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ON A TWO WEIGHTS ESTIMATE FOR THE COMMUTATOR

Daewon Chung

Abstract. We provide quantitative two weight estimates for the commutator of the Hilbert transform under certain conditions on a pair of weights (u, v) and b in $Carl_{u,v}$. In [10] and [11], Bloom's inequality is shown in a modern setting, and the boundedness of the commutators is provided by assuming both weights u, v are A_2 and $b \in BMO_\rho$. In the present paper we show that the condition on b can be replaced by $Carl_{u,v}$ by using the joint A_2^d condition.

1. Introduction and statement of Main results

The aim of this article is to establish quantitative two weight estimates for the commutator of the Hilbert transforms. The commutator operators were introduced by Coifman, Rochberg and Weiss in [8] as a tool to extend the classical factorization theorem for Hardy space in the unit circle to $\mathbb{R}^n$. It has been shown that the Calderón-Zgymund singular integral operator with smooth kernel $[b, T]f := bT(f) - T(bf)$ is a bounded operator on L^p, $1 < p < \infty$, when b is a BMO function. Weighted estimates for the commutator have been studied in [1], [4], [6], [7], and elsewhere. Also [6] and [7] provide the sharp version of one weighted estimates for the commutators. In [4] Bloom considers the commutator of the Hilbert transform $[b, H] : L^p(u) \to L^p(v)$: when the weight $u = v \in A_p$ then the boundedness is characterized by $b \in BMO$. Bloom also shows boundedness in case $u \neq v$ and $u, v \in A_p$. In this case, boundedness is characterized by the weighted BMO space, namely

$$\|b\|_{BMO_\rho} := \sup_I \left(\frac{1}{\rho(I)} \int_I |b(x) - m_I b|^2 dx \right)^{1/2},$$

where $\rho = (u/v)^{1/p}$. When $u = v$, the space $BMO_\rho = BMO$ a case which is well-known. The general case, however, is much more complicated, because there are three independent objects: the pair of weights (u, v) and the symbol b. Commutator operators are widely encountered and studied in many problems in

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PDEs, such as the div-curl estimates. Many of these topics remain unexplored in the setting of Bloom's inequality. Recently [10] and [11] give a modern proof of Bloom's result. However, there is a gap in applying these results to the weighted version of the div-curl estimate, because in the case of $u = v$, $BMO_\rho = BMO$.

In this paper, we provide quantitative estimates with a different condition on b, namely the two weight Carleson class denoted by $Carl_{u,v}$. This function class was introduced and studied in many papers, such as [1], [5] , [14], [15], and [16]. We now state our main results.

Theorem 1.1. *Let $(u, v) \in A_2^d$. If $u, v \in A_2^d$ and $b \in Carl_{u,v}$ then the commutator $[b, H^d]$ with the dyadic Hilbert transform is bounded from $L^2(u)$ into $L^2(v)$.*

In the Theorem 1.1, the dyadic Hilbert transform is defined by

$$H^d f(x) := \sum_{I \in D} \langle f, h_{\hat{I}} \rangle h_I(x) \,,$$

where $\hat{I}$ is the parent of I. These operators are often called the shifting operators. The first shifting operator was introduced in [12] where Petermichl proved that the norm of the Hilbert transform is bounded by the supremum of the norms of the commutator of shifting operators.

Definitions and frequently used theorems are collected in Section 2, including A_2^d, joint A_2^d, regular and weighted Haar functions, v-Carleson sequences, the class $Carl_{u,v}$, the weighted Carleson Lemma, and the known weighted estimates. The decomposition of the commutators are introduce in Section 3, and we provide a quantitative two weight estimate for each dyadic operator. Combining these estimates, we give the proof of our main theorem in Section 4, after which we give concluding remarks.

2. Notations and Useful known results

Throughout the paper a constant C will be a numerical constant that may change from line to line. Given a measurable set E in $\mathbb{R}$, $|E|$ will stand for its Lebesgue measure. We say a function $v : \mathbb{R} \to \mathbb{R}$ is a weight if v is an almost everywhere positive locally integrable function. For a given weight v, the v-measure of a measurable set E, denoted by $v(E)$, is $v(E) = \int_E v(x)dx$. Let us denote $\mathcal{D}$ and $\mathcal{D}(J)$ the collection of all dyadic intervals and the collection of all dyadic subintervals of J respectively.

We say that a weight v satisfies the A_2^d condition if and only if v is a weight, so v^{-1} is also a weight, and

$$[u]_{A_2^d} := \sup_{I \in \mathcal{D}} m_I(v^{-1}) m_I v < \infty \,,$$

where $m_I v$ denotes the integral average of a weight v over the interval I. Similarly, a weight v satisfies the A_p^d condition iff

$$[v]_{A_p^d} := \sup_{I\in\mathcal{D}} m_I\left(v^{-\frac{1}{p-1}}\right) m_I v < \infty\,.$$

We also say that a pair of weights (u, v) satisfies the joint A_2^d condition if and only if both v and u are weights and

$$[u, v]_{A_d^d} := \sup_{I\in\mathcal{D}} m_I(u^{-1}) m_I v < \infty\,.$$

For any interval $I \in \mathcal{D}$, there is a Haar function defined by

$$h_I(x) = \frac{1}{\sqrt{|I|}}\left(\chi_{I_+}(x) - \chi_{I_-}(x)\right)$$

where χ_I denotes the characteristic function of the interval I, and $I_+,\ I_-$ denote the right and left child of I respectively. It is a well known fact that the Haar systems $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2(\mathbb{R})$. The norm of $f \in L^2(v)$ is

$$\|f\|_{L^2(v)} := \left(\int_{\mathbb{R}} |f(x)|^2 v(x)dx\right)^2\,.$$

A positive sequence $\{\alpha_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence if there is a constant $C > 0$ such that for all dyadic intervals J

$$\sum_{I\in\mathcal{D}(J)} \alpha_I \le Cv(J)\,. \tag{2.1}$$

When $v = 1$ almost everywhere we say that the sequence is a Carleson sequence. The infimum among all C's that satisfy the inequality is called the intensity of the v-Carleson sequence $\{\alpha_I\}_{I\in\mathcal{D}}$. We now define a class of objects that will take the place of the BMO class in the two weighted case. It is called the two weight Carleson class.

Definition 1. Given a pair of weights (u, v), we say that a locally integrable function b belongs to the two weight Carleson class, $Carl_{u,v}$, if $\{b_I^2/m_I v\}_{I\in D}$ is a u^{-1}-Carleson sequence where $b_I = \langle b, h_I\rangle$.

We refer to [5] for the properties of the class $Carl_{u,v}$ and its relations to the BMO class. We now introduce the Weighted Carleson Lemma which will be used frequently throughout this paper. The lemma was stated first in [15]. One can also find proofs in [14].

Theorem 2.1. *[Weighted Carleson Lemma]. Let v be a weight, then $\{\alpha_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity B if and only if for all non-negative v-measurable function F on the line,*

$$\sum_{I\in\mathcal{D}} (\inf_{x\in I} F(x))\alpha_I \le B\int_{\mathbb{R}} F(x)v(x)dx\,. \tag{2.2}$$

Definition 2. We define the dyadic square function as follows

$$S^d f(x) := \left(\sum_{I \in D} |m_I f - m_{\hat{I}} f|^2 \chi_I(x) \right)^{1/2} .$$

The following two weight characterization was introduced by Wilson, see also [15].

Theorem 2.2 ([16])**.** *Let (u, v) be a pair of weights such that $(u, v) \in A_2^d$ and $\{|\Delta_I u^{-1}| |I| m_I v\}_{I \in \mathcal{D}}$ is a u^{-1}-Carleson sequence with intensity $C_{v^{-1}, u^{-1}}$. Then there is a constant $C > 0$ such that*

$$\|S^d\|_{L^2(u) \to L^2(v)} \le C([u, v]_{A_2^d} + \mathcal{C}_{v^{-1}, u^{-1}})^{1/2} .$$

We now list the known weighted estimates, such as the one weighted norm estimate for the Hilbert transform and weighted maximal function, and the two weighted norm estimates for the square function the dyadic paraproduct.

Definition 3. We define the dyadic weighted maximal function M_v^d as follows

$$M_v^d f(x) := \sup_{\substack{I \ni x \\ I \in \mathcal{D}}} \frac{1}{v(I)} \int_I |f(y)| \, v(y) dy .$$

The weighted maximal function M_v is defined analogously, only taking the supremum over all intervals not just dyadic intervals. A very important fact about the weighted maximal function is that the $L^p(v)$ norm of M_v^d only depends on $p' = p/(p-1)$ and not on the weight v .

Theorem 2.3. *Let v be a locally integrable function such that $v > 0$ a.e. Then for all $1 < p < \infty$, M_v^d is bounded in $L^p(v)$. Moreover, for all $f \in L^p(v)$*

$$\|M_v^d f\|_{L^p(v)} \le p' \|f\|_{L^p(v)} .$$

The sharp version of the one weighted norm estimate for the Hilbert transform was obtained by S. Petermichl ([13]).

Theorem 2.4. *Let v be a A_2^d weight. Then H^d is bounded in $L^2(v)$. Moreover, for all $f \in L^2(v)$*

$$\|H^d f\|_{L^2(v)} \le C[v]_{A_2^d} \|f\|_{L^2(v)} .$$

One can easily get a sharp $L^p(v)$ version by the sharp extrapolation([9]). Finally, we recall the definition of the dyadic paraproduct.

Definition 4. We formally define the dyadic paraproduct π_b associated to $b \in L^1_{loc}(\mathbb{R})$ as follows for functions f which are at least locally integrable:

$$\pi_b f(x) := \sum_{I \in \mathcal{D}} m_I f \, \langle b, h_I \rangle h_I(x).$$

It is a well know fact that the dyadic paraproduct is bounded not only oi $L^p(dx)$ but also in $L^p(v)$ when $b \in BMO^d$ and $v \in A_p^d$. Beznosova proved in [2] that the $L^2(v)$ norm of the dyadic paraproduct depends linearly on both $[v]_{A_2^d}$ and $\|b\|_{BMO^d}$. Recently, [5] provides the quantitative two weight estimate for the dyadic paraproduct as follows.

Theorem 2.5 ([5]). *Let (u, v) be a pair of weights such that v and u^{-1} are weights, $(u, v) \in \mathcal{A}_2^d$, and $\{|\Delta_I v|^2 |I| m_I(u^{-1})\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{C}_{u,v}$. Then π_b is bounded from $L^2(u)$ into $L^2(v)$ if $b \in Carl_{u,v}$. Moreover, if $\mathcal{B}_{u,v}$ is the intensity of the u^{-1}-Carleson sequence $\{|b_I|^2/m_I v\}_{I\in\mathcal{D}}$ then there exist $C > 0$ such that for all $f \in L^2(u)$*

$$\|\pi_b f\|_{L^2(v)} \leq C\sqrt{[u,v]_{\mathcal{A}_2^d}\mathcal{B}_{u,v}}\left(\sqrt{[u,v]_{\mathcal{A}_2^d}} + \sqrt{\mathcal{C}_{u,v}}\right)\|f\|_{L^2(u)}.$$

3. The Commutator of the dyadic Hilbert transform and its two-weight estimates

In this section we will prepare the proof of our main theorem which provides the two-weight estimates for the commutator of the dyadic Hilbert transform when $b \in Carl_{u,v}$. Note that, by definition, b is a locally integrable function, thus $b_I = \langle b, h_I\rangle$ is well defined. The commutator of the dyadic Hilbert transform has an explicit expansion in terms of the paraproduct and H^d.

$$[b, H^d] = [\pi_b^*, H^d] + [\pi_b, H^d] + [\lambda_b, H^d], \tag{3.1}$$

where π_b^* is the adjoint of the paraproduct and

$$\lambda_b(f) := \sum_{I\in D} m_I b\langle f, h_I\rangle h_I .$$

We refer to [6] for this decomposition. λ_b can't be a bounded operator in L^p; however $[\lambda_b, H^d]$ is bounded in L^p and is better behaved than $[b, H^d]$. The decomposition (3.1) was used to analyze the commutator with the Hilbert transform, first by Petermichl in [12] and then by the author in [6] to obtain the sharp bound for the commutators in the weighted Lebesgue space.

Before we estimate the term $[\lambda_b, H^d]$ in the following lemma, we first consider the term $[\lambda_b, H^d]$ and its subtle cancellation. We now rewrite $[\lambda_b, H^d]$ by using the definition.

$$\begin{aligned}
[\lambda_b, H^d](f) &= \lambda_b(H^d f) - H^d(\lambda_b f) \\
&= \sum_{I\in D} m_I b\langle H^d f, h_I\rangle h_I - \sum_{J\in D}\langle \lambda_b f, h_{\hat{J}}\rangle h_J \\
&= \sum_{I\in D}\sum_{J\in D} m_I b\langle f, h_{\hat{J}}\rangle\langle h_J, h_I\rangle h_I - \sum_{J\in D}\sum_{I\in D} m_I b\langle f, h_I\rangle\langle h_I, h_{\hat{J}}\rangle h_J \\
&= \sum_{I\in D} m_I b\langle f, h_{\hat{I}}\rangle h_I - \sum_{J\in D} m_{\hat{J}} b\langle f, h_{\hat{J}}\rangle h_J
\end{aligned}$$

$$= \sum_{I\in D}(m_I b - m_{\hat{I}} b)\langle f, h_{\hat{I}}\rangle h_I$$

$$(3.2) \qquad = \sum_{I\in D}\frac{\text{sgn}(I,\hat{I})}{\sqrt{|\hat{I}\,|}}\langle b, h_{\hat{I}}\rangle\langle f, h_{\hat{I}}\rangle h_I\,,$$

where $sgn(I,\hat{I}) = \pm 1$ if $I = \hat{I}_{\pm}$. Then we have the following estimate.

Lemma 3.1. *Let (u,v) be a pair of functions such that v and u^{-1} are weights, $(u,v)\in A_2^d$, and $u\in A_2^d$. Then $[\lambda_b, H^d]$ is bounded from $L^2(u)$ into $L^2(v)$ if $b\in Carl_{u,v}$, and $\{\Delta_I v|^2|I|m_I(u^{-1})\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{C}_{u,v}$. Moreover, if $\mathcal{B}_{u,v}$ is the intensity of the u^{-1}-Carleson sequence $\{|b_I|^2/m_I v\}_{I\in D}$ then there exist $C>0$ such that for all $f\in L^2(u)$*

$$\|[\lambda_b, H^d]\|_{L^2(v)} \le C\sqrt{[u,v]_{A_2^d}\mathcal{B}_{u,v}\left([u,v]_{A_2^d} + \mathcal{C}_{u,v}\right)}\|f\|_{L^2(u)}\,.$$

Proof. In order to use the duality argument, let us fix $f\in L^2(u)$ and $g\in L^2(v^{-1})$. Using the observation (3.2), we have

$$\big|\langle[\lambda_b, H^d]f, g\rangle\big| = \left|\sum_{I\in D}\frac{\text{sgn}(I,\hat{I})}{\sqrt{|\hat{I}\,|}}\langle b, h_{\hat{I}}\rangle\langle f, h_{\hat{I}}\rangle\langle g, h_I\rangle\right|$$

$$(3.3) \qquad \le \sum_{I\in D}|\langle b, h_{\hat{I}}\rangle||m_{\hat{I}}|f|||\langle g, h_I\rangle|$$

$$(3.4) \qquad \le \left(\sum_{I\in D}\frac{m_{\hat{I}}^2|f|\,b_{\hat{I}}^2}{m_I u^{-1}}\right)^{1/2}\left(\sum_{I\in D}|\langle g, h_I\rangle|^2 m_I u^{-1}\right)^{1/2},$$

where the inequality (3.3) is due to

$$\frac{1}{\sqrt{|\hat{I}|}}|\langle f, h_{\hat{I}}\rangle| \le \frac{1}{|\hat{I}\,|}\langle |f|, \chi_{\hat{I}}\rangle = m_{\hat{I}}|f|$$

and we use the Cauchy-Schwarz inequality for (3.4).

Since $(u,v)\in A_2^d$ and $\{|\Delta_I v|^2 m_I(u^{-1})|\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{C}_{u,v}$, S^d is a bounded operator from $L^2(u^{-1})$ into $L^2(v^{-1})$ by Theorem 2.2. Therefore there is a constant C such that

$$\left(\sum_{I\in\mathcal{D}}|\langle g, h_I\rangle|^2 m_I u^{-1}\right)^{1/2} = \|S^d g\|_{L^2(u^{-1})}$$

$$(3.5) \qquad \le C([u,v]_{A_2^d} + \mathcal{C}_{u,v}))^{1/2}\|g\|_{L^2(v^{-1})}\,.$$

On the other hand we have the following estimate by using Theorem 2.3 and the Weighted Carleson Lemma.

$$\sum_{I\in\mathcal{D}} \frac{m_{\hat{I}}^2|f|\, b_{\hat{I}}^2}{m_I u^{-1}} = \sum_{I\in\mathcal{D}} \left(\frac{m_{\hat{I}}(|f|uu^{-1})}{m_{\hat{I}}u^{-1}}\right)^2 b_{\hat{I}}^2 \frac{m_{\hat{I}}^2 u^{-1}}{m_I u^{-1}}$$

$$\leq C[u]_{A_2^d}[u,v]_{A_2^d} \sum_{I\in D} \left(m_I^{u^{-1}}(|f|u)\right)^2 \frac{b_I^2}{m_I v} \tag{3.6}$$

$$\leq C[u]_{A_2^d}[u,v]_{A_2^d} \sum_{I\in D} \inf M_{u^{-1}}^2 f(u) \frac{b_I^2}{m_I v}$$

$$\leq C[u]_{A_2^d}[u,v]_{A_2^d}\mathcal{B}_{u,v}\|M_{u^{-1}}(fu)\|_{L^2(u^{-1})}^2$$

$$\leq C[u]_{A_2^d}[u,v]_{A_2^d}\mathcal{B}_{u,v}\|fu\|_{L^2(u^{-1})}^2$$

$$= C[u]_{A_2^d}[u,v]_{A_2^d}\mathcal{B}_{u,v}\|f\|_{L^2(u)}^2 . \tag{3.7}$$

Combining the inequalities (3.4), (3.5), and (3.7), we get the following estimate which completes the proof by the usual duality argument.

$$|\langle[\lambda_b, H^d]f, g\rangle| \leq C\sqrt{[u]_{A_2^d}[u,v]_{A_2^d}\mathcal{B}_{u,v}\left([u,v]_{A_2^d} + \mathcal{C}_{u,v}\right)}\|f\|_{L^2(u)}\|g\|_{L^2(v^{-1})} .$$

□

For $f \in L^2(u)$ and $g \in L^2(v^{-1})$, we have the following.

$$\begin{aligned}
\left|\langle H^d\pi_b f, g\rangle\right| &= \left|\langle \pi_b f, (H^d)^* g\rangle\right| \\
&= \left|\left\langle \sum_{I\in\mathcal{D}} m_I f\langle b, h_I\rangle h_I, \sum_{J\in\mathcal{D}} \langle g, h_J\rangle h_{\hat{J}} \right\rangle\right| \\
&\leq \left|\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} m_I f\langle b, h_I\rangle\langle g, h_J\rangle\langle h_I, h_{\hat{J}}\rangle\right| \\
&= \left|\sum_{J\in\mathcal{D}} m_{\hat{J}} f\langle b, h_{\hat{J}}\rangle\langle g, h_J\rangle\right| .
\end{aligned}$$

Thus by using a similar estimate with (3.4) one can easily get the following lemma.

Lemma 3.2. *Let (u,v) be a pair of functions such that v and u^{-1} are weights and $(u,v)\in A_2^d$. Then $H^d\pi_b$ is bounded from $L^2(u)$ into $L^2(v)$ if $b \in Carl_{u,v}$ and $\{\Delta_I v|^2|I|m_I(u^{-1})\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{C}_{u,v}$. Moreover, if $\mathcal{B}_{u,v}$ is the intensity of the u^{-1}-Carleson sequence $\{|b_I|^2/m_I v\}_{I\in D}$ then there exist $C>0$ such that for all $f\in L^2(u)$*

$$\|H^d\pi_b f\|_{L^2(v)} \leq C\sqrt{[u,v]_{A_2^d}\mathcal{B}_{u,v}\left([u,v]_{A_2^d} + \mathcal{C}_{u,v}\right)}\|f\|_{L^2(u)} .$$

For the term $\pi_b^* H^d$, we have the following lemma.

Lemma 3.3. *Let (u,v) be a pair of functions such that v and u^{-1} are weights and $(u,v)\in A_2^d$. Then $\pi_b^* H^d$ is bounded from $L^2(u)$ into $L^2(v)$ if $b\in Carl_{v^{-1},u^{-1}}$ and $\{\Delta_I(u^{-1})|^2|I|m_I v\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence with intensity $\mathcal{C}_{v^{-1},u^{-1}}$. Moreover, if $\mathcal{B}_{v^{-1},u^{-1}}$ is the intensity of the u^{-1}-Carleson sequence $\{|b_I|^2/m_I v\}_{I\in D}$ then there exists $C>0$ such that for all $f\in L^2(u)$*

$$\|\pi_b^* H^d f\|_{L^2(v)} \le C\sqrt{[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\left([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}}\right)}\|f\|_{L^2(u)}\,.$$

Proof. Similarly to the previous lemma, let us fix $f\in L^2(u)$ and $g\in L^2(v^{-1})$. Then we have that

$$\begin{aligned}
|\langle \pi_b^* H^d f, g\rangle| &= |\langle H^d f, \pi_b g\rangle| \\
&= \left|\left\langle \sum_{I\in\mathcal{D}}\langle f, h_{\hat{I}}\rangle h_I, \sum_{J\in\mathcal{D}} m_J g\langle b, h_J\rangle h_J\right\rangle\right| \\
&= \left|\sum_{I\in\mathcal{D}}\langle f, h_{\hat{I}}\rangle m_I g\langle b, h_I\rangle\right| \\
&\le \left(\sum_{I\in\mathcal{D}}\frac{m_I^2 g b_I^2}{m_{\hat{I}} v}\right)^{1/2}\left(\sum_{I\in\mathcal{D}}|\langle f, h_{\hat{I}}\rangle|^2 m_{\hat{I}} v\right)^{1/2}.
\end{aligned}$$

We also get the following estimates:

$$\begin{aligned}
\sum_{I\in\mathcal{D}}\frac{m_I^2 g b_I^2}{m_{\hat{I}} v} &\le 2\sum_{I\in\mathcal{D}}\left(\frac{m_I(gvv^{-1})}{m_I v}\right)^2 b_I^2 m_I v \\
&\le 2[u,v]_{A_2^d}\sum_{I\in\mathcal{D}}\left(m_I^v(gv^{-1})\right)^2\frac{b_I^2}{m_I u^{-1}} \\
&\le 2[u,v]_{A_2^d}\sum_{I\in\mathcal{D}}\inf M_v^2(gv^{-1})\frac{b_I^2}{m_I u^{-1}} \\
&\le 2[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\sum_{I\in\mathcal{D}}\|M_v(gv^{-1})\|_{L^2(v)}^2 \\
&\le 2[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\sum_{I\in\mathcal{D}}\|gv^{-1}\|_{L^2(v)}^2 \\
&\le 2[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\sum_{I\in\mathcal{D}}\|g\|_{L^2(v^{-1})}^2
\end{aligned}$$

and

$$\left(\sum_{I\in\mathcal{D}}|\langle f, h_{\hat{I}}\rangle|^2 m_{\hat{I}} v\right)^{1/2} \le C([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}})^{1/2}\|f\|_{L^2(u)}\,.$$

Combining these two estimates, we get the following

$$\left|\langle \pi_b^* H^d f, g\rangle\right| \le C\sqrt{[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\left([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}}\right)}\|f\|_{L^2(u)}\|g\|_{L^2(v^{-1})}\,.$$

We, therefore, can conclude there exist a constant C such that

$$\|\pi_b^* H^d f\|_{L^2(v)} \le C\sqrt{[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\left([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}}\right)}\|f\|_{L^2(u)}\,.$$

□

4. Proof of the main result and remarks

In this section, we prove Theorem 1.1. In order to prove the Theorem, we decompose the commutator into three commutators using (3.1), and estimate each term (except for the terms $H^d\pi_b^*$ and $\pi_b H^d$) using Lemmas in Section 3. It is shown in [6] that the terms $H^d\pi_b^*$ and $\pi_b H^d$ have more complicated behavior than the other terms. Indeed, in the case of one weight estimate, while the first terms obey a linear estimate in terms of A_2 weight characteristic, the boundedness of $H^d\pi_b^*$ and $\pi_b H^d$ depend quadratically on the A_2 weight characteristic. To overcome this difficulty, we will simply use the previous known estimates such as, Theorem 2.4 and Theorem 2.5. The followings are immediate applications of these Theorems.

$$\begin{aligned}\|H^d\pi_b^* f\|_{L^2(v)} &\le C[v]_{A_2}\|\pi_b^* f\|_{L^2(v)} \le C[v]_{A_2}\|\pi_b^*\|_{L^2(u)\to L^2(v)}\|f\|_{L^2(u)}\\ &\le [v]_{A_2}\sqrt{[u,v]_{A_2^d}\mathcal{B}_{v^{-1},u^{-1}}\left([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}}\right)}\|f\|_{L^2(v)}\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\|\pi_b H^d f\|_{L^2(v)} &\le C\|\pi_b\|_{L^2(u)\to L^2(v)}\|H^d f\|_{L^2(u)} \le C[u]_{A_2^d}\|\pi_b\|_{L^2(u)\to L^2(v)}\|f\|_{L^2(u)}\\ &\le [u]_{A_2}\sqrt{[u,v]_{A_2^d}\mathcal{B}_{u,v}\left([u,v]_{A_2^d}+\mathcal{C}_{v^{-1},u^{-1}}\right)}\|f\|_{L^2(v)}\,.\end{aligned}\tag{4.2}$$

Let us now prove Theorem 1.1.

Proof of Theorem 1.1. Let (u,v) be a pair of weights such that $(u,v)\in A_2^d$ and $u,v\in A_2^d$. Then we can easily see the following estimate.

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\Delta_I v|^2|I|m_I(u^{-1}) &\le [u,v]_{A_2^d}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\frac{|\Delta_I v|^2}{m_I v}|I|\\ &\le [u,v]_{A_2^d}[v]_{A_2^d}m_J v\,.\end{aligned}\tag{4.3}$$

Here the inequality (4.3) is due to Buckley's inequality, but one can find a sharper version of the inequality in [3]. Therefore $\{|\Delta_I v|^2|I|m_I(u^{-1})\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\mathcal{C}_{u,v}\le[u,v]_{A_2}[v]_{A_2}$. Similarly one can check that $\{|\Delta_I(u^{-1})|^2|I|m_I v\}_{I\in\mathcal{D}}$ is a u^{-1}-Carleson sequence with intensity $\mathcal{C}_{v^{-1},u^{-1}}\le[u,v]_{A_2}[u]_{A_2}$. On the other hand, it is proved in Theorem 4.7 of

[5] that $Carl_{u,v} = Carl_{v^{-1},u^{-1}}$ when $u, v \in A_2^d$. Use Lemma 3.3 and (4.1) to estimate the term $[\pi_b^*, H^d]$ then

$$\begin{aligned} \|[\pi_b^*, H^d]f\|_{L^2(v)} &= \|\pi_b^* H^d f - H^d \pi_b^* f\|_{L^2(v)} \\ &\le \|\pi_b^* H^d f\|_{L^2(v)} + \|H^d \pi_b^* f\|_{L^2(v)} \\ &\le C[u,v]_{A_2^d}[v]_{A_2^d}\sqrt{[u]_{A_2^d}\mathcal{B}_{u,v}}\|f\|_{L^2(v)} \end{aligned} \tag{4.4}$$

Similarly, using Lemma 3.2 and (4.2) we get

$$\|[\pi_b, H^d]\|_{L^2(v)} \le C[u,v]_{A_2^d}[u]_{A_2^d}\sqrt{[v]_{A_2^d}\mathcal{B}_{u,v}}\|f\|_{L^2(v)} . \tag{4.5}$$

Finally, combining Lemma 3.1, (4.4), and (4.5), we have

$$\|[b, H^d]\|_{L^2(v)} \le C[u,v]_{A_2^d}\left(\sqrt{[v]_{A_2^d}} + \sqrt{[u]_{A_2^d}}\right)\sqrt{[v]_{A_2^d}[u]_{A_2^d}\mathcal{B}_{u,v}}\|f\|_{L^2(v)}$$

which completes the proof. □

Remark 1. The weight conditions $u, v \in A_2^d$ and $(u, v) \in A_2^d$ in Theorem 1.1 are independent. The joint A_2^d condition characterizes the relationship between u and v, but $u \in A_2^d$ and $v \in A_2^d$ only describe the weight itself. However, if we assume that $(u, v) \in A_2^d$ and $(v, u) \in A_2^d$ then these two conditions immediately imply u and v satisfy the A_2^d condition respectively. On the other hand, the weight conditions $u, v \in A_2^d$ and $(u, v) \in A_2^d$ do not imply that $(v, u) \in A_2^d$.

Remark 2. The one weight A_2^d conditions appear in Section 3, so that we may apply the doubling property of A_2^d weight and for the boundedness of the Hilbert transform in the Section 4. Thus, if one has the two weight estimate of the Hilbert transform under similar assumptions in Theorem 2.5, it might be possible to get rid of one or both of these conditions from the estimate. In the subsequent paper, this argument will be discussed and used to obtain the boundedness of the commutator under weaker conditions.

Remark 3. We use the fact of $Carl_{u,v} = Carl_{v^{-1},u^{-1}}$ when $u, v \in A_2^d$. However, it will be more interesting to compare the condition $Carl_{u,v}$ and BMO_ρ when $u, v \in A_2^d$.

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Daewon Chung
Faculty of Basic Sciences, Mathematics Major, Keimyung University, 1095 Dalgubeol-Daero, Daegu, 704-701, Daegu, Korea
E-mail address: dwchung@kmu.ac.kr