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A REMARK ON CIRCULANT DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS BY GREGARIOUS CYCLES

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Abstract. Let k, m and t be positive integers with $m \geq 4$ and even. It is shown that $K_{km+1(2t)}$ has a decomposition into gregarious m-cycles. Also, it is shown that $K_{km(2t)}$ has a decomposition into gregarious m-cycles if $\frac{(m-1)^2+3}{4m} < k$. In this article, we make a remark that the decompositions can be circulant in the sense that it is preserved by the cyclic permutation of the partite sets, and we will exhibit it by examples.

1. Introduction

Decompositions of graphs into edge-disjoint cycles has been an active research area for many years. Especially, decompositions by cycles of a fixed length has been considered in many different ways. It is shown that a complete graph of odd orderdegree, or a complete graph of even order minus a 1-factor, has a decomposition into k-cycles if k divides the number of edges (see [1], [14] and [15] as well as their references). The key factor for all these works was the decomposition of complete bipartite graphs obtained by Sotteau ([19]). Then, many authors began to consider cycle decompositions with special properties ([4], [5], [12], [13]). Especially, Billington and Hoffman ([2]) introduced the notion of *gregarious cycles* in tripartite graphs. However, the definition of gregarious cycles has been modified in later research articles ([2], [4], [8]).

In this article, we will adopt the notations and the terminology used in [6]. Let $K_{n(t)}$ denote the complete multipartite graph with n partite sets of size t. We call a cycle in a multipartite graph *gregarious* if it involves at most one vertex from any particular partite set. For simplicity, by γ_m*-cycle* we will mean a gregarious cycle of length m, and by γ_m*-decomposition* a decomposition by γ_m-cycles.

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Billington and Hoffman ([3]) and Cho and et el. ([8]) independently showed that $K_{n(2t)}$ has a γ_4-decomposition for $n \geq 4$ if and only if the number of edges is divided by 4. In [9], Cho and Gould showed that $K_{n(2t)}$ also has a γ_6-decomposition if and only if the number of edges is divided by 6. Then, similar decompositions of $K_{n(t)}$ by gregarious cycles of various fixed length followed ([16], [17], [18]).

We say that a decomposition is *circulant* if it is preserved by the cyclic permutation of the partite sets. That is, if the graph is drawn with the n partite sets placed on a circle (or an n-gon), then the graph is invariant under the rotation by $\frac{2\pi}{n}$. It will be clearly understood in later explanations and examples later.

In this article, we remark that the decompositions in [7] and [11] are circulant, and exhibit some decompositions by examples.

Because of the following theorem, we may only consider $K_{km(2)}$ and $K_{km+1(2)}$ instead of $K_{km(2t)}$ and $K_{km+1(2t)}$.

Theorem 1.1. *Let t be positive integers, m an even integer with $m \geq 4$, and $n \geq m$. If $K_{n(2)}$ has a circulant γ_m-decomposition, then so does $K_{n(2t)}$.*

Proof. We adopt the folklore "blow up" method used in [5] and [10]. We blow up each vertex a of $K_{n(2)}$ by replacing it with t new vertices and label them $a_1, a_2, \ldots, a_t$. We now join the vertex a_i to the vertex b_j if ab is an edge in $K_{n(2)}$. Obviously, this new graph is $K_{m(2t)}$. Let Φ be a circulant γ_m-decomposition of $K_{n(2)}$. If $\lambda = \langle a^{(1)}, a^{(2)}, \ldots, a^{(m)} \rangle$ is a γ_m-cycle in Φ then, for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, t$,

$$\lambda_{ij} = \langle a_i^{(1)}, a_j^{(2)}, a_i^{(3)}, a_j^{(4)}, \ldots, a_i^{(m-1)}, a_j^{(m)} \rangle,$$

are t^2 edge-disjoint γ_m-cycles of $K_{n(2t)}$. The collection of all such cycles of $K_{n(2t)}$ obtained in this way constitutes a circulant γ_m-decomposition of $K_{n(2t)}$. □

2. Cycles from feasible sequences of differences

Throughout the article, m is even with $m \geq 4$.

If $n = km + 1$, let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ and use the arithmetic modulo n. Then $\mathrm{D}_n = \{\pm 1, \pm 2, \ldots, \pm\frac{n-1}{2}\}$ is a complete set of differences of two distinct elements in $\mathbb{Z}_n$. In this case, let the partite sets of $K_{n(2)}$ be $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, $\cdots$, $A_{n-1} = \{n-1, \overline{n-1}\}$, and put $V = \cup_{i=0}^{n-1} A_i$.

If $n = km$, let $\mathbb{Z}_{n-1}^{\infty} = \{\infty, 0, 1, \ldots, n-2\}$. Extending the arithmetic of $\mathbb{Z}_{n-1} = \{0, 1, \ldots, n-2\}$ to $\mathbb{Z}_{n-1}^{\infty}$, we define $a \pm \infty = \infty \pm a = \infty$ for $a \in \mathbb{Z}_{n-1}$ and $\infty \pm \infty = 0$. Then, since n is even, the set $\mathrm{D}_n = \{\infty, \pm 1, \pm 2, \ldots, \pm\frac{n-2}{2}\}$ is a complete set of differences of two distinct elements in $\mathbb{Z}_{n-1}^{\infty}$. In this case, let the partite sets of $K_{n(2)}$ be $A_\infty = \{\infty, \overline{\infty}\}$, $A_0 = \{0, \overline{0}\}$, $A_1 = \{1, \overline{1}\}$, $\cdots$, $A_{n-2} = \{n-2, \overline{n-2}\}$, an put $V = A_\infty \cup \left(\cup_{i=0}^{n-2} A_i \right)$.

When $n = km+1$, we draw $K_{n(2)}$ on a circle, evenly arranging the partite sets. When $n = km$, we draw $K_{n(2)}$ on a circular cone, by putting A_∞ at the top vertex of the cone and arranging $A_0, A_1, \cdots, A_{n-2}$ at the circle of the cone.

An edge between a vertex in A_i and a vertex in A_j is called an *edge of distance* d if $i-j = \pm d$ for some $\pm d$ in D_n. In particular, if $d = \infty$ the edge is called an edge of *infinite distance* because of the obvious reason. For example, in $K_{13(2)}$, the edges $0\,\overline{4}$ and $\overline{11}\,2$ are edges of distance 4. In $K_{12(2)}$, the edge $10\,\overline{2}$ is an edge of distance 3, while $\infty\overline{3}$ is an edge of infinite distance.

Let $\rho = (r_1, r_2, \ldots, r_m)$ a sequence of elements in D_n. The *sequence of initial sums*, or the *s-sequence* for short, of ρ is the $\sigma_\rho = (s_0, s_1, s_2, \ldots, s_{m-1})$ defined by $s_0 = 0$ and $s_i = \sum_{j=1}^{i} r_j$ for $i = 1, 2, \ldots, m-1$. Note that all entries of σ_ρ belong to $\mathbb{Z}_n$ or all to $\mathbb{Z}_{n-1}^{\infty}$, and that $s_i = s_{i-1} + r_i$ for each $i = 1, 2, \ldots, m-1$.

Let $\rho = (r_1, r_2, \ldots, r_m)$ be a sequence of elements in D_n. We assume that, when $n = km$ and ρ involves ∞, ρ is of the form $(r_1, r_2, \ldots, r_{m-2}, \infty, \infty)$ with none of $r_1, r_2, \ldots, r_{m-2}$ being ∞. Then, ρ is called a *feasible sequence* or an *f-sequence* for short, if

(i) $\sum_{i=1}^{m} r_i = 0$, that is, the total sum of the terms of the sequence is zero, and

(ii) $\sum_{i=j}^{k} r_i \neq 0$ for all j, k with $1 < j$ or $k < m$, that is, any proper partial sum of consecutive entries is nonzero.

We may consider an s-sequence σ_ρ as an ordering of partite sets involved in a trail or circuit, and if ρ is an f-sequence then the trail or circuit is a γ_m-cycle of $K_{km(2)}$.

Let ϕ^+ and ϕ^- be mappings of $\mathbb{Z}_n$ or $\mathbb{Z}_{n-1}^{\infty}$ into V defined by $\phi^+(a) = a$ and $\phi^-(a) = \overline{a}$ for all a in $\mathbb{Z}_n$ or $\mathbb{Z}_{n-1}^{\infty}$. A *flag* is a sequence $\phi^* = (\phi_0, \phi_1, \ldots, \phi_{m-1})$ of ϕ^+ and ϕ^-. Given a flag ϕ^*, we also use the same notation ϕ^* to denote the mapping defined on $(\mathbb{Z}_{n-1}^{\infty})^m$ by

$$\phi^*(a_0, a_1, \ldots, a_{m-1}) = \langle \phi_0(a_0), \phi_1(a_1), \ldots, \phi_{m-1}(a_{m-1}) \rangle.$$

Let $\tau : V \to V$ be the mapping defined by $\tau(a) = a+1$ and $\tau(\overline{a}) = \overline{a+1}$ for a in $\mathbb{Z}_n$ or $\mathbb{Z}_{n-1}^{\infty}$. That is, τ is the permutation on the vertex set V, defined by a product of cycles as

$$\tau = (0, 1, 2, \cdots, n-1)(\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-1}), \text{ when } n = km+1,$$

or the permutation

$$\tau = (0, 1, 2, \cdots, n-2)(\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-2})(\infty)(\overline{\infty}), \text{ when } n = km.$$

Thus, τ can be regraded as a permutation of partite sets as well.

Now, we define a mapping τ_* on the set of γ_m-cycles by

$$\tau_*(\langle \alpha_0, \alpha_1, \ldots, \alpha_{m-1} \rangle) = \langle \tau(\alpha_0), \tau(\alpha_1), \ldots, \tau(\alpha_{m-1}) \rangle,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are elements of V.

Now, if a pair (ρ, ϕ^*) of an f-sequence and a flag is given, we can generate a class of γ_m-cycles. $\{\tau_*^i(\phi^*(\sigma_\rho)) \mid i \in \mathbb{Z}_n\}$ if $n = km+1$ or $\{\tau_*^i(\phi^*(\sigma_\rho)) \mid i \in \mathbb{Z}_{n-1}\}$ if $n = km$. Note that both classes are invariant under τ_*. We call a decomposition *circulant* if the decomposition is invariant under τ_*.

The above procedure is the method to produce a γ_m-decomposition of $K_{n(2)}$ or $K_{n+1(2)}$. The remaining problem then is how to choose pairs of f-sequences and flags so that, in the γ_m-cycles produced by these pairs, each of the edges $p\,q$, $\overline{p}\,q$, $p\,\overline{q}$ and $\overline{p}\,\overline{q}$ of distance d appears exactly once for every possible distance d.

Note that, in the above procedure, a γ_m-decomposition is obtained from a set of specified γ_m-cycles by applying τ_* repeatedly. Therefore, the decomposition is circulant.

We will also see a basic difference between the γ_m-decompositions when $n = km+1$ and $n = km$.

3. Examples when $n = km$

In this section, m is even with $m \geq 4$ and $\frac{(m-1)^2+3}{4m} < k$. Put $n = km$. The number of edges in $K_{km(2)}$ is $2km(km-1) = 2km(n-1)$. The author of [11] obtained a γ_m-decomposition by producing $2k(n-1)$ edge-disjoint γ_m-cycles in $2k$ classes, each containing $n-1$ γ_m-cycles.

Given $K_{km(2)}$, the procedure to produce pairs of f-sequences and flags is explained in [11]. We present two examples in this section following the procedure.

Example 3.1. (m is not divisible by 4.) Let $m = 6$ and $k = 2$. We have $n = km = 12$ and $\mathrm{D}_{12} = \{\infty, \pm1, \pm2, \pm3, \pm4, \pm5\}$. Following [11], we have two f-sequences

$$\rho = (1, 2, 1, 2, \infty, -\infty) \quad \text{and} \quad \lambda = (3, -4, 5, 4, -3, -5).$$

Then the corresponding s-sequences are

$$\sigma_\rho = (0, 1, 3, 4, 6, \infty) \quad \text{and} \quad \sigma_\lambda = (0, 3, 10, 4, 8, 5).$$

Apply two flags

$$\phi_1^* = (\phi^+, \phi^+, \phi^-, \phi^-, \phi^-, \phi^-) \quad \text{and} \quad \phi_2^* = (\phi^-, \phi^+, \phi^+, \phi^-, \phi^+, \phi^+)$$

specified in [11] to σ_ρ, and we obtain two γ_6-cycles

$$\phi_1^*(\sigma_\rho) = \langle 0, 1, \overline{3}, \overline{4}, \overline{6}, \overline{\infty} \rangle \quad \text{and} \quad \phi_2^*(\sigma_\rho) = \langle \overline{0}, 1, 3, \overline{4}, 6, \infty \rangle.$$

Apply two flags

$$\psi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^-) \quad \text{and} \quad \psi_2^* = (\phi^-, \phi^+, \phi^-, \phi^-, \phi^-, \phi^+)$$

specified in [11] to σ_λ, and we obtain two γ_6-cycles

$$\psi_1^*(\sigma_\lambda) = \langle 0, 3, 10, 4, \overline{8}, \overline{5} \rangle \quad \text{and} \quad \psi_2^*(\sigma_\lambda) = \langle \overline{0}, 3, \overline{10}, \overline{4}, \overline{8}, 5 \rangle.$$

Now, we apply τ_*^i for $i = 0, 1, \cdots, 10$ to each of the above four γ_6-cycles, and then obtain the circulant γ_6-decomposition

$$\{\tau_*^i(\phi_1^*(\sigma_\rho)),\ \tau_*^i(\phi_2^*(\sigma_\rho)),\ \tau_*^i(\psi_1^*(\sigma_\lambda)),\ \tau_*^i(\psi_2^*(\sigma_\lambda)) \mid 0 \le i \le 10\},$$

which can be partitioned into four classes, each with 11 γ_6-cycles. We list them as below. In Figure A, two γ_6-cycles $\phi_1^*(\sigma_\rho) = \langle 0, 1, \bar{3}, \bar{4}, \bar{6}, \overline{\infty}\rangle$ and $\tau_*^5(\phi_1^*(\sigma_\rho)) = \langle 5, 6, \bar{8}, \bar{9}, \bar{0}, \overline{\infty}\rangle$ of $K_{12(2)}$ are exhibited. Note that $\tau_*^5(\phi_1^*(\sigma_\rho))$ is obtained by rotating vertices of $\phi_1^*(\sigma_\rho)$ on the circle by angle $4 \cdot \frac{2\pi}{11}$ counterclockwise while fixing the vertex $\overline{\infty}$.

$$\begin{array}{llll}
\langle 0, 1, \bar{3}, \bar{4}, \bar{6}, \overline{\infty}\rangle, & \langle \bar{0}, 1, 3, \bar{4}, 6, \infty\rangle, & \langle 0, 3, 10, 4, \bar{8}, \bar{5}\rangle, & \langle \bar{0}, 3, \overline{10}, \bar{4}, \bar{8}, 5\rangle, \\
\langle 1, 2, \bar{4}, \bar{5}, \bar{7}, \overline{\infty}\rangle, & \langle \bar{1}, 2, 4, \bar{5}, 7, \infty\rangle, & \langle 1, 4, 0, 5, \bar{9}, \bar{6}\rangle, & \langle \bar{1}, 4, \bar{0}, \bar{5}, \bar{9}, 6\rangle, \\
\langle 2, 3, \bar{5}, \bar{6}, \bar{8}, \overline{\infty}\rangle, & \langle \bar{2}, 3, 5, \bar{6}, 8, \infty\rangle, & \langle 2, 5, 1, 6, \overline{10}, \bar{7}\rangle, & \langle \bar{2}, 5, \bar{1}, \bar{6}, \overline{10}, 7\rangle, \\
\langle 3, 4, \bar{6}, \bar{7}, \bar{9}, \overline{\infty}\rangle, & \langle \bar{3}, 4, 6, \bar{7}, 9, \infty\rangle, & \langle 3, 6, 2, 7, \bar{0}, \bar{8}\rangle, & \langle \bar{3}, 6, \bar{2}, \bar{7}, \bar{0}, 8\rangle, \\
\langle 4, 5, \bar{7}, \bar{8}, \overline{10}, \overline{\infty}\rangle, & \langle \bar{4}, 5, 7, \bar{8}, 10, \infty\rangle, & \langle 4, 7, 3, 8, \bar{1}, \bar{9}\rangle, & \langle \bar{4}, 7, \bar{3}, \bar{8}, \bar{1}, 9\rangle, \\
\langle 5, 6, \bar{8}, \bar{9}, \bar{0}, \overline{\infty}\rangle, & \langle \bar{5}, 6, 8, \bar{9}, 0, \infty\rangle, & \langle 5, 8, 4, 9, \bar{2}, \overline{10}\rangle, & \langle \bar{5}, 8, \bar{4}, \bar{9}, \bar{2}, 10\rangle, \\
\langle 6, 7, \bar{9}, \overline{10}, \bar{1}, \overline{\infty}\rangle, & \langle \bar{6}, 7, 9, \overline{10}, 1, \infty\rangle, & \langle 6, 9, 5, 10, \bar{3}, \bar{0}\rangle, & \langle \bar{6}, 9, \bar{5}, \overline{10}, \bar{3}, 0\rangle, \\
\langle 7, 8, \overline{10}, \bar{0}, \bar{2}, \overline{\infty}\rangle, & \langle \bar{7}, 8, 10, \bar{0}, 2, \infty\rangle, & \langle 7, 10, 6, 0, \bar{4}, \bar{1}\rangle, & \langle \bar{7}, 10, \bar{6}, \bar{0}, \bar{4}, 1\rangle, \\
\langle 8, 9, \bar{0}, \bar{1}, \bar{3}, \overline{\infty}\rangle, & \langle \bar{8}, 9, 0, \bar{1}, 3, \infty\rangle, & \langle 8, 0, 7, 1, \bar{5}, \bar{2}\rangle, & \langle \bar{8}, 0, \bar{7}, \bar{1}, \bar{5}, 2\rangle, \\
\langle 9, 10, \bar{1}, \bar{2}, \bar{4}, \overline{\infty}\rangle, & \langle \bar{9}, 10, 1, \bar{2}, 4, \infty\rangle, & \langle 9, 1, 8, 2, \bar{6}, \bar{3}\rangle, & \langle \bar{9}, 1, \bar{8}, \bar{2}, \bar{6}, 3\rangle, \\
\langle 10, 0, \bar{2}, \bar{3}, \bar{5}, \overline{\infty}\rangle, & \langle \overline{10}, 0, 2, \bar{3}, 5, \infty\rangle, & \langle 10, 2, 9, 3, \bar{7}, \bar{4}\rangle, & \langle \overline{10}, 2, \bar{9}, \bar{3}, \bar{7}, 4\rangle.
\end{array}$$

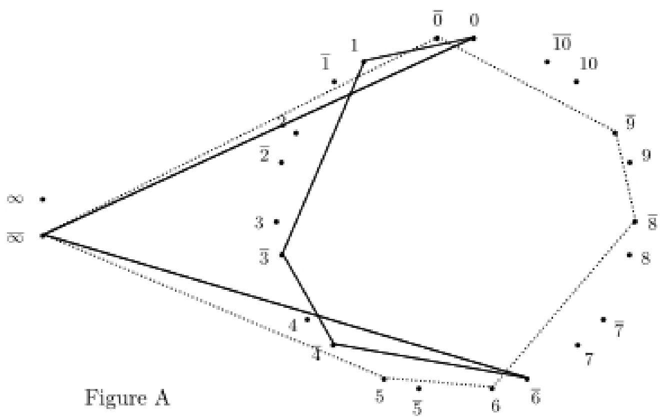

Figure A

Example 3.2. (m is divisible by 4.) Let $m = 8$ and $k = 3$. Then $n = km = 24$ and $D_{24} = \{\infty, \pm 1, \pm 2, \ldots, \pm 11\}$. Following [11], we have three f-sequences

$$\rho = (1, 2, 3, 1, 2, 3, \infty, -\infty), \qquad \lambda = (4, -5, 6, -7, -6, 5, -4, 7)$$
$$\text{and} \qquad \eta = (8, -9, 10, -11, -10, 9, -8, 11),$$

and the corresponding s-sequences are

$$\sigma_\rho = (0, 1, 3, 6, 7, 9, 12, \infty), \qquad \sigma_\lambda = (0, 4, 22, 5, 21, 15, 20, 16)$$
$$\text{and} \qquad \sigma_\eta = (0, 8, 22, 9, 21, 11, 20, 12),$$

respectively. Applying two flags

$$\phi_1^* = (\phi^+, \phi^+, \phi^-, \phi^-, \phi^-, \phi^+, \phi^-, \phi^-) \quad \text{and} \quad \phi_2^* = (\phi^-, \phi^+, \phi^+, \phi^+, \phi^-, \phi^-, \phi^+, \phi^+)$$

specified in [11] to σ_ρ, we obtain two γ_8-cycles

$$\phi_1^*(\sigma_\rho) = \langle 0, 1, \overline{3}, \overline{6}, \overline{7}, 9, \overline{12}, \overline{\infty} \rangle \quad \text{and} \quad \phi_2^*(\sigma_\rho) = \langle \overline{0}, 1, 3, 6, \overline{7}, \overline{9}, 12, \infty \rangle.$$

Applying another two flags

$$\psi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^+, \phi^-) \text{ and } \psi_2^* = (\phi^-, \phi^-, \phi^-, \phi^-, \phi^-, \phi^+, \phi^-, \phi^+)$$

specified in [11] to both σ_λ and σ_η, we obtain four γ_8-cycles

$$\psi_1^*(\sigma_\lambda) = \langle 0, 4, 22, 5, 21, \overline{15}, 20, \overline{16} \rangle, \quad \psi_2^*(\sigma_\lambda) = \langle \overline{0}, \overline{4}, \overline{22}, \overline{5}, \overline{21}, 15, \overline{20}, 16 \rangle,$$
$$\psi_1^*(\sigma_\eta) = \langle 0, 8, 22, 9, 21, \overline{11}, 20, \overline{12} \rangle, \quad \psi_2^*(\sigma_\eta) = \langle \overline{0}, \overline{8}, \overline{22}, \overline{9}, \overline{21}, 11, \overline{20}, 12 \rangle.$$

Applying τ_*^i for $i = 0, 1, \cdots, 22$ to each of the above six γ_8-cycles, we obtain six classes, each with 23 γ_8-cycles. These constitute a circulant γ_8-decomposition of $K_{24(2)}$.

4. Examples when $n = km + 1$

In this section, m is even with $m \geq 4$. Put $n = km + 1$. The number of edges in $K_{km+1(2)}$ is $2(km+1)km = 2kmn$. The author of [7] obtained a γ_m-decomposition by producing $2kn$ edge-disjoint γ_m-cycles in $2k$ classes, each containing n γ_m-cycles.

Given $K_{km+1(2)}$, the procedure to produce pairs of f-sequences and flags is explained in [7]. We present two examples in this section following the procedure.

Example 4.1. (m is not divisible by 4.) Let $m = 10$ and $k = 2$. Then, $n = km + 1 = 21$ and $\mathrm{D}_{21} = \{\pm 1, \pm 2, \ldots, \pm 10\}$. Following [7], we have two f-sequences

$$\rho = (1,-2,3,-4,5,4,-3,2,-1,-5) \text{ and } \lambda = (6,-7,8,-9,10,9,-8,7,-6,-10).$$

Then, the corresponding s-sequences are

$$\sigma_\rho = (0, 1, 20, 2, 19, 3, 7, 4, 6, 5) \text{ and } \sigma_\lambda = (0, 6, 20, 7, 19, 8, 17, 9, 16, 10),$$

respectively. Applying two flags

$$\phi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^+, \phi^-, \phi^-) \text{ and } \phi_2^* = (\phi^-, \phi^+, \phi^-, \phi^+, \phi^-, \phi^-, \phi^-, \phi^-, \phi^-, \phi^+)$$

specified in [11] to both σ_ρ and σ_λ, we obtain the following four starter cycles.

$$\phi_1^*(\sigma_\rho) = \langle 0, 1, 20, 2, 19, 3, \overline{7}, 4, \overline{6}, \overline{5} \rangle, \quad \phi_2^*(\sigma_\rho) = \langle \overline{0}, 1, \overline{20}, 2, \overline{19}, \overline{3}, \overline{7}, \overline{4}, \overline{6}, 5 \rangle,$$
$$\phi_1^*(\sigma_\lambda) = \langle 0, 6, 20, 7, 19, 8, \overline{17}, 9, \overline{16}, \overline{10} \rangle, \quad \phi_2^*(\sigma_\lambda) = \langle \overline{0}, 6, \overline{20}, 7, \overline{19}, \overline{8}, \overline{17}, \overline{9}, \overline{16}, 10 \rangle.$$

Applying τ_*^i for $i = 0, 1, \cdots, 20$ to each of the above γ_{10}-cycles, we obtain four classes, each with 21 gregarious 10-cycles. These constitute a circulant γ_{10}-decomposition of $K_{21(2)}$. In Figure B, two γ_{10}-cycles $\phi_1^*(\sigma_\rho)$ and $\tau_*^9(\phi_1^*(\sigma_\rho)) = \langle 9,10,8,11,7,12,\overline{16},13,\overline{15},\overline{14} \rangle$ of $K_{21(2)}$ are exhibited. Note that, $\tau_*^9(\phi_1^*(\sigma_\rho))$ is obtained by rotating $\phi_1^*(\sigma_\rho)$ by angle $8 \cdot \frac{2\pi}{21}$ counterclockwise.

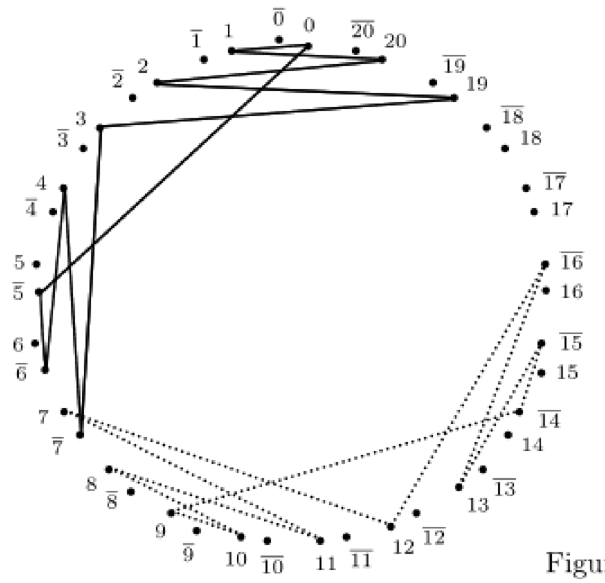

Figure B

Example 4.2. (m is divisible by 4.) Let $m = 8$ and $k = 3$. Then, $n = km+1 = 25$ and $\mathrm{D}_{25} = \{\pm 1, \pm 2, \ldots, \pm 12\}$. Following [7], we have three f-sequences

$$\rho = (1,-2,3,-4,-3,2,-1,4), \qquad \lambda = (5,-6,7,-8,-7,6,-5,8)$$
$$\text{and} \qquad \eta = (9,-10,11,-12,-11,10,-9,12).$$

The corresponding s-sequences are

$$\sigma_\rho = (0,1,24,2,23,20,22,21), \qquad \sigma_\lambda = (0,5,24,6,23,16,22,17)$$
$$\text{and} \qquad \sigma_\eta = (0,9,24,10,23,12,22,13),$$

respectively. Applying two flags

$$\phi_1^* = (\phi^+, \phi^+, \phi^+, \phi^+, \phi^+, \phi^-, \phi^+, \phi^-) \text{ and } \phi_2^* = (\phi^-, \phi^-, \phi^-, \phi^-, \phi^-, \phi^+, \phi^-, \phi^+)$$

specified in [7] to each of the three s-sequences, we obtain the following six starter cycles.

$$\phi_1^*(\sigma_\rho) = \langle 0,1,24,\ 2,\ 23,\overline{20},22,\overline{21}\rangle, \quad \phi_2^*(\sigma_\rho) = \langle \overline{0},\overline{1},\overline{24},\ \overline{2},\ \overline{23},20,\overline{22},21\rangle,$$
$$\phi_1^*(\sigma_\lambda) = \langle 0,5,24,\ 6,\ 23,\overline{16},22,\overline{17}\rangle, \quad \phi_2^*(\sigma_\lambda) = \langle \overline{0},\overline{5},\overline{24},\ \overline{6},\ \overline{23},16,\overline{22},17\rangle,$$
$$\phi_1^*(\sigma_\eta) = \langle 0,9,24,10,23,\overline{12},22,\overline{13}\rangle, \quad \phi_2^*(\sigma_\eta) = \langle \overline{0},\overline{9},\overline{24},\overline{10},\overline{23},12,\overline{22},13\rangle.$$

Applying τ_*^i for $i = 0, 1, \cdots, 24$ to each of the above γ_8-cycles, we obtain six classes, each with 25 gregarious 8-cycles. These constitute a circulant γ_8-decomposition of $K_{25(2)}$.

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