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ON OPTIMALITY OF GENERALIZED OPTIMIZATION PROBLEMS ASSOCIATED WITH OPERATOR AND EXISTENCE OF $(T_\eta;\xi_\theta)$-INVEX FUNCTIONS

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Abstract. The main purpose of this paper is to introduce a pair new class of primal and dual problem associated with an operator. We prove the sufficient optimality theorem, weak duality theorem and strong duality theorem for these problems. The equivalence between the generalized optimization problems and the generalized variational inequality problems is studied in ordered topological vector space modeled in Hilbert spaces. We introduce the concept of partial differential associated (PDA)-operator, PDA- vector function and PDA-antisymmetric function to show the existence of a new class of function called, $(T_\eta;\xi_\theta)$-invex functions. We discuss first and second kind of $(T_\eta;\xi_\theta)$-invex functions and establish their existence theorems in ordered topological vector spaces.

1. Introduction

The notion of invexity was introduced and studied in the optimization by Hanson [15] in 1981 for a generalized concept of convex function. He utilized the property invexity of the function in the place of convexity to analyze optimality of the optimization problems.

Later Kaul and Kaur [16] called these functions η-convex and defined η-pseudoconvex and η-quasiconvex functions. As an extension, the concept of ρ-(η,θ)-invexity was introduced by Zalmai [19] which is generalization of investiy.

The class of convex functions have also been further extended to the class of B-invex functions, introduced by Bector et al. [1, 2]. A class of pseudo B-invex and quasi B-invex functions are also studied by Bector et al. [2], which are generalization of pseudoinvex and quasiinvex functions respectively. Bector et al. [2] have introduced the sufficient optimality conditions and duality results for a nonlinear programming problem using B-invex functions. For reference we refer Behera et al. [4, 5], Ben and Mond [6] to name only a few.

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In 2006, Behera and Das [3] have introduced T-η-invex function as an invex function associated with respect to an operator T to generalize the invexity concept of a smooth function, introduced by Hanson [15] and nonsmooth function, introduced by Craven [11]. Later the properties of T-η-invex functions and η-invex sets are studied by Behera and Das [14].

For our need, we recall the definition of T-η-invex function and η-invex set. Let X be topological vector space and K be a nonempty subset of X. Let (Y, P) be an ordered topological vector space equipped with the closed convex pointed cone such that $intP \neq \varnothing$. Let $L(X, Y)$ be the set of linear continuous functionals from X to Y. Let the pair $\langle f, x\rangle$ denote the value of $f \in L(X, Y)$ at $x \in X$. Let $T : K \to L(X, Y)$ be any map.

Definition 1. [15] Let K be any subset of the vector space X. Let $\eta : K \times K \to X$ be a continuous vector valued mapping. The set K is said to be η-invex if for all $x, u \in K$ and for all $t \in (0,\ 1)$, we have $u + t\eta(x, u) \in K$, by the rule

$$u + t\eta(x,u) = \begin{cases} u, & \text{if } t = 0; \\ z \in I(x,u), & \text{if } 0 < t < 1; \\ x, & \text{if } t = 1, \end{cases}$$

where $I(x, y) \subset int\, K$ is the path joining x and y.

Definition 2. [3] A mapping $F : K \to Y$ is T-η-invex on $K \subset X$ if there exists a vector valued mapping $\eta : K \times K \to X$ such that

$$F(x) - F(x') - \langle T(x'), \eta(x, x')\rangle \notin -intP \quad \text{for all } x, x' \in K.$$

Definition 3. [3] The mapping $T : K \subset X \to L(X, Y)$ is said to be η-*monotone* if there exists a vector function $\eta : K \times K \to X$ such that

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin intP$$

for all $x, x' \in K$. For strictly η-*monotone* case,

$$\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin -int\, P \cup int\, P$$

for $x = x'$ only.

The concept of invexity is broadly used in the theory of variational inequality problems and complementarity problems. To study the generalized complementarities directly from its related generalized variational inequalities, condition C_0 (introduced by Behera and Das [3]) has played an important role. In fact, the concept of η-invex cone in both positive and negative orthants (i.e., sides) is induced by the concept of condition C_0 defined as follows.

Definition 4. [3] Let X be a topological vector space. Let $K \subset X$ with $intK \neq \emptyset$. A vector function $\eta : K \times K \to X$ is said to satisfy *condition* C_0 if the following hold:

(a) $z = u + \eta(x, u) \in K$ and $\eta(z, u) + \eta(u, z) = 0$ for all $x, u \in K$,
(b) $\eta(u + t\eta(x, u), u) + t\eta(x, u) = 0$ for all $x, u \in K$ and $t \in (0, 1)$.

Various phenomena which occur in physical and economical sciences are mathematically formulated as nonsmooth variational inequalities or as optimization problems, where some nonsmooth constraints have to be taken into account. The nonsmooth mixed variational inequalities include classical variational inequalities as well as the nonsmooth convex optimization problems.

To make the paper self contained, we recall the concept of subdifferential of a nonsmooth convex function in sense of Clarke [9]. Let $F : K \to \mathbb{R}$ be Lipschitz (not necessarily differentiable) near a given point $x \in K$ and let v be any point in X. The generalized directional derivative of F at x in the direction v, i.e., $F^\circ(x; v)$ is defined by

$$F^\circ(x;v) = \limsup_{y \to x,\ t \downarrow 0} \frac{F(y+tv) - F(y)}{t}.$$

The generalized gradient (or subdifferential) of F at x, denoted $\partial F(x)$ which is the subset of X^* is given by

$$\partial F(x) = \{\xi \in X^* : F^\circ(x;v) \geq \langle \xi, v\rangle\}.$$

Through out this paper, X is considered as topological vector space and K is a nonempty subset of X. Let $\mathbb{H}$ be a Hilbert space. Let $F : X \to \mathbb{H}$ be any mapping. Let $Y = (Ran(F), \tau)$ where $Ran(F)$ be the range of F and τ be the induced topology of $Ran(F)$ obtained by F. Let (Y, P) be a topological vector space of dimension n in an Hilbert space $\mathbb{H}$ equipped with the closed convex pointed cone with nonempty interior, i.e., $intP \neq \varnothing$. Let $F : X \to Y$ be a mapping defined by

$$F(x) = (F_1(x), F_2(x), \cdots, F_n(x))$$

where $F_i : K \to \mathbb{R}$ are not necessarily smooth but Lipschitz continuous for each $x \in X$. Let $T : K \to L(X,Y)$ be any map and $\xi : K \to \partial F(K)$ be the operator functional defined by

$$\xi(x) \in \partial F(x)$$

for each $x \in K$ if and only if

$$(\xi_1(x), \xi_2(x), \cdots, \xi_n(x)) \in \prod_{i=1}^{n} \partial F_i(x).$$

2. Optimality of Generalized OP and its Equivalence Theorem with GVIP

In 2006, Matrino and Xu [17] have defined the variational inequality problem **(*VIP*)** in Hilbert space $\mathbb{H}$ as follows. The problem is to :
find $z_0 \in M \subset \mathbb{H}$ such that

$$\langle A(z_0), z - z_0\rangle \geq \gamma \langle \xi(z_0), z - z_0\rangle \quad \text{for all } z \in M. \qquad (VIP{:}T{,}F)$$

They used this problem to show the optimality condition for the minimization problem

$$\min_{z\in M} \frac{1}{2}\langle A(z), z\rangle - F(z),$$

in a Hilbert space $\mathbb{H}$, where $A : M \to L(\mathbb{H})$, $\xi : \mathbb{H} \to L(\mathbb{H})$ and F is the potential function for $\gamma\xi$ (i.e., $F'(z) = \gamma\xi(z)$ for all $z \in M \subset \mathbb{H}$).

For our need, we make the following definition.

Definition 5. Let K be any subset of the vector space X such that $0 \in K$. Let $\eta : K \times K \to X$ be continuous vector valued mapping. Let K be η-invex. The set $\mathrm{Ker}_\eta(K)$ called as *Kernel* of K with respect to η is defined by

$$\mathrm{Ker}_\eta(K) = \{u \in K : \eta(u + t\eta(z,u), u) + t\eta(z,u) = 0,\ t \in [0,1],\ z \in X\}.$$

We consider the primal problem (**P**) which is a minimization problem defined by

$$\inf_{x\in \mathrm{Ker}_\eta(K)} \frac{1}{2}\langle T(x), \eta(z,x)\rangle - F(x), \tag{P}$$

subject to:

$$z \in K, \quad \text{and} \quad T(x) \in K^{\ominus} = \{f \in L(X,Y) : \langle f, v\rangle \notin intP,\ v \in X\}$$

where X is a topological vector space, $T : K \to L(X,Y)$, and $\xi : K \to L(X,Y)$ and F is the potential function for $\gamma\xi$ (i.e., $F'(x) = \gamma\xi(x)$ for all $x \in K$).

The corresponding dual problem of (**P**) is the maximization problem (**D**) defined by

$$\sup_{y\in \mathrm{Ker}_\eta(K)} \frac{1}{2}\langle T(y), \eta(z,y)\rangle - F(y), \tag{D}$$

subject to:

$$z \in K, \quad \text{and} \quad T(y) \in K^{\oplus} = \{f \in L(X,Y) : \langle f, v\rangle \notin -intP,\ v \in X\}.$$

In this section, we consider the generalized F-variational inequality problem (*GVIP:T,F*) is to:

find $y \in K \subset X$ such that

$$\langle T(y), \eta(x,y)\rangle \geq \gamma \langle \xi(y), \eta(x,y)\rangle \quad \text{for all } x \in K. \tag{GVIP:T,F}$$

Remark 1. The problem (*GVIP:T,F*) coincides with (*VIP:T,F*) if $\eta(z,x)$ is replaced by x in the minimization problem.

For our need, we consider the following condition.

Condition 1 (PPC)**.** For the primal problem, the condition is

$$\langle T(x), \eta(z,x)\rangle + \langle \nabla F(x), \eta(z,x)\rangle \notin -int\,P \cup int\,P \tag{1a}$$

$$\langle \nabla F(x), \eta(p,x)\rangle \notin -int\,P \cup int\,P \tag{1b}$$

for all $p \in K$ and $z \in I(x,y)$.

For simplicity, we use the following notations [3].

Note 1. For simplicity, we use the following terminologies:

(a) $y \notin -intP$ if and only if $y \in P$ if and only if $y \geq_P 0$;
(b) $y \in intP$ if and only if $y >_P 0$;
(c) $y \notin intP$ if and only if $y \in -P$ if and only if $y \leq_P 0$;
(d) $y \in -intP$ if and only if $y <_P 0$;
(e) $y - z \notin -intP$ if and only if $y - z \geq_P 0$ (i.e., $y \geq_P z$);
(f) $y - z \notin intP$ if and only if $y - z \leq_P 0$ (*i.e.*, $y \leq_P z$);
(g) $y - z \notin (-intP \bigcup intP)$ if and only if *if* $y - z \in (-P \bigcap P)$ if and only if $y - z =_P 0$, (*i.e.*, $y =_P z$).

We also use the following terminologies as and when required:

(A) $y - z \notin -P$ and $z \notin -intP$ imply $y \notin -intP$;
(B) $y - z \notin -intP$ and $z \notin -intP$ imply $y \notin -intP$;
(C) $y - z \notin -P$ and $y \in -intP$ imply $z \in -intP$;
(D) $y - z \notin -intP$ and $y \notin -intP$ imply $z \notin -intP$;
(E) $y - z \in -intP$ and $z \in -intP$ imply $y \in -intP$;
(F) $y \notin -intP$ if and only if $-y \notin intP$;
(G) $y \notin -intP$ and $z \notin -intP$ imply $y + z \notin -intP$.

Lemma 2.1. *Let* $K \subset X$ *and* P *be a closed convex pointed cone in* Y. *Let* K *be a* η*-invex set. Let* T *be* η*-monotone on* $\mathrm{Ker}_\eta(K)$. *Let* $x \in \mathrm{Ker}_\eta(K)$ *be a feasible solution of* (**P**) *and* $y \in \mathrm{Ker}_\eta(K)$ *be a feasible solution of* (**D**). *Then*

(a) $\langle T(y), \eta(z, y)\rangle - \langle T(y), \eta(x, y)\rangle \notin int\, P$,
(b) $\langle T(x), \eta(z, x)\rangle + \langle T(x), \eta(y, x)\rangle \notin int\, P$

for all $z \in I(x, y)$ *the invex path joining* $x, y \in \mathrm{Ker}_\eta(K)$.

Proof. T is η-monotone at $y \in \mathrm{Ker}_\eta(K)$, i.e.,

$$\langle T(y), \eta(x, y)\rangle + \langle T(x), \eta(y, x)\rangle \leq_P 0$$

for all $x \in \mathrm{Ker}_\eta(K)$. We claim that

(a) $\langle T(y), \eta(z, y)\rangle - \langle T(y), \eta(x, y)\rangle \notin int\, P$,
(b) $\langle T(x), \eta(z, x)\rangle + \langle T(x), \eta(y, x)\rangle \notin int\, P$

for all $z \in I(x, y)$, $x, y \in \mathrm{Ker}_\eta(K)$. By the condition of the dual problem **D**, we get $T(y) \in K^{\oplus}$, i.e.,

$$\langle T(y), \eta(x, y)\rangle \notin -int\, P$$

for all $x \in \mathrm{Ker}_\eta(K)$, implying

$$\langle T(x), \eta(y, x)\rangle \notin int\, P$$

for all $x, y \in \mathrm{Ker}_\eta(K)$. By definition of the set $\mathrm{Ker}_\eta(K)$, η satisfies condition C_0 at y. Since $z \in I(x, y)$, we have $z = y + t\eta(x, y)$ for $x, y \in \mathrm{Ker}_\eta(K)$ and

$t \in (0,1)$. Now Replacing z by $z = y + t\eta(x,y)$ in the L.H.S. of (a), we get

$$\begin{aligned}\langle T(y), \eta(z,y)\rangle - \langle T(y), \eta(x,y)\rangle &=_P \langle T(y), \eta(y + t\eta(x,y), y)\rangle - \langle T(y), \eta(x,y)\rangle \\ &=_P -t\,\langle T(y), \eta(x,y)\rangle - \langle T(y), \eta(x,y)\rangle \\ &=_P -(1+t)\,\langle T(y), \eta(x,y)\rangle \\ &\leq_P 0\end{aligned}$$

for $x, y \in \mathrm{Ker}_\eta(K)$ and $t \in (0,1)$. This proves (a). By the condition of the dual problem **P**, we get $T(x) \in K^{\ominus}$, i.e.,

$$\langle T(x), \eta(y,x)\rangle \notin int\, P$$

for all $y \in \mathrm{Ker}_\eta(K)$. Replacing z by $z = x + \lambda\eta(y,x)$ in the L.H.S. of (b), we get

$$\begin{aligned}\langle T(x), \eta(z,x)\rangle + \langle T(x), \eta(y,x)\rangle &= \langle T(x), \eta(x + \lambda\eta(y,x), x)\rangle + \langle T(x), \eta(y,x)\rangle \\ &=_P -\lambda\,\langle T(x), \eta(y,x)\rangle + \langle T(x), \eta(y,x)\rangle \\ &=_P (1-\lambda)\,\langle T(x), \eta(y,x)\rangle \\ &\leq_P 0\end{aligned}$$

for $x, y \in \mathrm{Ker}_\eta(K)$ and $\lambda \in (0,1)$. This proves (b). This completes the proof. □

In the following theorem, we establish the sufficient optimality of the (Primal) problem.

Theorem 2.2 (**Sufficient Optimality Theorem**). *Let $K \subset X$ and P be a closed convex pointed cone in Y. Let K be a η-invex set. Let T be η-monotone on $\mathrm{Ker}_\eta(K)$. Let F be smooth on $\mathrm{Ker}_\eta(K)$. Let $y \in \mathrm{Ker}_\eta(K)$ be a feasible solution of* (**P**) *at which (PPC) conditions are satisfied. Then y is an optimal solution for the problem* (**P**).

Proof. Since $y \in \mathrm{Ker}_\eta(K)$ is the feasible solution of (**P**), $T(y) \in K^{\oplus}$, implying

$$\langle T(y), \eta(x,y)\rangle \notin -int\, P$$

for all $x \in K$. By (PPC) conditions,

$$\langle \nabla F(y), \eta(x,y)\rangle \notin int\, P$$

for all $x \in K$, implying

$$F(x) - F(y) \notin int\, P$$

for all $x \in K$ as K is η-invex. Using condition C_0 of η, we get

$$\begin{aligned}
\frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x) &=_P \frac{1}{2}\langle T(x),\eta(x+\lambda\eta(y,x),x)\rangle - F(x)\\
&=_P -\frac{\lambda}{2}\langle T(x),\eta(y,x)\rangle - F(y)\\
&\geq_P -\frac{1}{2}\langle T(x),\eta(y,x)\rangle - F(y)\\
&\geq_P \frac{1}{2}\langle T(y),\eta(x,y)\rangle - F(y)\\
&\geq_P \frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y)
\end{aligned}$$

for all $x \in \mathrm{Ker}_\eta(K)$, $z \in I(x,y)$, and $\lambda \in (0,1)$, i.e.,

$$\inf_{x\in \mathrm{Ker}_\eta(K)} \frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x) \geq_P \sup_{y\in \mathrm{Ker}_\eta(K)} \frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y),$$

for all $z \in I(x,y)$. Hence y is an optimal solution for the problem (**P**). This completes the proof of the theorem. □

The following theorem establishes the existence of weak duality theorem under certain conditions.

Theorem 2.3 (**Weak Duality Theorem**). *Let $K \subset X$ and P be a closed convex pointed cone in Y. Let K be η-invex set. Let $x \in \mathrm{Ker}_\eta(K)$ be feasible solution of* (**P**) *and $y \in \mathrm{Ker}_\eta(K)$ be a feasible solution of* (**D**)*. Let F be smooth on K and T-η-invex at $y \in \mathrm{Ker}_\eta(K)$. Assume that*

$$\langle T(x),\eta(z,x)\rangle + \langle T(x),\eta(y,x)\rangle =_P 0$$

for all $z \in I(x,y)$. Then

$$\left[\frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x)\right] - \left[\frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y)\right] \notin -int\,P.$$

Proof. Since F is T-η-invex on $\mathrm{Ker}_\eta(K)$, then by Theorem 2.13 [3], T is η-monotone on $\mathrm{Ker}_\eta(K)$. Since all the conditions of Lemma 2.1 are satisfied, we have

(*a*) $\langle T(y),\eta(z,y)\rangle - \langle T(y),\eta(x,y)\rangle \notin int\,P$,
(*b*) $\langle T(x),\eta(z,x)\rangle + \langle T(x),\eta(y,x)\rangle \notin int\,P$

for all $z \in I(x,y)$. Since $y \in \mathrm{Ker}_\eta(K)$ is the feasible solution of (**D**), we have

$$\langle \nabla F(y),\eta(x,y)\rangle \notin int\,P$$

for all $x \in K$, implying

$$F(x) - F(y) \notin int\,P$$

for all $x \in K$ as K is η-invex. Taking

$$\langle T(x),\eta(z,x)\rangle + \langle T(x),\eta(y,x)\rangle =_P 0$$

for all $z \in I(x,y)$, we get

$$\begin{aligned}\frac{1}{2}\langle T(y),\eta(z,y)\rangle + F(x) &\leq_P \frac{1}{2}\langle T(y),\eta(x,y)\rangle + F(x)\\ &\leq_P -\frac{1}{2}\langle T(x),\eta(y,x)\rangle + F(y)\\ &=_P \frac{1}{2}\langle T(x),\eta(z,x)\rangle + F(y),\end{aligned}$$

implying

$$\frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y) \leq_P \frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x)$$

for all $z \in I(x,y)$. Thus

$$\left[\frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x)\right] - \left[\frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y)\right] \notin -int\, P$$

for all $z \in I(x,y)$. This completes the proof of the theorem. □

Theorem 2.4 (**Strong Duality Theorem**). *Let F, T and η satisfy all the condition of Weak Duality Theorem 2.3. Let F be smooth and T-η-invex on* $\mathrm{Ker}_\eta(K)$. *Let $x^* \in \mathrm{Ker}_\eta(K)$ be the optimal solution of* (**P**), *then $x^* \in \mathrm{Ker}_\eta(K)$ is the optimal solution of* (**D**). *The optimal solution x^* is unique if T is strictly η-monotone on K.*

Proof. Since $x^* \in \mathrm{Ker}_\eta(K)$ is the optimal solution of (**P**), we have

$$\inf_{x\in \mathrm{Ker}_\eta(K)}\left[\frac{1}{2}\langle T(x),\eta(z,x)\rangle - F(x)\right] =_P \frac{1}{2}\langle T(x^*),\eta(z,x^*)\rangle - F(x^*)$$

for all $z \in I(x^*,y^*)$, implying $T(x^*) \in K^{\ominus}$, i.e.,

$$\langle T(x^*),\eta(z,x^*)\rangle \notin int\, P$$

for all $z \in K$. Replacing z by $x^* + t\eta(x,x^*)$, $t \in (0,1)$ in the above equation and applying condition C_0, we get

$$\langle T(x^*),\eta(z,x^*)\rangle \notin -int\, P$$

for all $z \in K$, implying

$$\langle T(x^*),\eta(x,x^*)\rangle \notin -int\, P \cup int\, P$$

for all $x \in K$. Hence $T(x^*) \in K^{\oplus}$ which is a condition of (**D**). By Weak Duality Theorem 2.3, $x^* \in \mathrm{Ker}_\eta(K)$ is the optimal solution of (**D**), i.e.,

$$\sup_{y\in \mathrm{Ker}_\eta(K)}\left[\frac{1}{2}\langle T(y),\eta(z,y)\rangle - F(y)\right] =_P \frac{1}{2}\langle T(x^*),\eta(z,x^*)\rangle - F(x^*)$$

for all $z \in K$. Next to prove x^* is unique. If not, let y^* be another optimal solution of (**D**). Since F is T-η-invex on $\mathrm{Ker}_\eta(K)$, we have

$$\langle T(y),\eta(x,y)\rangle + \langle T(x),\eta(y,x)\rangle \leq_P 0$$

for $x, y \in \mathrm{Ker}_\eta(K)$. By strictly monotonicity of T, the inequality holds for $x^* \neq y^*$ and equality holds for $x^* = y^*$, i.e.,

$$\langle T(y^*), \eta(x^*, y^*)\rangle + \langle T(x^*), \eta(y^*, x^*)\rangle =_P 0,$$

implying $x^* = y^*$. Hence the optimal solution of (**P**) is unique. This completes the proof of the theorem. □

The following theorem establishes the existence of the result obtained from the Weak duality Theorem 2.3 for the problems (**P**) and (**D**) is equivalent to find the solution of the problem (*GVIP:T,F*) under certain conditions.

Theorem 2.5. *Let F, T and η satisfy all the condition of Weak Duality Theorem 2.3. Let F be smooth on $\mathrm{Ker}_\eta(K)$. Let $x \in \mathrm{Ker}_\eta(K)$ be the optimal solution of* (**P**) *and $y \in \mathrm{Ker}_\eta(K)$ be the optimal solution of* (**D**), *then y solves the problem (GVIP:T,F) where $\xi(y) = \nabla F(y)$, the Hadamard differential of F at y and $\gamma = 2$.*

Proof. Since $y \in \mathrm{Ker}_\eta(K)$, η satisfies condition C_0 at y. By Weak Duality Theorem 2.3, we have

$$\frac{1}{2}\langle T(x), \eta(z, x)\rangle - F(x) \geq \frac{1}{2}\langle T(y), \eta(z, y)\rangle - F(y)$$

for all $z \in I(x, y)$, $x \in \mathrm{Ker}_\eta(K)$. For $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in \mathrm{Ker}_\eta(K)$ for all $x \in \mathrm{Ker}_\eta(K)$. Replacing z by $y + \lambda\eta(x, y)$ and using condition C_0, we get

$$\frac{1}{2}\langle T(x), -(1+\lambda)\eta(x, y)\rangle - F(x) \geq \frac{1}{2}\langle T(y), -\lambda\eta(x, y)\rangle - F(y),$$

i.e.,

$$-\frac{1+\lambda}{2}\langle T(x), \eta(x, y)\rangle + \frac{\lambda}{2}\langle T(y), \eta(x, y)\rangle \geq F(x) - F(y),$$

for all $x \in K$, $y \in \mathrm{Ker}_\eta(K)$ and $\lambda \in [0, 1]$. At $\lambda = 1$, we get

$$-\langle T(x), \eta(x, y)\rangle + \frac{1}{2}\langle T(y), \eta(x, y)\rangle \geq F(x) - F(y),$$

for all $x \in K$ and $y \in \mathrm{Ker}_\eta(K)$. Again replacing x by $y + t\eta(x, y)$ and using condition C_0, we get

$$t\langle T(y + t\eta(x, y)), \eta(x, y)\rangle - \frac{t}{2}\langle T(y), \eta(x, y)\rangle \geq F(y + t\eta(x, y)) - F(y),$$

for all $x \in K$ and $y \in \mathrm{Ker}_\eta(K)$. Dividing both sides by t and taking limit as $t \to 0$, we get

$$\langle T(y), \eta(x, y)\rangle - \frac{1}{2}\langle T(y), \eta(x, y)\rangle \geq \langle \nabla F(y), \eta(x, y)\rangle,$$

implying,

$$\begin{aligned}\langle T(y), \eta(x, y)\rangle &\geq 2\langle \nabla F(y), \eta(x, y)\rangle \\ &= \langle \gamma\xi(y), \eta(x, y)\rangle\end{aligned}$$

for all $x \in K$ and $y \in \mathrm{Ker}_\eta(K)$ where $\xi(y) = \nabla F(y)$, i.e., $\gamma = 2$. This completes the proof of the theorem. □

3. *PDA*-Operator and its Example

We introduce the concept of partial differential associated (*PDA*)-operator, *PDA*-vector function and *PDA*-antisymmetric function as follows.

3.1. Definition and Properties

We introduce the new operator $\mathcal{D}$, called *PDA-operator* which is a symmetrical partial differential operator. *PDA*-operator induces a weakly antisymmetric function, called *PDA-antisymmetric function* associated with a function φ. For our need, we use the following notations.

Definition 6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function (not necessarily smooth) such that $f_{xy} = f_{yx}$. Let $\mathcal{P}_f : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector valued mapping defined by*

$$\mathcal{P}_f(x, y) = [f(x, y), f(y, x)].$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real valued function. The PDA-operator

$$\mathcal{D}^\varphi_{yx} : dom(\mathcal{P}_f) \to \mathbb{R}$$

is defined by

$$\mathcal{D}^\varphi_{yx}(\mathcal{P}_f(x, y)) = \mathcal{D}^\varphi_{yx} \cdot \mathcal{P}_f(x, y).$$

Thus

$$\begin{aligned} \mathcal{D}^\varphi_{yx}(\mathcal{P}_f(x, y)) &= \left[\varphi(y)\partial^2_{xy},\ \varphi(x)\partial^2_{yx}\right] \cdot [f(x, y), f(y, x)] \\ &= \varphi(y)\partial^2_{xy} f(x, y) + \varphi(x)\partial^2_{yx} f(y, x) \end{aligned} \tag{2}$$

for $x, y \in \mathbb{R}$. Interchanging x and y in the above equation, we get

$$\begin{aligned} \mathcal{D}^\varphi_{xy}(\mathcal{P}_f(y, x)) &= \varphi(x)\partial^2_{yx} f(y, x) + \varphi(y)\partial^2_{xy} f(x, y) \\ &= \varphi(y)\partial^2_{xy} f(x, y) + \varphi(x)\partial^2_{yx} f(y, x) \\ &= \mathcal{D}^\varphi_{yx}(\mathcal{P}_f(x, y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$. Thus

$$\mathcal{D}^\varphi_{xy}(\mathcal{P}_f(y, x)) = \mathcal{D}^\varphi_{yx}(\mathcal{P}_f(x, y)) = \varphi(y)\partial^2_{xy} f(x, y) + \varphi(x)\partial^2_{yx} f(y, x)$$

for all $x, y \in \mathbb{R}$. Sine $f_{xy} = f_{yx}$, i.e., $\partial^2_{xy} f = \partial^2_{yx} f$, we get

$$\mathcal{D}^\varphi_{yx}(\mathcal{P}_f(x, y)) = \varphi(y)\partial^2_{yx} f(x, y) + \varphi(x)\partial^2_{yx} f(y, x)$$

for all x, y, implying

$$\mathcal{D}^\varphi_{yx} = [\varphi(y), \varphi(x)]\, \partial^2_{yx}$$

and

$$\begin{aligned}\mathcal{D}^{\varphi}_{xy}(\mathcal{P}_f(y,x)) &= \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(x,y))\\ &= \varphi(y)\partial^2_{xy}f(x,y)+\varphi(x)\partial^2_{yx}f(y,x)\\ &= \varphi(y)\partial^2_{yx}f(x,y)+\varphi(x)\partial^2_{xy}f(y,x)\\ &= \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(y,x)) \qquad (3)\end{aligned}$$

for all x, y, implying $\mathcal{D}^{\varphi}_{xy}(\mathcal{P}_f(y,x))$ and $\mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(y,x))$ are of two functions having asymptotes are symmetrical. Thus

$$\mathcal{D}^{\varphi}_{yx} \asymp_{\mathcal{P}_f} \mathcal{D}^{\varphi}_{xy};$$

where $\asymp_{\mathcal{P}_f}$ denotes "asymptotically symmetrical" with respect to the function

$$\mathcal{P}_f(x,y) = [f(x,y),\ f(y,x)]$$

and the *PDA*-operator $\mathcal{D}^{\varphi}_{yx}$ is said to be weakly symmetric with respect to the vector function

$$\mathcal{P}_f(x,y) = [f(x,y),\ f(y,x)] \quad \text{if } f \text{ is continuous.}$$

3.1.1. *Property of PDA-Operator.* The *PDA*-operator $\mathcal{D}^{\varphi}_{yx}$ satisfies

$$f \equiv g \Rightarrow \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f) = \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_g),$$

i.e., for all $x, y \in \mathbb{R}$,

(i) if $f \equiv g$, then $\mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f) = \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_g)$,
(ii) if $\mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f) = \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_g)$, then $f \not\equiv g$.

Definition 7 (*PDA*-Vector Function). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. The vector function $\mathcal{P}_f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathcal{P}_f(x,y) = [f(x,y),\ f(y,x)]$$

is said to be *PDA-vector function* with respect to $f(x,y)$ associated with $\varphi : \mathbb{R} \to \mathbb{R}$, if

(i) $f(x,y)$ is a weakly antisymmetric function (or weakly symmetric function) with respect to the PDA-operator $\mathcal{D}_{yx}$
(ii) for all x, y, we have

$$\mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(x,y)) = 0,$$

i.e.,

$$\varphi(y)\partial^2_{yx}f(x,y)+\varphi(x)\partial^2_{yx}f(y,x) = 0.$$

For simplicity, we call φ as *weighted function* for the equilibrium and f as *PDA-antisymmetric function.*

From the definition of *PDA*-antisymmetric function, we get

$$\mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(x,y)) = 0.$$

The above expression can be written as,

$$\frac{1}{\varphi(x)}\partial^2_{yx} f(x,y) + \frac{1}{\varphi(y)}\partial^2_{yx} f(y,x) = 0,$$

equivalently,

$$\frac{1}{\varphi(x)} f_{xy}(x,y) + \frac{1}{\varphi(y)} f_{xy}(y,x) = 0$$

or

$$\frac{1}{\varphi(x)} f_{yx}(x,y) + \frac{1}{\varphi(y)} f_{yx}(y,x) = 0.$$

We prove the existence theorem for *PDA*-vector function relative to $f(x,y)$ as follows.

Theorem 3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$ be any two continuous functions. Let the PDA-antisymmetric function $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined by*

$$f_{yx}(x,y) = \varphi(x)[p(x) - p(y)]$$

for all $x, y \in \mathbb{R}$. Then the function $\mathcal{P}_f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathcal{P}_f(x,y) = [f(x,y),\ f(y,x)]$$

is a PDA-vector function with respect to the PDA-operator

$$\mathcal{D}^{\varphi}_{yx} = \left[\varphi(y)\partial^2_{xy},\ \varphi(x)\partial^2_{yx}\right]$$

associated with the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ and relative to f.

Proof. To prove the vector function $\mathcal{P}_f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathcal{P}_f(x,y) = [f(x,y),\ f(y,x)]$$

is a *PDA*-vector function with respect to the *PDA*-operator $\mathcal{D}^{\varphi}_{yx}$, we show

$$\frac{1}{\varphi(x)}\partial^2_{yx} f(x,y) + \frac{1}{\varphi(y)}\partial^2_{yx} f(y,x) = 0,$$

i.e.,

$$\frac{1}{\varphi(x)} f_{xy}(x,y) + \frac{1}{\varphi(y)} f_{xy}(y,x) = 0.$$

Since $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_{yx}(x,y) = \varphi(x)\,[\ p(x) - p(y)]$$

for all x, y is continuous; interchanging x and y in the above expression, we get

$$f_{xy}(y,x) = \varphi(y)\,[\ p(y) - p(x)]\,,$$

i.e.,

$$f_{yx}(y,x) = \varphi(y)\,[\ p(y) - p(x)]$$

for all x, y. Hence

$$\begin{aligned}\frac{1}{\varphi(x)} f_{yx}(x,y) + \frac{1}{\varphi(y)} f_{yx}(y,x) &= \frac{1}{\varphi(x)}\varphi(x)\,[\,p(x)-p(y)] \\ &\quad + \frac{1}{\varphi(y)}\varphi(y)\,[\,p(y)-p(x)] \\ &= p(x)-p(y)+p(y)-p(x) \\ &= 0\end{aligned}$$

for all x, y, showing $\mathcal{P}_f(x,y)$ is *PDA*-vector function. This completes the proof of the theorem. □

The following example illustrates the existence of *PDA*-vector function $\mathcal{P}_f(x,y)$ with respect to the *PDA*-operator $\mathcal{D}^{\varphi}_{yx}$ associated with φ relative to *PDA*-antisymmetric function $f(x,y)$.

Example 1. Let the mappings $f : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^3}{3} e^{-y} - [1+(1+x)^2] y e^{-x},$$

and

$$\varphi(x) = x^2$$

respectively. We prove $\mathcal{P}_f(x,y)$ is *PDA*-vector function with respect to the *PDA*-operator

$$\mathcal{D}^{\varphi}_{yx} = \left[\varphi(y)\partial^2_{xy},\ \varphi(x)\partial^2_{yx}\right]$$

associated with the mapping $\varphi(x)$, and relative to *PDA*-antisymmetric function $f(x,y)$, that is,

$$D^{\varphi}_{yx}(\mathcal{P}_f(x,y)) = 0,$$

i.e.,

$$\varphi(y) f_{yx}(x,y) + \varphi(x) f_{yx}(y,x) = 0,$$

i.e.,

$$y^2 f_{yx}(x,y) + x^2 f_{yx}(y,x) = 0.$$

We have

$$\begin{aligned}f_{yx}(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y) &= \frac{\partial^2}{\partial x \partial y}\left(\frac{x^3}{3} e^{-y} - [1+(1+x)^2] y e^{-x}\right) \\ &= x^2(e^{-x} - e^{-y}).\end{aligned}$$

Now $f_{xy}(y,x) = f_{yx}(y,x)$, since

$$\begin{aligned}f_{yx}(y,x) = \frac{\partial^2}{\partial x \partial y} f(y,x) &= \frac{\partial^2}{\partial x \partial y}\left(\frac{y^3}{3} e^{-x} - [1+(1+y)^2] x e^{-y}\right) \\ &= y^2(e^{-y} - e^{-x})\end{aligned}$$

and

$$f_{xy}(y,x) = \frac{\partial^2}{\partial y \partial x} f(y,x) \;=\; \frac{\partial^2}{\partial y \partial x}\left(\frac{y^3}{3}e^{-x} - [1+(1+y)^2]xe^{-y}\right)$$
$$= y^2(e^{-y} - e^{-x}).$$

Hence

$$\begin{aligned} D^{\varphi}_{yx}(\mathcal{P}_f(x,y)) &= y^2 f_{yx}(x,y) + x^2 f_{yx}(y,x) \\ &= y^2x^2(e^{-x} - e^{-y}) + x^2y^2(e^{-y} - e^{-x}) \\ &= x^2y^2(e^{-x} - e^{-y} + e^{-y} - e^{-x}) \\ &= 0 \end{aligned}$$

for all x, y. Hence, $\mathcal{P}_f(x,y)$ is *PDA*-vector function.

The *PDA*-antisymmetric function

$$f(x,y) = \frac{x^3}{3}e^{-y} - [1+(1+x)^2]ye^{-x}$$

given in Example 1 is used in Example 2 to show the existence of $(T_\eta;\xi_\theta)$-invex function of both first kind and second kind.

Theorem 3.2. *Let $W = K \times K$ be a contractible domain in $\mathbb{R}^2$ where $K \subset \mathbb{R}$. Let $\theta : W \to \mathbb{R}$ be a continuous antisymmetric function, i.e., $\theta(x,y)+\theta(y,x) = 0$ for all $x, y \in K$. Let $\varphi : K \to \mathbb{R}$ and $f : W \to \mathbb{R}$ be two functions given by the relation,*

$$f_{yx}(x,y) = \varphi(x)\theta(x,y)$$

for all $x, y \in K$. Then $\mathcal{P}_f(x,y)$ is a PDA-vector vector function with respect to the PDA-operator $\mathcal{D}^{\varphi}_{yx}$ associated with φ and relative to PDA-antisymmetric function f.

Furthermore if $\theta : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\theta(x,y) = p(x) - p(y)$$

where p is a linear skew projective map, then $f(x,y)$ satisfies condition C_0 weakly with respect to the operator $\frac{1}{\varphi(x)}\partial^2_{yx}$.

Proof. Since $\theta : W \to \mathbb{R}$ is antisymmetric, we get

$$\theta(x,y) + \theta(y,x) = 0$$

and

$$\begin{aligned} \mathcal{D}^{\varphi}_{yx}(\mathcal{P}_f(x,y)) &= \varphi(y)\partial^2_{xy}f(x,y) + \varphi(x)\partial^2_{yx}f(y,x) \\ &= \varphi(y)\partial^2_{yx}f(x,y) + \varphi(x)\partial^2_{yx}f(y,x) \\ &= \varphi(y)f_{xy}(x,y) + \varphi(x)f_{yx}(y,x) \\ &= \varphi(y)\varphi(x)\theta(x,y) + \varphi(x)\varphi(y)\theta(y,x) \\ &= \varphi(x)\varphi(y)\left[\theta(x,y) + \theta(y,x)\right] \\ &= 0, \end{aligned}$$

for all $x, y \in \mathbb{K}$, so $\mathcal{P}_f(x,y)$ is *PDA*-vector function with respect to the *PDA*-operator $\mathcal{D}^{\varphi}_{yx}$ associated with the mapping φ, and relative to *PDA*-antisymmetric function $f(x,y)$.

Now, we have

$$f_{yx}(x,y) = \varphi(x)\theta(x,y),$$

implying

$$\frac{1}{\varphi(x)} f_{yx}(x,y) = \theta(x,y),$$

i.e.,

$$\frac{1}{\varphi(x)} \partial^2_{yx} f(x,y) = \theta(x,y)$$

for all $x, y \in \mathbb{R}$. Since p is a linear skew projective, (*i.e.*, $p^2 = -p$) and

$$\theta(x,y) = p(x) - p(y),$$

θ satisfies condition C_0. Hence $f(x,y)$ satisfies condition C_0 weakly with respect to the operator $\frac{1}{\varphi(x)}\partial^2_{yx}$. This completes the proof of the theorem. □

4. $(T_\eta;\xi_\theta)$-Invex Function

We introduce the the concept of $(T_\eta;\xi_\theta)$-invex functions of two kinds in ordered topological vector space Y where X and Y are defined in previous section.

Definition 8. The mapping $F : K \to Y$ is said to be $(T_\eta;\ \xi_\theta)$-*invex function of first kind* on K if there exists a vector function $\theta : K \times K \to X$ such that for all $x, x' \in K$,

$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle - \langle \xi(x'), \theta(x,x')\rangle \geq_P 0.$$

Definition 9. The mapping $F : K \to Y$ is said to be $(T_\eta;\ \xi_\theta)$-*invex function of second kind* on K if there exists a vector function $\theta : K \times K \to X$ such that for all $x, x' \in K$,

$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle + \langle \xi(x'), \theta(x,x')\rangle \geq_P 0.$$

Remark 2.
(1) Let $Y = \mathbb{R}$. Let for some $x' \in K$, $\xi(x') \equiv 0$ then both the $(T_\eta;\ \xi_\theta)$-invexity of first kind and second kind of F coincide with the definition of T-η-invexity function of F [3].
(2) Let X be a Hilbert space, $Y = \mathbb{R}$, and $\xi(x') = \xi \in \partial F(x')$, the subdifferential of F at x' then the definition of $(T_\eta;\ \xi_\theta)$-invex function of first kind coincides with the definition of θ-invex function defined in nonsmooth analysis.
(3) If $\xi \neq T$ and for all $x, x' \in K$,

$$\langle T(x'), \eta(x,x')\rangle \geq_P 0$$

then the definition of $(T_\eta;\ \xi_\theta)$-invex function of first kind reduces to nonsmooth θ-invex function [11].

Remark 3. Let for each $x' \in K$, $\langle \xi(x'), \theta(x,x') \rangle \geq_P 0$ for all $x \in K$, then for any map $F : K \to Y$, we have

$$F(x) - F(x') - \langle T(x'), \eta(x,x') \rangle + \langle \xi(x'), \theta(x,x') \rangle \\ \geq_P F(x) - F(x') - \langle T(x'), \eta(x,x') \rangle - \langle \xi(x'), \theta(x,x') \rangle$$

for all $x \in K$. If for each $x' \in K$,

$$F(x) - F(x') - \langle T(x'), \eta(x,x') \rangle - \langle \xi(x'), \theta(x,x') \rangle \geq_P 0,$$

then

$$F(x) - F(x') - \langle T(x'), \eta(x,x') \rangle + \langle \xi(x'), \theta(x,x') \rangle \geq_P 0$$

for all $x \in K$ which establishes the relation between $(T_\eta;\, \xi_\theta)$-invex function of first kind and second kind.

For each $x \in K$, let $K_\theta^+(x)$, $K_\theta^-(x)$ and $K_\theta^0(x)$ be the subsets of K defined by

$$K_\theta^+(x) = \{x' \in K : \langle \xi(x'), \theta(x,x') \rangle \geq_P 0\},$$
$$K_\theta^-(x) = \{x' \in K : \langle \xi(x'), \theta(x,x') \rangle \leq_P 0\}$$

and

$$K_\theta^0(x) = \{x' \in K : \langle \xi(x'), \theta(x,x') \rangle =_P 0\},$$

then

$$K_\theta^+(x) \bigcup K_\theta^-(x) = K$$

and

$$K_\theta^+(x) \bigcap K_\theta^-(x) = K_\theta^0(x).$$

Proposition 4.1. *Let $F : K \subset X \to Y$ be a nonsooth Lipschitz continuous map on K. Let $T : K \to L(X,Y)$ and $\xi : K \to \partial F(K)$ be two functionals. Let $\eta : K \times K \to X$ and $\theta : K \times K \to X$ be any vector valued functions. Then the following are true:*

(1) *If for each $x \in K$, F is $(T_\eta;\, \xi_\theta)$-invex function of first kind at $x' \in K_\theta^+(x)$, then F is $(T_\eta;\, \xi_\theta)$-invex function of second kind and F is T-η-invex at $x' \in K_\theta^+(x)$, but not conversely, that is, if F is $(T_\eta;\, \xi_\theta)$-invex function of second kind at $x' \in K_\theta^+(x)$, then F may or may not be $(T_\eta;\, \xi_\theta)$-invex function of first kind on $K_\theta^+(x)$ or F is T-η-invex at $x' \in K_\theta^+(x)$.*

(2) *If for each $x \in K$, F is $(T_\eta;\, \xi_\theta)$-invex function of second kind at $x' \in K_\theta^-(x)$, then F is T-η-invex at $x' \in K_\theta^-(x)$, and F is $(T_\eta;\, \xi_\theta)$-invex function of first kind at $x' \in K_\theta^-(x)$, but not conversely.*

(3) *Let $x \in K$. Then for all $x' \in K_\theta^0(x)$, $(T_\eta;\, \xi_\theta)$-invexity function of first kind, second kind and T-η-invexity of F coincide with each other.*

Proof. **Case (1):** Let F be $(T_\eta;\, \xi_\theta)$-invex function of first kind at $x' \in K_\theta^+(x)$. Then, at $x' \in K_\theta^+(x)$, we have

$$\langle \xi(x'), \theta(x,x') \rangle \geq_P 0,$$

and

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle-\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

i.e.,

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle \geq_P \langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

implying,

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle \geq_P 0$$

for each $x\in K$, that is, F is T-η-invex at $x'\in K_\theta^+(x)$. Again adding the above two inequalities, we obtain

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle+\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

that is, F is $(T_\eta;\,\xi_\theta)$-invex function of second kind at $x'\in K_\theta^+(x)$.

Conversely, for $x'\in K_\theta^+(x)$, we have

$$\langle \xi(x'),\theta(x,x')\rangle \geq_P 0$$

for all $x\in K$. Since F is $(T_\eta;\,\xi_\theta)$-invex function of second kind at $x'\in K_\theta^+(x)$, we have

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle+\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

i.e.,

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle \geq_P -\langle \xi(x'),\theta(x,x')\rangle,$$

implying

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle-\langle \xi(x'),\theta(x,x')\rangle \geq_P -2\langle \xi(x'),\theta(x,x')\rangle$$

for each $x\in K$, this does not confirm about the $(T_\eta;\,\xi_\theta)$-invexity of first kind or T-η-invexity of F at $x'\in K_\theta^+(x)$.

Case (2): For $x'\in K_\theta^-(x)$, we have

$$\langle \xi(x'),\theta(x,x')\rangle \leq_P 0$$

for all $x\in K$. Since F is $(T_\eta;\,\xi_\theta)$-invex function of second kind at $x'\in K_\theta^-(x)$, we have

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle+\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

i.e.,

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle \geq_P -\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

implying

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle \geq_P 0$$

for each $x\in K$, that is, F is T-η-invex at $x'\in K_\theta^-(x)$, again adding the above two inequalities, we obtain

$$F(x)-F(x')-\langle T(x'),\eta(x,x')\rangle-\langle \xi(x'),\theta(x,x')\rangle \geq_P 0,$$

for each $x\in K$, that is, F is $(T_\eta;\,\xi_\theta)$-invex function of first kind at $x'\in K_\theta^+(x)$.

Conversely, let F is $(T_\eta;\,\xi_\theta)$-invex function of first kind at $x' \in K_\theta^-(x)$. Then, at $x' \in K_\theta^-(x)$, we have
$$\langle \xi(x'), \theta(x,x')\rangle \leq_P 0,$$
and
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle - \langle \xi(x'), \theta(x,x')\rangle \geq_P 0,$$
i.e.,
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle \geq_P \langle \xi(x'), \theta(x,x')\rangle,$$
implying
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle + \langle \xi(x'), \theta(x,x')\rangle \geq_P 2\langle \xi(x'), \theta(x,x')\rangle$$
for each $x \in K$, this does not confirm about the $(T_\eta;\,\xi_\theta)$-invexity of second kind or T-η-invexity of F at $x' \in K_\theta^-(x)$.

Case (3): Let
$$x' \in K_\theta^0(x) = \{x' \in K : x \in K_\theta^+(x) \bigcap K_\theta^-(x)\}.$$
Then
$$\langle \xi(x'), \theta(x,x')\rangle =_P 0,$$
for all $x \in K$. If F is T-η-invex at $x' \in K_\theta^0(x)$, then
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle \geq_P 0$$
for each $x \in K$, which can be written as
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle - \langle \xi(x'), \theta(x,x')\rangle \geq_P 0,$$
and also
$$F(x) - F(x') - \langle T(x'), \eta(x,x')\rangle + \langle \xi(x'), \theta(x,x')\rangle \geq_P 0,$$
for each $x \in K$. For each $x \in K$, F is both $(T_\eta;\,\xi_\theta)$-invex function of first kind and second kind at $x' \in K_\theta^0(x)$. Hence, for all $x' \in K_\theta^0(x)$, $(T_\eta;\,\xi_\theta)$-invexity function of first kind, second kind and T-η-invexity of F coincide with each other. This completes the proof of the proposition. □

The following Example illustrates the existence of $(T_\eta;\,\xi_\theta)$-invex function of first kind and $(T_\eta;\,\xi_\theta)$-invex function of second kind.

Example 2. Let $X = Y = \mathbb{R}$,
$$K = \{x \in [-a,b] : a \geq 0, b > 0\}.$$
Let $F : K \to \mathbb{R}$ be a mapping defined by
$$F(x) = x^2 e^{-x}$$
for all $x \in K$. Let $f : K \times K \to Y$ be any function defined by
$$f(x,u) = \frac{x^3}{3} e^{-u} - [1 + (1+x)^2] u e^{-x}$$

for all $x, u \in K$. Let $T : K \to X^*$ and $\eta : K \times K \to X$ be two mapping defined by

$$T(x) = -2xe^{-x},$$

$$\eta(x,u) = x + u$$

for all $x, u \in K$ such that

$$\langle T(u), \eta(x,u)\rangle = T(u)\cdot\eta(x,u) - f_{ux}(x,u)$$

Let $\xi : K \to X^*$ and $\theta : K \times K \to X$ be two mapping defined by

$$\xi(x) = e^{-x},$$

$$\theta(x,u) = x - u$$

for all $x, u \in K$ such that

$$\langle \xi(u), z\rangle = \xi(u)\cdot z^2$$

for all $z \in X$. Since

$$\begin{aligned} f_{ux}(x,u) = \frac{\partial^2}{\partial x\partial u} f(x,u) \qquad &= \frac{\partial^2}{\partial x\partial u}\left(\frac{x^3}{3}e^{-u} - [1 + (1+x)^2]ue^{-x}\right) \\ &= x^2(e^{-x} - e^{-u}) \end{aligned}$$

for all $x, u \in K$, we have

$$\begin{aligned} F(x) - F(u) \quad &- \quad \langle T(u), \eta(x,u)\rangle - \langle \xi(u), \theta(x,u)\rangle \\ &= \quad x^2e^{-x} - u^2e^{-u} - \left[(-2ue^{-u})(x+u) - f_{ux}(x,u)\right] - e^{-u}(x-u)^2 \\ &= \quad 4x^2u^2e^{-u} \geq 0 \end{aligned}$$

for all $x, u \in K$, implying F is $(T_\eta;\, \xi_\theta)$-invex function of first kind on K. Again we have

$$\begin{aligned} F(x) - F(u) \quad &- \quad \langle T(u), \eta(x,u)\rangle + \langle \xi(u), \theta(x,u)\rangle \\ &= \quad x^2e^{-x} - u^2e^{-u} - [(-2ue^{-u})(x+u) - f_{ux}(x,u) + e^{-u}(x-u)^2 \\ &= \quad 2(x^2 + u^2)e^{-u} \geq 0 \end{aligned}$$

for all $x, u \in K$, implying F is $(T_\eta;\, \xi_\theta)$-invex function of second kind on K. Furthermore, since

$$\langle \xi(u), \theta(x,u)\rangle = \xi(u)\cdot\theta^2(x,u) = e^{-u}(x-u)^2 \geq 0$$

for all $x, u \in K$, by Proposition 4.1, case (1), F is T-η-invex on K.

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