theorem, we need to resolve any crossing of the P-crossings to obtain $D(*0)$ and $D(*1)$. It is clear that $D(*0)$ is a diagram of the Kanenobu knot $K(p+1,0)$ and $D(*1)$ is a diagram of unlink of two components. Since p is a negative integer, the theorem will be proved by induction on $|P|$.

Now we will need several lemmas.

Lemma 3.9. *$K(p,0)$ is homologically thin over $\mathbb{F}_2$.*

Proof. From the induction hypothesis, it follows that $K(p+1,0)$ is homologically thin. Therefore, $H^{i,j}(K(p+1,0))$ is supported in two diagonal lines $j-2i = \sigma(K(p+1,0)) \pm 1 = \pm 1$. We conclude that the $K(p,0)$ is homologically thin as its cohomology is supported in two lines $j - 2i = -4 \pm 1$. □

Lemma 3.10. *Let $D(*1)$ be a diagram of unlink of two components, its Khovanov cohomology groups are given as follows.*

$$H^{i,j}(D(*1)) = \begin{cases} F_2, & \{(i,j) = (3-p, 4-2p), (3-p, -2p)\}; \\ F_2 \oplus F_2, & \{(i,j) = (3-p, 2-2p)\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Since $x(D(*1)) = 4 - (p+1) = 3 - p$ and $y(D(*1)) = 4$, it follows that $H^{i,j}(D(*1)) = \overline{H}^{i+3-p, j+[6-2p-4]}(D(*1)) = \overline{H}^{i+3-p, j+2-p}(D(*1))$.

Let $i + 3 - p = i'$, $j + 2 - p = j'$, we have

$$\begin{aligned} j' - 2i' &= j + 2 - 2p - 2(i + 3 - p) = j + 2 - 2p - 2i - 6 + 2p \\ &= j - 2i - 4 = -4 \pm 1. \end{aligned}$$

It can be easily seen that the cohomology of $(D(*1))$ is

$$\overline{H}^{i,j}(D(*1)) = \begin{cases} F_2, & \{(i,j) = (0,2), (0,-2)\}; \\ F_2 \oplus F_2, & \{(i,j) = (0,0)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$H^{i,j}(D(*1)) = \begin{cases} F_2, & \{(i,j) = (3-p, 4-2p), (3-p, -2p)\}; \\ F_2 \oplus F_2, & \{(i,j) = (3-p, 2-2p)\}; \\ 0, & \text{otherwise.} \end{cases}$$

□

Lemma 3.11. *The following formulas are established.*

$$\begin{aligned} &\overline{H}^{4-p,5-2p}(K(p,0)) = \overline{H}^{4-p,5-2p}(K(p+1,0)) \oplus \mathbb{F}_2, \\ &\overline{H}^{3-p,1-2p}(K(p,0)) \oplus \mathbb{F}_2 = \overline{H}^{3-p,1-2p}(K(p+1,0)), \\ &\overline{H}^{4-p,1-2p}(K(p,0)) = 0, \\ &\overline{H}^{3-p,3-2p}(K(p,0)) \oplus \mathbb{F}_2 = \overline{H}^{3-p,3-2p}(K(p+1,0)), \end{aligned}$$

$$\overline{H}^{4-p,3-2p}(K(p,0)) = \overline{H}^{4-p,3-2p}(K(p+1,0)) \oplus \mathbb{F}_2.$$

Proof. By the long exact sequence on cohomology

$$\cdots \to \overline{H}^{i-1,j}(D*0) \to \overline{H}^{i-1,j-1}(D*1) \to \overline{H}^{i,j}(D) \to \overline{H}^{i,j}(D*0)$$
$$\to \overline{H}^{i,j-1}(D*1) \to \cdots$$

we obtain $\overline{H}^{i,j}(K(p,0)) = \overline{H}^{i,j}(K(p+1,0))$ except in the following three cases:

Case 1. By Lemma 3.10, it follows that $\overline{H}^{3-p,4-2p}(D*1) = \mathbb{F}_2$. Then the long exact sequence is given as follows.

$$0 \to \mathbb{F}_2 \to \overline{H}^{4-p,5-2p}(K(p,0)) \to \overline{H}^{4-p,5-2p}(K(p+1,0)) \to 0.$$

Therefore, $\overline{H}^{4-p,5-2p}(K(p,0)) = \overline{H}^{4-p,5-2p}(K(p+1,0)) \oplus \mathbb{F}_2$.

Case 2. By Lemma 3.10, it follows that $\overline{H}^{3-p,-2p}(D*1) = \mathbb{F}_2$. Therefore, we obtain the following long exact sequence.

$$0 \to \overline{H}^{3-p,1-2p}(K(p,0)) \to \overline{H}^{3-p,1-2p}(K(p+1,0)) \to \mathbb{F}_2$$
$$\to \overline{H}^{4-p,1-2p}(K(p,0) \to 0.$$

We have two subcases and only the second holds.

Case 2-(1). By Lemma 3.3, it follows that

$$\overline{H}^{3-p,1-2p}(K(p+1,0)) = \overline{H}^{4-p,5-2p}(K(p+1,0)) \oplus \mathbb{F}_2.$$

Suppose $\overline{H}^{3-p,1-2p}(K(p,0)) = \overline{H}^{3-p,1-2p}(K(p+1,0))$ and $\overline{H}^{4-p,1-2p}(K(p,0)) = \mathbb{F}_2$, then $\overline{H}^{3-p,1-2p}(K(p,0)) = \overline{H}^{4-p,5-2p}(K(p,0))$. In the spectral sequence,

$$0 \to \overline{H}^{3-p,1-2p}(K(p,0)) \to \overline{H}^{4-p,5-2p}(K(p,0)) \to 0.$$

The differential has to be injective and not surjective, since in the E_∞ page one copy of $\mathbb{F}_2$ survives at the j grading 1 instead of the j grading -3. This contradicts the fact that the domain and the codomain have the same dimension.

Case 2-(2). Consequently, we infer that

$$\overline{H}^{3-p,1-2p}(K(p,0)) \oplus \mathbb{F}_2 = \overline{H}^{3-p,1-2p}(K(p+1,0)),$$
$$\overline{H}^{4-p,1-2p}(K(p,0)) = 0.$$

Case 3. In the following long exact sequence

$$\cdots \to \overline{H}^{1-p,-2-2p}(D(*1)) \to \overline{H}^{2-p,-1-2p}(K(p,0))$$
$$\to \overline{H}^{2-p,-1-2p}(K(p+1,0)) \to \overline{H}^{2-p,-2-2p}(D(*1)) \to \cdots$$

by Lemma 3.10, we obtain

$$\overline{H}^{1-p,-2-2p}(D(*1)) = \overline{H}^{2-p,-2-2p}(D(*1)) = 0.$$

Therefore,

$$\overline{H}^{2-p,-1-2p}(K(p,0)) = \overline{H}^{2-p,-1-2p}(K(p+1,0)).$$

By Lemma 3.10, it follows that $\overline{H}^{3-p,2-2p}(D*1) = \mathbb{F}_2 \oplus \mathbb{F}_2$. Then the long exact sequence is given as follows.

$$\begin{aligned} 0 &\to \overline{H}^{3-p,3-2p}(K(p,0)) \to \overline{H}^{3-p,3-2p}(K(p+1,0)) \to \mathbb{F}_2 \oplus \mathbb{F}_2 \\ &\to \overline{H}^{4-p,3-2p}(K(p,0)) \to \overline{H}^{4-p,3-2p}(K(p+1,0)) \to 0. \end{aligned}$$

As a result of the statement of lemma 3.3, it can be easily seen that

$$\begin{aligned} &\overline{H}^{3-p,3-2p}(K(p+1,0)) = \overline{H}^{2-p,-1-2p}(K(p+1,0)) \oplus F_2, \\ &\overline{H}^{4-p,3-2p}(K(p,0)) = \overline{H}^{5-p,7-2p}(K(p,0)) \oplus F_2, \\ &\overline{H}^{4-p,5-2p}(K(p,0)) = \overline{H}^{3-p,1-2p}(K(p,0)) \oplus F_2. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \overline{H}^{3-p,3-2p}(K(p,0)) \oplus \mathbb{F}_2 &= \overline{H}^{i+(4-p),j+(4-2p)}(K(p,0)) \oplus \mathbb{F}_2 \\ &= H^{-1,-1}(K(p,0)) = H^{-2,-5}(K(p,0)) \oplus \mathbb{F}_2 \\ &= \overline{H}^{2-p,-1-2p}(K(p,0)) \oplus \mathbb{F}_2 \\ &= \overline{H}^{2-p,-1-2p}(K(p+1,0)) \oplus \mathbb{F}_2 \\ &= \overline{H}^{3-p,3-2p}(K(p+1,0)) \end{aligned}$$

and $\overline{H}^{4-p,3-2p}(K(p,0)) = \overline{H}^{4-p,3-2p}(K(p+1,0)) \oplus \mathbb{F}_2$. □

It can be easily seen that $x(K(0,0)) = 4$, $y(K(0,0)) = 4$, $2x(K(0,0)) - y(K(0,0)) = 4$; $x(K(p,0)) = 4-p$, $y(K(p,0)) = 4$, $2x(K(p,0)) - y(K(p,0)) = 4-2p$; $x(K(p+1,0)) = 3-p$, $y(K(p+1,0)) = 4$, $2x(K(p+1,0) - y(K(p+1,0)) = 2-2p$.

These results will be used in the following lemmas.

Lemma 3.12.

$$H^{0,\pm 1}(K(p,0)) = H^{\pm 1,\pm 3}(K(p,0)) \oplus \mathbb{F}_2,$$

$$H^{i,j}(K(p,0)) = H^{i+1,j+4}(K(p,0)), (i \neq 0).$$

Proof. By Lemma 3.3, it follows that

$$\begin{aligned} H^{0,-1}(K(p,0)) &= \overline{H}^{x(K(p,0)),2x(K(p,0))-y(K(p,0))-1}(K(p,0)) \\ &= \overline{H}^{x(K(p,0))+1,2x(K(p,0))-y(K(p,0))+3}(K(p,0)) \oplus \mathbb{F}_2 \\ &= H^{1,3}(K(p,0)) \oplus \mathbb{F}_2. \end{aligned}$$

$$\begin{aligned} H^{0,1}(K(p,0)) &= \overline{H}^{x(K(p,0)),2x(K(p,0))-y(K(p,0))+1}(K(p,0)) \\ &= \overline{H}^{x(K(p,0))-1,2x(K(p,0))-y(K(p,0))-3}(K(p,0)) \oplus \mathbb{F}_2 \\ &= H^{-1,-3}(K(p,0)) \oplus \mathbb{F}_2. \end{aligned}$$

Therefore,

$$\begin{aligned} &H^{0,\pm1}(K(p,0)) = H^{\pm1,\pm3}(K(p,0)) \oplus \mathbb{F}_2, \\ &H^{i,j}(K(p,0)) = H^{i+1,j+4}(K(p,0)), (i \neq 0). \end{aligned}$$

□

Let us now finish the proof of Theorem 3.8. Thus we shall give the calculating formula of $H^{i,j}((K(p,q))$ over $\mathbb{F}_2$.

Proof. By Lemma 3.11, we will prove this in eight cases.

(a) In the following long exact sequence

$$\begin{aligned} \cdots &\to \overline{H}^{3-p,4-2p}(D(*1)) \to \overline{H}^{4-p,5-2p}(K(p+1,0)) \\ &\to \overline{H}^{4-p,5-2p}(K(p+2,0)) \to \overline{H}^{3-p,5-2p}(D(*1)) \to \cdots \end{aligned}$$

by Lemma 3.10, we have

$$\overline{H}^{3-p,4-2p}(D(*1)) = \overline{H}^{3-p,5-2p}(D(*1) = 0.$$

Then $\overline{H}^{4-p,5-2p}(K(p+1,0)) = \overline{H}^{4-p,5-2p}(K(p+2,0))$. Therefore,

$$\begin{aligned} H^{0,1}(K(p,0)) &= \overline{H}^{4-p,5-2p}(K(p,0)) = \overline{H}^{4-p,5-2p}(K(p+1,0)) \oplus \mathbb{F}_2 \\ &= \overline{H}^{4-p,5-2p}(K(p+2,0)) \oplus \mathbb{F}_2 = \cdots \\ &= \overline{H}^{4-p,5-2p}(K(0,0)) \oplus \mathbb{F}_2 = H^{i+4,j+4}(K(0,0)) \oplus \mathbb{F}_2 \\ &= H^{-p,1-2p}(K(0,0)) \oplus \mathbb{F}_2. \end{aligned}$$

(b) In the following long exact sequence

$$\begin{aligned} \cdots &\to \overline{H}^{3-p,2-2p}(D(*1)) \to \overline{H}^{4-p,3-2p}(K(p+1,0)) \\ &\to \overline{H}^{4-p,3-2p}(K(p+2,0)) \to \overline{H}^{5-p,3-2p}(D(*1)) \to \cdots \end{aligned}$$

by Lemma 3.10, we have

$$\overline{H}^{3-p,2-2p}(D(*1)) = \overline{H}^{5-p,3-2p}(D(*1)) = 0.$$

Therefore, $\overline{H}^{4-p,3-2p}(K(p+1,0)) = \overline{H}^{4-p,3-2p}(K(p+2,0))$. Thus

$$\begin{aligned} H^{0,-1}(K(p,0)) &= \overline{H}^{4-p,3-2p}(K(p,0)) = \overline{H}^{4-p,3-2p}(K(p+1,0)) \oplus F_2 \\ &= \overline{H}^{4-p,3-2p}(K(p+2,0)) \oplus F_2 \\ &= \cdots = \overline{H}^{4-p,3-2p}(K(0,0)) \oplus F_2 \\ &= H^{i+4,j+4}(K(0,0)) \oplus F_2 = H^{-p,-1-2p}(K(0,0)) \oplus F_2. \end{aligned}$$

(c) By Lemma 3.3, it follows that

$$\begin{aligned} H^{0,1}(K(0,0)) &= \overline{H}^{i+4,j+4}(K(0,0)) = \overline{H}^{4,5}(K(0,0)) \\ &= \overline{H}^{3,1}(K(0,0)) \oplus \mathbb{F}_2 = \overline{H}^{3,1}(K(p,0)) \oplus \mathbb{F}_2 \\ &= H^{p-1,2p-3}(K(p,0)) \oplus \mathbb{F}_2 = H^{p,2p+1}(K(p,0)) \oplus \mathbb{F}_2. \end{aligned}$$

(d) By Lemma 3.3, it follows that

$$\begin{aligned} H^{0,-1}(K(0,0)) &= \overline{H}^{i+4,j+4}(K(0,0)) = \overline{H}^{4,3}(K(0,0)) \\ &= \overline{H}^{5,7}(K(p,0)) \oplus \mathbb{F}_2 = H^{p+1,2p+3}(K(p,0)) \oplus \mathbb{F}_2 \\ &= H^{p,2p-1}(K(p,0)) \oplus \mathbb{F}_2. \end{aligned}$$

(e) If $p \neq -1$, by Lemma 3.11, we obtain that

$$\begin{aligned} H^{-1,-1}(K(p,0)) &= \overline{H}^{3-p,3-2p}(K(p,0)) = \overline{H}^{3-p,3-2p}(K(p+1,0)) = \cdots \\ &= \overline{H}^{3-p,3-2p}(K(0,0)) = H^{4+i,j+4}(K(0,0)) \\ &= H^{1-p,-2p-1}(K(0,0)). \end{aligned}$$

(f) If $p \neq -1$, by Lemma 3.11, we obtain that

$$\begin{aligned} H^{-1,-3}(K(p,0)) \oplus F_2 &= \overline{H}^{3-p,1-2p}(K(p,0)) \oplus F_2 \\ &= \overline{H}^{3-p,1-2p}(K(p+1,0)) \\ &= \cdots = \overline{H}^{3-p,1-2p}(K(0,0)) \\ &= H^{4+i,j+4}(K(0,0)) = H^{-1-p,-2p-3}(K(0,0)). \end{aligned}$$

(g) If $i \neq 0$, we obtain that

$$\begin{aligned} H^{i,j}(K(p,0)) &= \overline{H}^{i+4-p,j+4-2p}(K(p,0)) = \overline{H}^{(i-p)+4,(j-2p)+4}(K(p+1,0)) \\ &= \cdots = \overline{H}^{(i-p)+4,(j-2p)+4}(K(0,0)) = H^{i-p,j-2p}(K(0,0)). \end{aligned}$$

The proof is completed. □

Lemma 3.13 ([9, Corollary 11])**.** *For an oriented link L and integers i, j, there are equalities of isomorphism classes of abelian groups.*

$$H^{i,j}(L^!;\mathbb{Z}) \otimes \mathbb{Q} = H^{-i,-j}(L;\mathbb{Z}) \otimes \mathbb{Q},$$
$$\mathrm{Tor}(H^{i,j}(L^!;\mathbb{Z})) = \mathrm{Tor}(H^{1-i,-j}(L;\mathbb{Z}))$$

where $L^!$ denotes the mirror image of L.

By the universal coefficient theorem, Lemmas 2.3 and 3.13, we have the following corollary.

Corollary 3.14. *For a positive integer p, the Kanenobu knot $K(p,0)$ is homologically thin over $\mathbb{F}_2$ and its Khovanov cohomology groups are given by*

$$H^{i,j}(K^!(p,0)) = H^{-i,-j}(K(-p,0)).$$

We now come to a position to prove the following theorem.

Theorem 3.15. *For any $p, q \in \mathbb{Z}$, we have*

$$H^{i,j}(K(p,q)) = H^{i,j}(K(p+q,0)).$$

Proof. We proceed to prove this theorem in two steps.

First, $p < 0$, $|p| > q$, that is, $p + q < 0$.

Let $D_1(*1)$, $D_2(*1)$ be the unlink of the two components obtained by resolving one of the p-th and $(p+q)$-th crossings in the $K(p,q)$ and $K(p+q,0)$ respectively. We obtain $\overline{H}^{i,j}(D_1(*1)) = \overline{H}^{i,j}(D_2(*1))$.

Since $x(K(p,q)) = x(K(p+q,0))$ and $y(K(p,q)) = y(K(p+q,0))$, $x(K(p+1,q)) = x(K(p+q+1,0))$ and $y(K(p+1,q)) = y(K(p+q+1,0))$, it is sufficient to show that $\overline{H}^{i,j}(K(p,q)) = \overline{H}^{i,j}(K(p+q,0))$. In fact, the long exact sequences on cohomology for $K(p,q)$ and $K(p+q,0)$ are isomorphic, as seen in the following commutative diagram using the five lemma.

$$\cdots \to \overline{H}^{i-1,j}(K(p+1,q)) \to \overline{H}^{i-1,j-1}(D_1(*1)) \to \overline{H}^{i,j}(K(p,q))$$
$$\to \overline{H}^{i,j}(K(p+1,q)) \to \overline{H}^{i,j-1}(D_1(*1)) \to \cdots$$
$$\cdots \to \overline{H}^{i-1,j}(K(p+q+1,0)) \to \overline{H}^{i-1,j-1}(D_2(*1)) \to \overline{H}^{i,j}(K(p+q,0))$$
$$\to \overline{H}^{i,j}(K(p+q+1,0)) \to \overline{H}^{i,j-1}(D_2(*1)) \to \cdots$$

where

$$\overline{H}^{i-1,j}(K(p+1,q)) = \overline{H}^{i-1,j}(K(p+q+1,0)),$$
$$\overline{H}^{i,j}(K(p+1,q)) = \overline{H}^{i,j}(K(p+q+1,0)),$$
$$\overline{H}^{i-1,j-1}(D_1(*1)) = \overline{H}^{i-1,j-1}(D_2(*1)),$$
$$\overline{H}^{i,j-1}(D_1(*1)) = \overline{H}^{i,j-1}(D_2(*1)).$$

Next, $p + q > 0$. It follows from Corollary 3.14 that

$$H^{i,j}(K(p+q,0)) = H^{-i,-j}(K^!(-p-q,0)) = H^{i,j}(K(p,q)). \qquad \square$$

As an immediate consequence, we obtain the following corollary.

Corollary 3.16. *For any $p, q \in \mathbb{Z}$, we have*

$$H^{i,j}(K(p,q)) = H^{i,j}(K(r,s)) \quad \textit{whenever } p+q = r+s.$$

Remark 3.17. The ranks of the free parts and the torsion parts of the cohomology groups of $K(p,q)$ are dependent on $p+q$. The type of torsion that can occur in these groups is limited to the type of torsion that occurs in $K(0,0)$ since no new torsion is generated as p, q increases. Therefore the rank of the Khovanov cohomology of $K(p,q)$ is an invariant of $p+q$.

Note that H^{odd} satisfies the same skein exact sequence as above, we shall adopt the same procedure as in the proof of Theorem 3.8.

Theorem 3.18. *For any $p, q \in \mathbb{Z}$, we have*

$$H^{odd}(K(p,q)) = H^{odd}(K(p+q,0)).$$

Similar to the reasoning Corollary 3.16, we have the following corollary.

Corollary 3.19. *For any $p, q \in \mathbb{Z}$, we have*

$$H^{odd}(K(p,q)) = H^{odd}(K(r,s)) \quad \textit{whenever} \quad p+q = r+s.$$

Remark 3.20. By combining Theorems 3.8 and 3.15, we have obtained a calculating formula for the Khovanov cohomology over $\mathbb{F}_2$ for any Kanenobu knots. It is natural to wonder what the formulas of the Khovanov-Rozansky homology and the Heegaard Floer homology of Kanenobu knots are. This might worth further study.

It was claimed in [10, 11] that Khovanov-Rozansky homology, HKR_N, is a categorification of HOMFLY polynomial. As shown in [14], A. Lobb showed the following fact:

$$HKR_N(K(p,q)) = HKR_N(K(p+2,q-2)) \quad \text{for all } p,q \in \mathbb{Z}.$$

Similar to the method used in this paper, by iterating the above formula, we can see that we only need to consider the parity of q to calculate $HKR_N(K(p,0))$ and $HKR_N(K(p,1))$. Then how to calculate these cohomology? In [4], N. Carqueville and D. Murfet presented the first method to directly compute HKR_N for arbitrary links. Therefore, it might be feasible to give a formula to $HKR_N(K(p,q))$.

On the other hand, the Heegaard Floer homology, $\widehat{HFk}$, was introduced in [17, 20]. It provided a categorification of the Alexander polynomial. As in [6], J. Greene and L. Watson obtained the skein exact sequence in knot Floer homology and established the following result:

$$\widehat{HFK}(K(p,q)) = \widehat{HFK}(K(p+2,q)) = \widehat{HFK}(K(p,q+2)) \text{ for all } p,q \in \mathbb{Z}.$$

Similarly, with the recursive version, it is sufficient to consider the parity of q to calculate $\widehat{HFK}(K(p,0))$ and $\widehat{HFK}(K(p,1))$. In general, it is difficult to calculate $\widehat{HFK}$ explicitly. However, we might refer to [1, 24] for a combinatorial method for the calculation of the above Heegaard Floer homology.

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