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IRREDUCIBLE REPRESENTATIONS OF SOME METACYCLIC GROUPS WITH AN APPLICATION

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Abstract. Motivated by the problem of determining all right ideals of a group algebra FG for a finite group G over a finite field F, we explicitly determine the faithful irreducible representations of some finite metacylic groups over finite fields. By using that result, we determine the structure of all right ideals of the group algebra for the symmetric group S_3 over a finite field F, as an example.

1. Introduction

This paper is motivated by the problem of determining all right ideals of a group algebra FG for a finite group G over a finite field F. The problem is in fact equal to that of determining all submodules of the right regular module FG.

For the purpose, it is required to determine the equivalence types of faithful irreducible representations of metacyclic groups G over a finite field F. For the nonabelian metacyclic groups of order pq for two prime numbers p and q, it was completely answered for the ordinary cases in [1]. For every metacyclic group whose largest normal 2-subgroup is of nilpotency class at most 2, in [7] it was answered in principle in terms of the center of the fitting subgroup. In this paper, by utilizing the result in [7] we strengthen the result for our specific case so explicitly enough for our motivating problem.

We also give an example in which the structure of all right ideals of the group algebra for the symmetric group S_3 over a finite field F are determined by using our results.

We here summarize some basic facts and set up some notation and terminology, which will be used frequently in this paper.

For positive integers m and n with $\gcd(m, n) = 1$, *the multiplicative order of m modulo n* is the smallest positive integer a such that $m^a \equiv 1 \bmod n$.

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Let G be a group. The order of G is denoted by $|G|$. The index of a subgroup H in G is denoted by $|G : H|$. The *Fitting subgroup* of G is the largest nilpotent normal subgroup of G, which is denoted by $\mathrm{Fit}G$. For a prime number p, the largest normal p-subgroup of G is denoted by $\mathbf{O}_p(G)$. The automorphism group of G is denoted by $\mathrm{Aut}(G)$.

By a *metacyclic group* we mean a group G that has a cyclic normal subgroup K such that G/K is cyclic. It is well-known that every finite metacyclic group has a presentation of the form:

$$\langle\, x, y \,|\, x^m = y^s,\, y^n = 1,\, x^{-1}yx = y^r \,\rangle$$

where $0 < m, n$, $r^m \equiv 1 \bmod n$, $s(r-1) \equiv 0 \bmod n$. Conversely, a presentation above defines a metacyclic group of order mn.

Let F be a field and let G be a finite group. The group algebra of G over F is denoted by FG. By an FG-module, we shall mean a right FG-module of finite dimension over F.

The right regular module FG can be decomposed into a direct sum of some indecomposable submodules, and the quotient $FG/\mathrm{Rad}FG$ by the radical can be decomposed into a direct sum of irreducible modules. Moreover, every irreducible FG-module V is isomorphic to one of the irreducible direct summands of $FG/\mathrm{Rad}FG$, and the multiplicity of V as an irreducible constituent of $FG/\mathrm{Rad}FG$ is equal to $\dim_F V/\dim_F \mathrm{End}_{FG}V$.

Let V be an FG-module. We denote by $\ker V$ the kernel of V; namely $\ker V$ is the subgroup defined by $\{g \in G : vg = v \text{ for all } v \in V\}$.

Let H be a subgroup of G. The FH-module obtained by restricting the operators on V to FH is denoted by V_H. Suppose that H is normal in G. Let W be an FH-module. Let W^G denote the induced FG-module $W \otimes_{FH} FG$, and let $T(W)$ denote the inertia subgroup $\{g \in G : Wg \cong W\}$ of W, where Wg denotes the conjugate of W by g, which is isomorphic with $W \otimes g$ as FH-modules.

The notation and terminology that are not given in this paper may be standard and most general concepts of the representations of groups which are used without reference in this paper can be found in Huppert and Blackburn [3], for example.

Finally, we briefly present some of the main results from the representation theory of finite cyclic groups over arbitrary fields without proof. Most of the results are basic and well-known; for the detailed proofs, see Theorem 9.8 in [2] for example.

Let G be a cyclic group of order n, and let F be a field. Let $p(x)$ be a monic irreducible polynomial of degree d in $F[x]$ that divides $x^n - 1$. Then $V := F[x]/\langle\, p(x)\,\rangle$ is an extension field of degree d over F, and $V = F(\alpha)$ for some zero α of $p(x)$. Moreover, V can be an irreducible FG-module via the natural action of a generator g of G defined by $vg = v\alpha$ for every v in V. Conversely, every irreducible FG-module V, there exists a monic irreducible polynomial $p(x)$ in $F[x]$ that divides $x^n - 1$ such that $V \cong F[x]/\langle\, p(x)\,\rangle$. In fact, the set of all isomorphism types of irreducible FG-modules is bijective with that of all monic irreducible polynomials in $F[x]$ that divide $x^n - 1$. In particular, if F is a finite field whose characteristic does not divide n, then the isomorphism types of all faithful irreducible FG-modules correspond to the irreducible divisors of the nth cyclotomic polynomial. All faithful irreducible FG-modules have the same degree d, and d is the multiplicative order of $|F|$ modulo n. Moreover, the endomorphism rings of the faithful irreducible modules have dimension d over F.

2. Representation theory of metacyclic groups

Let G be a finite metacyclic group, and let K be a cyclic normal subgroup such that G/K is cyclic. We here assume that G is nonabelian and G/K is of a prime order. Since G is nonabelian, the centerizer $\mathbf{C}_G(K)$ of K in G is K and so K contains the center $\mathbf{Z}(\mathrm{Fit}G)$ of the Fitting subgroup of G from Lemma 2.5 in [7].

Let F be a field. For every irreducible FK-module W, it follows that W has a core-free kernel if and only if W is faithful, since every normal subgroup of the cyclic group K is also normal in G.

We now assume that $\mathbf{O}_2(G)$ is of nilpotency class at most 2. We then have the following result as a consequence of Theorem 3.1 in [7].

Theorem 2.1. *Let F be a field and let G be a nonabelian finite metacyclic group with a cyclic normal subgroup K of a prime index in G. If $\mathbf{O}_2(G)$ is of nilpotency class at most 2, then for every faithful irreducible FK-module W, there exists a faithful irreducible FG-module V such that $W^G \cong V \oplus \cdots \oplus V$, and hence there exists a one-to-one correspondence between the isomorphism types of faithful irreducible FG-modules and the G-conjugacy classes of isomorphism types of faithful irreducible FK-modules.*

Let G and K be the groups in Theorem 2.1. Then from Lemma 3.2 in [6], all faithful irreducible FK-modules have the same inertia group and hence we denote the common inertia subgroup by T. Moreover it is known from Theorem 3.3 in [6] that all faithful irreducible FG-modules have the same dimension, provided that F is of prime characteristic. The common dimension is equal to $|G : T|d$ where d is the common dimension of the faithful irreducible FK-modules.

We then have the following consequence of the above theorem.

Theorem 2.2. *Let F be a finite field and let G and K be the groups as given in Theorem 2.1. Assume that $\mathbf{O}_2(G)$ is of nilpotency class at most 2. Then for every faithful irreducible FK-module W, there exists a faithful irreducible FG-module V such that W^G is isomorphic to the direct sum of $|T : K|$ copies of V.*

Let G be a finite nonabelian metacyclic group with a cyclic normal subgroup K such that G/K is of a prime order q. There exists a cyclic subgroup S such that $G = SK$. Let $S = \langle x \rangle, K = \langle y \rangle$ and $[K| = n$. Then of course, G is of order qn and G has a presentation of the form

$$\langle x, y \,|\, x^q = y^s,\, y^n = 1,\, x^{-1}yx = y^r \rangle$$

for some nonnegative integers r, s with conditions:

$$r^q \equiv 1 \mod n,\; r \not\equiv 1 \mod n,\; s(r-1) \equiv 0 \bmod n.$$

Write $G(q, s, n, r)$ for the group defined by the above presentation with the numerical conditions above. We then prove the following result:

Theorem 2.3. *Let F be a finite field and let $G = G(q, s, n, r)$ and K be the groups as defined above with a prime q. Assume that* Aut(K) *is cyclic. Let a be the multiplicative order of $|F|$ modulo n. Then*

(1) *if q does not divide a then for every faithful irreducible FK-modu le W, there exists a faithful irreducible FG-module V with dimension aq such that $V \cong W^G$;*

(2) *if q divide a then every faithful irreducible FK-module W, there exists a faithful irreducible FG-module V with dimension a such that W^G is isomorphic to the direct sum of q copies of V.*

Moreover every faithful irreducible FG-module V is a direct summand of FG with multiplicity q.

Proof. Let W be a faithful irreducible FK-module. We want to determine the common inertia subgroup T for all faithful irreducible FK-modules. Let a be the multiplicative order of $|F|$ modulo n, namely the smallest positive integer a such that $|F|^a \equiv 1 \bmod n$. From the well-known representation theory of finite cyclic groups, the common degree of the faithful irreducible FK-modules is a, and there exist exactly $\phi(n)/a$ isomorphism types of faithful irreducible FK-modules.

The automorphism group Aut(K) acts transitively on the isomorphism types of faithful irreducible FK-modules. Therefore, the stabilizer in Aut(K) of each isomorphism type of such FK-modules is a subgroup of order a. Let α be the automorphism of K that maps y to $y^{|F|}$. Since the order $\langle \alpha \rangle$ is a and α acts trivially on each isomorphism type, the subgroup $\langle \alpha \rangle$ is the stabilizer in Aut(K) of each isomorphism type of faithful irreducible FK-modules.

For our group G, either $T = K$ or $T = G$ since $|G : K| = q$ is a prime. Since x induces the automorphism that maps y to y^r, and $T = G$ if and only if $x \in T$, it follows that $T = G$ if and only if $r \equiv |F|^s \bmod n$ for some positive integer s. Moreover, if $T = G$ then $r^a \equiv 1 \bmod n$ and so q divides a. Hence if q does not divide the multiplicative order a of $|F|$ modulo n, then $T = K$.

It is now straightforward to see that $T = G$ if and only if q divides a when Aut(K) is cyclic. We also note that $\mathbf{O}_2(G)$ is of nilpotency class at most 2 in this case. The results (1) and (2) now follow from Theorem 2.2.

Let V be a faithful irreducible FG-module. Then the characteristic of F does not divide the order of the cyclic normal subgroup K. Thus FK is completely reducible and so $FK \cong W_0 \oplus W_1 \oplus \cdots \oplus W_k$, where $k = \phi(n)/a$ and W_1, ..., W_k

are pairwise non-isomorphic faithful irreducible FK-modules. Therefore, $FG \cong W_0^G \oplus W_1^G \oplus \cdots \oplus W_k^G$. If q does not divide a then $V = W_i^G$ for some $i = 1, 2, \ldots k$, and precisely q conjugates W_j induce the modules isomorphic to V. If q divides a then W_i^G is isomorphic to exactly q copies of V; in this case W_i^G and W_j^G are non-isomorphic for different i and j. Consequently, each faithful irreducible FG-module V is a direct summand of FG with multiplicity q. □

3. Right ideals of the group algebra of S_3 over a finite field

It is a basic fact from the representation theory that the set of all right ideals of a group algebra FG is equal to that of all submodules of the right regular module FG. Therefore, to describe the ideal structure of a group algebra FG it is natural to ask the submodule structure of the module FG. In [4], Remak gives a description of the subgroups of direct product of two finite groups. The results for groups can easily carry over to submodules of the right regular module FG for a finite group algebra FG.

We summarizes the main results as follow.

Let M and N be FG-modules, and $M \oplus N$ the direct sum of M and N. Let η and κ be the projections from $M \oplus N$ onto M and N, respectively. We may regard M and N as submodules of $M \oplus N$.

Lemma 3.1. *Let V be a submodule of M, and let W be a submodule of N. For each FG-homomorphism θ from V to N/W, define $U = \{h + k : h \in V,\ k + W = \theta(h)\}$. Then U is a submodule of $M \oplus N$ such that $\eta(U) = V$, $\theta(V) = \kappa(U)/W$, $U \cap M = \mathrm{Ker}(\theta)$ and $U \cap N = W$.*

Then we have the main result, which is essentially due to Remak [4].

Theorem 3.2. *There exists one-to-one correspondence between the submodules of $M \oplus N$ and the triples (V, W, θ) such that $V \leq M$, $W \leq N$ and θ in $\mathrm{Hom}_{FG}(V, N/W)$.*

The proof can be found in [5], [8], for example.

From the first homomorphism theorem, each homomorphism θ from V into N/W yields an isomorphism from $V/\mathrm{Ker}(\theta)$ onto $\theta(V)$, and so the above theorem can be rephrased as follows.

Corollary 3.3. *There is a one-to-one correspondence between the submodules of $M \oplus N$ and the FG-isomorphisms from each section V_1/V_2 of M onto each section W_1/W_2 of N.*

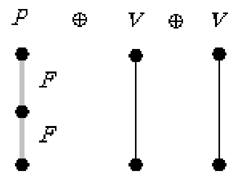
We shall now present an application: by using the above result with irreducible modules determined in Section 2, we here determine the lattice of right ideals of the group algebra FS_3 for the symmetric group S_3 over a finite field F.

Let $G = S_3$ be the symmetric group of degree 3. Then G can be presented with $\langle x, y \,|\, x^2 = 1,\ y^3 = 1,\ x^{-1}yx = y^2 \rangle$, and let $K = \langle y \rangle$. Let F be a finite field. The multiplicative order of $|F|$ modulo $|K| = 3$ is denoted by a.

We divide our discussion into three cases depending on the characteristic of F.

CASE 1) $\mathrm{char}(F) = 2$:

We see that $a = 1$ if $|F|$ is a power of 4, and $a = 2$ otherwise. So in any case, every faithful irreducible FK-module is of dimension 2, and there exists exactly one G-conjugacy class of isomorphism types of faithful irreducible FK-modules. Therefore, there exists exactly one isomorphism type of faithful irreducible FG-modules from Theorem 2.3. Since $|G/K| = 2$, every non-faithful irreducible FG-module is isomorphic to the trivial FG-module F. Let P be the projective indecomposable module of which quotient by the radical is the trivial module F. Then $P \cong F(G/K)$. Let V be the unique faithful irreducible FG-module up to isomorphism. Then $FG \cong P \oplus V \oplus V$.



Moreover $\mathrm{End}_{FG}V \cong F$ since $\dim_F V/\dim_F \mathrm{End}_{FG}V = 2$. Therefore, F is a splitting field for G. From Corollary 3.3, $V \oplus V$ has precisely $3+|F|$ submodules and hence FG has $(3+|F|)3 = 9+3|F|$ submodules.

The submodule lattice of FS_3 for $|F| = 2$ is shown in Figure 1.

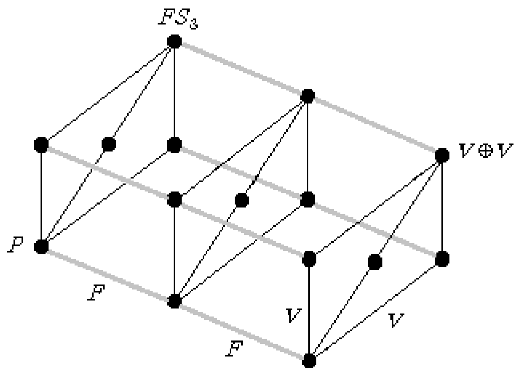
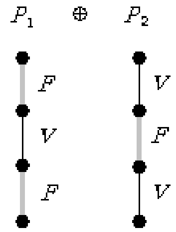


FIGURE 1. The submodule lattice of FS_3 for $|F| = 2$

CASE 2) $\mathrm{char}(F) = 3$:

Since $|F| \equiv 1 \bmod 2$, F is a splitting field for G. In this case, all irreducible FG-module are the trivial module F and the unique irreducible FG-module V with kernel K, up to isomorphism. Let P_1 and P_2 be the projective indecomposable modules of which quotients by the radicals are isomorphic to the trivial module F and V, respectively. Then $P_2 \cong P_1 \otimes_F V$. Since the Sylow 3-subgroup is a cyclic normal subgroup, both P_1 and P_2 are uniserial of dimension 3. Then $FG \cong P_1 \oplus P_2$. Moreover the quotient modules of Loewy series of P_1 are F, V, F, and those of P_2 are V, F, V. Since F is a splitting field for G and both of $\mathrm{End}_{FG}(\mathrm{Rad}P_1)$ and $\mathrm{End}_{FG}(\mathrm{Rad}P_2)$ are local, the number of FG-automorphisms of $\mathrm{Rad}P_i$ is equal to $|F|-1$ for each $i = 1, 2$.



It is now straightforward from Corollary 3.3 to see that FG has precisely $16 + 6(|F| - 1) = 10 + 6|F|$ submodules.

The submodule lattice of FS_3 for $|F| = 3$ is shown in Figure 2.

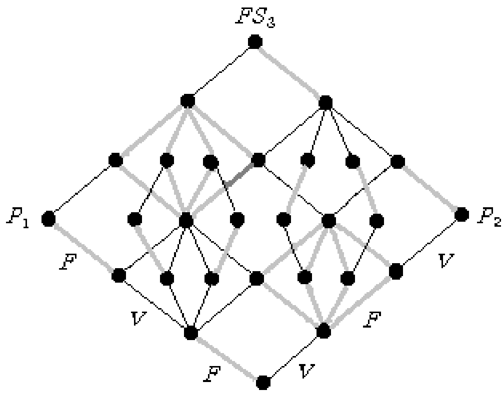
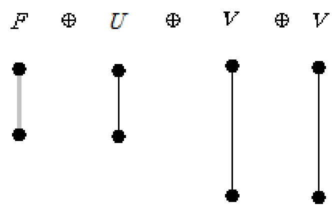


FIGURE 2. The submodule lattice of FS_3 for $|F| = 3$

CASE 3) $\mathrm{char}(F) \neq 2, 3$:

There are exactly two non-faithful irreducible FG-modules: the trivial module F and the module U of dimension 1 with kernel K. From Theorem 2.3, up to isomorphism there exists one faithful irreducible FG-module V, of which dimension is equal to 2. Therefore, we have $FG \cong F \oplus U \oplus V \oplus V$ from Theorem 2.3.



Moreover $\mathrm{End}_{FG}V \cong F$ since $\dim_F V / \dim_F \mathrm{End}_{FG}V = 2$. It follows from Corollary 3.3 that FG has precisely $2 \times 2 \times (4 + |F| - 1) = 4|F| + 12$ submodules. Even if it is now possible to determine the submodule lattice of FG, we omit it because it is too complicated to draw it here.

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