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BERGMAN KERNEL ESTIMATES FOR GENERALIZED FOCK SPACES

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Abstract. We will prove size estimates of the Bergman kernel for the generalized Fock space $\mathcal{F}^2_\varphi$, where φ belongs to the class $\mathcal{W}$. The main tool for the proof is to use the estimate on the canonical solution to the $\bar\partial$-equation. We use Delin's weighted L^2-estimate ([3], [6]) for it.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $dA(z)$ be the area measure on $\mathbb{C}$. $H(\mathbb{C})$ denotes the space of all entire functions in $\mathbb{C}$. Let $\varphi \in C^2(\mathbb{C})$ be a radial function (i.e., $\varphi(z) = \varphi(|z|)$, $\forall z \in \mathbb{C}$) such that $\Delta\varphi(z) > 0$, where Δ is the Laplace operator. We consider certain generalized Fock spaces

$$\mathcal{F}^p_\varphi = \left\{ f \in H(\mathbb{C}) : \|f\|^p_{p,\varphi} = \int_{\mathbb{C}} \left|f(z)e^{-\varphi(z)}\right|^p dA(z) < \infty \right\}, \quad 1 \le p < \infty,$$

and

$$\mathcal{F}^\infty_\varphi = \left\{ f \in H(\mathbb{C}) : \|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} \right\}.$$

The space $\mathcal{F}^p_\varphi$ is the closed subspace of $L^p_\varphi := L^p(\mathbb{C}, e^{-p\varphi} dA)$ consisting of entire functions. Since the space $\mathcal{F}^2_\varphi$ is a reproducing kernel Hilbert space, for each $z \in \mathbb{C}$, there is a function $K_z \in \mathcal{F}^2_\varphi$ with $f(z) = \langle f, K_z\rangle$, where $\langle \cdot, \cdot\rangle$ is the usual inner product in L^2_φ. The orthogonal projection from L^2_φ to $\mathcal{F}^2_\varphi$ is given by

$$P_\varphi f(z) = \int_{\mathbb{C}} f(w)K(z, w)e^{-2\varphi} dA(w),$$

where $K(z, w) = \overline{K_z(w)}$.

Definition 1. A positive function τ on $\mathbb{C}$ is said to belong to the class $\mathcal{L}$ if it satisfies the following two properties:

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(a) τ is bounded on $\mathbb{C}$;
(b) There is a constant C_1 such that

$$|\tau(z) - \tau(w)| \le C_1 |z - w|, \tag{1}$$

for any $z, w \in \mathbb{C}$.

The following notation is frequently used:

$$m_\tau = \frac{1}{4} \min\left\{1, \frac{1}{C_1}\right\},$$

where C_1 is the constant in (1).

Given $z \in \mathbb{C}$ and $r > 0$, we write $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ for the Euclidean disc centered at z with radius r. Throughout this paper, we use the notation $D^\rho(z) := D(z, \rho\tau(z))$.

Definition 2. We say that a weight function $\varphi \in C^2(\mathbb{C})$ is in the class $\mathcal{W}$ if it satisfies the following properties:

(a) φ is radial;
(b) $\Delta\varphi > 0$;
(c) $(\Delta\varphi(z))^{-\frac{1}{2}} \sim \tau(z), |z| \ge 1$, with $\tau(z)$ being a function in the class $\mathcal{L}$.

The class $\mathcal{W}$ includes the power functions $\varphi(r) = r^\alpha$ with $\alpha \ge 2$ and exponential type functions such as $\varphi(r) = e^{\beta r}, \beta > 0$ or $\varphi(r) = e^{e^r}$ [4]. Since the classical Fock space is induced by $\varphi(r) = r^2$, the classical Fock space is covered by the generalized Fock spaces.

For $z, w \in \mathbb{C}$, the distance d_φ induced by the metric $\tau(z)^{-2} dz \otimes d\bar{z}$ is given by

$$d_\varphi(z, w) = \inf_\gamma \int_0^1 \frac{|\gamma'(t)|}{\tau(\gamma(t))}\, dt,$$

where $\gamma : [0, 1] \to \mathbb{C}$ is a parametrization of a piecewise C^1 curve with $\gamma(0) = z$ and $\gamma(1) = w$.

We will prove size estimates of the Bergman kernel for the generalized Fock space $\mathcal{F}^2_\varphi$, where φ belongs to the class $\mathcal{W}$. The following is our main theorem.

Theorem 1.1. *Let $\varphi \in \mathcal{W}$, then there exist positive constants C and σ such that*

$$|K(z, w)| \le C \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)} \exp\left(-\sigma d_\varphi(z, w)\right), \tag{2}$$

for $z, w \in \mathbb{C}$.

The Bergman kernel size estimates have been already studied on various Bergman type spaces with exponential type weights ([1], [2], [7]). For the generalized Fock spaces, J. Marzo and J. Ortega-Cerdà [9] obtained similar estimates under the hypothesis that φ is a subharmonic function whose Laplacian $\Delta\varphi$

is a doubling measure (see [8], [9]). In our paper, we prove the size estimates without the doubling condition.

For the Bergman spaces with certain exponential type weights, S. Asserda and A. Hichame [2] proved that the estimate (2) holds. We follow a similar argument as [2] and [9]. In fact, the main tool for the proof is to use the estimate on the canonical solution to the $\bar{\partial}$-equation. We use Delin's weighted L^2-estimate ([3], [6]) for it.

The expression $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \sim g$ means that both $f \lesssim g$ and $g \lesssim f$ hold.

2. Preliminaries

There are two lemmas which follow previous definition. These lemmas will be used many times.

Lemma 2.1. *Let $\tau \in \mathcal{L}$, $0 < \alpha \leq m_\tau$, and $w \in \mathbb{C}$. Then*

$$\frac{3}{4}\tau(w) \leq \tau(z) \leq \frac{5}{4}\tau(w),$$

for any $z \in D^\alpha(w)$.

Proof. Fix $w \in \mathbb{C}$. As τ is Lipschitz, τ satisfies (1). For $0 < \alpha \leq m_\tau$, $z \in D^\alpha(w)$ implies $|z - w| \leq \alpha\tau(w) \leq \frac{1}{4C_1}\tau(w)$. Hence

$$|\tau(z) - \tau(w)| \leq C_1|z - w| \leq \frac{1}{4}\tau(w).$$

It is equivalent to

$$-\frac{1}{4}\tau(w) \leq \tau(z) - \tau(w) \leq \frac{1}{4}\tau(w).$$

By adding $\tau(w)$, we get the result. □

Corollary 2.2. *Let $\tau \in \mathcal{L}$, $0 < \alpha, \beta \leq m_\tau$, and $z, w \in \mathbb{C}$. Suppose that $D^\alpha(z) \cap D^\beta(w) \neq \emptyset$. Then*

(a) *$\tau(z) \sim \tau(w)$.*
(b) *$d_\varphi(z, w) \lesssim 1$.*

Proof. (a) is immediate from Lemma 2.1. In case (b) take $\gamma(t) = (1 - t)z + tw$, $t \in [0, 1]$. Then γ is a parametrization of a curve from z to w. By definition of d_φ and previous (a), we obtain

$$d_\varphi(z, w) \leq \int_0^1 \frac{|\gamma'(t)|}{\tau(\gamma(t))}dt \sim \int_0^1 \frac{|z - w|}{\tau(z)}dt = \frac{|z - w|}{\tau(z)}. \tag{3}$$

But $|z - w| < \alpha\tau(z) + \alpha\tau(w) \sim \tau(z)$. Hence we get the result. □

Lemma 2.3. *Let $\tau \in \mathcal{L}$ and $z \in \mathbb{C}$. We define a function $h_z(\zeta) = d_\varphi(\zeta, z)$. Then h_z is locally Lipschitz.*

Proof. Let $\alpha \in (0, m_\tau]$ be a constant. Let $\zeta_0 \in \mathbb{C}$, then for any point $\zeta \in D^\alpha(\zeta_0)$, we obtain the followings from (3):

$$\begin{aligned} |h_z(\zeta) - h_z(\zeta_0)| &= |d_\varphi(\zeta, z) - d_\varphi(\zeta_0, z)| \\ &\leq d_\varphi(\zeta, \zeta_0) \\ &\leq C\frac{|\zeta - \zeta_0|}{\tau(\zeta_0)} \\ &= \delta|\zeta - \zeta_0|, \end{aligned}$$

where $\delta = C/\tau(\zeta_0)$. □

Next lemma is obtained in [10]. This sub-mean-value property will be used often.

Lemma 2.4 ([4], [10]). *Suppose that φ is a subharmonic function and τ is Lipschitz such that $\tau(z)^2\Delta\varphi(z) \leq M$ for some constant $M > 0$. Let $0 < p < \infty$ and $s \in \mathbb{R}$. Then for small $\alpha \in (0, m_\tau]$, there exists constant $C > 0$ depending on α such that*

$$|f(a)|^p e^{-s\varphi(a)} \leq C\frac{1}{\tau(a)^2}\int_{D^\alpha(a)} |f|^p e^{-s\varphi}\, dA,$$

for any $f \in H(\mathbb{C})$ and $a \in \mathbb{C}$.

In [6], Delin gave the improved L^2-estimates for the canonical solution of $\bar\partial u = f$ in L^2_φ. It is essential for the proof of main theorem.

Lemma 2.5 ([3], [6]). *Suppose that $\Delta\phi > 0$ on $\Omega \subseteq \mathbb{C}$. Let $\omega \in C^\infty(\Omega)$ be a weighted function on Ω satisfying $\tau(z)|\partial\omega| \leq \mu\omega$, where $0 < \mu < \sqrt{2}$. Let $\tau = (\Delta\phi)^{-\frac{1}{2}}$ and u be the canonical solution of $\bar\partial u = f$ in L^2_φ. Then*

$$\int_\Omega |u|^2 e^{-\phi}\omega\, dA \leq \frac{2}{(\sqrt{2} - \mu)^2}\int_\Omega \tau^2|f|^2 e^{-\phi}\omega\, dA.$$

3. Bergman kernel estimates

Before proving the main estimate, we show an estimate of the Bergman kernel which is more rough than (2). It is caused by the sub-mean-value property easily.

Proposition 3.1. *Let $\varphi \in \mathcal{W}$. Then there is a constant C such that*

$$|K(z, w)| \leq C\frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)},$$

where $z, w \in \mathbb{C}$.

Proof. For $z \in \mathbb{C}$, $K_z(w)$ is an entire function on $\mathbb{C}$ and φ is subharmonic. By Lemma 2.4 for small $\alpha > 0$, and basic argument, we get the followings:

$$\begin{aligned}|K(z,w)|^2 e^{-2\varphi(w)} &= |K_z(w)|^2 e^{-2\varphi(w)} \\ &\lesssim \frac{1}{\tau(w)^2} \int_{D^\alpha(w)} |K_z(\zeta)|^2 e^{-2\varphi(\zeta)} dA(\zeta) \\ &= \frac{1}{\tau(w)^2} \int_{D^\alpha(w)} K_z(\zeta) K(z,\zeta) e^{-2\varphi(\zeta)} dA(\zeta) \\ &\le \frac{1}{\tau(w)^2} \int_{\mathbb{C}} K_z(\zeta) K(z,\zeta) e^{-2\varphi(\zeta)} dA(\zeta).\end{aligned}$$

By reproducing property,

$$\int_{\mathbb{C}} K_z(\zeta) K(z,\zeta) e^{-2\varphi(\zeta)} dA(\zeta) = K_z(z).$$

Hence,

$$|K(z,w)|^2 \lesssim \frac{e^{2\varphi(w)}}{\tau(w)^2} K_z(z).$$

By taking $w = z$, we obtain

$$K(z,z) \lesssim \frac{e^{2\varphi(z)}}{\tau(z)^2}.$$

Therefore,

$$|K(z,w)| \le \sqrt{K(z,z)K(w,w)} \lesssim \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)}.$$

□

Theorem 3.2. *Let $\varphi \in \mathcal{W}$, then there exist positive constants C and σ such that*

$$|K(z,w)| \le C \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)} \exp\left(-\sigma d_\varphi(z,w)\right),$$

for $z, w \in \mathbb{C}$.

Proof. Let $\beta \in (0, m_\tau]$ be a constant. First, we assume $D^\beta(z) \cap D^\beta(w) \neq \emptyset$. By Proposition 3.1, for every $z, w \in \mathbb{C}$, we have

$$|K(z,w)| \lesssim \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)}.$$

By Lemma 2.2, we have $d_\varphi(z,w) \lesssim 1$ and then $1 \lesssim \exp\left(-\sigma d_\varphi(z,w)\right)$. Hence we get the following estimate:

$$|K(z,w)| \lesssim \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)} \exp\left(-\sigma d_\varphi(z,w)\right),$$

where $D^\beta(z) \cap D^\beta(w) \neq \emptyset$.

Next, we assume $D^\beta(z)\cap D^\beta(w)=\emptyset$. We choose a cut-off function $\chi\in C_0^\infty(\mathbb{C})$ such that $\operatorname{supp}\chi\subset D^\beta(w), 0<\chi<1, \chi=1$ on $D^{\beta/2}(w)$ and $|\partial\chi|^2\lesssim\frac{\chi}{\tau(w)^2}$. By Lemma 2.4, we obtain

$$\begin{aligned}|K(z,w)|^2e^{-2\varphi(w)} &\lesssim \frac{1}{\tau(w)^2}\int_{D^{\frac{\beta}{2}}(w)}|K_z(\zeta)|^2e^{-2\varphi(\zeta)}\,dA(\zeta)\\ &=\frac{1}{\tau(w)^2}\int_{D^{\frac{\beta}{2}}(w)}\chi(\zeta)|K_z(\zeta)|^2e^{-2\varphi(\zeta)}\,dA(\zeta)\\ &\lesssim\frac{1}{\tau(w)^2}\|K_z\|^2_{L^2(\chi e^{-2\varphi}dA)}.\end{aligned}$$

The norm of $K_z\in L^2(\chi e^{-2\varphi}dA)$ is given by

$$\|K_z\|^2_{L^2(\chi e^{-2\varphi}dA)}=\sup_f\left|\langle f,K_z\rangle_{L^2(\chi e^{-2\varphi}dA)}\right|,$$

where f is holomorphic on $D^\beta(w)$ with $\|f\|_{L^2(\chi e^{-2\varphi}dA)}=1$. Because $f\chi\in L^2(e^{-2\varphi}dA)$, we have

$$\langle f,K_z\rangle_{L^2(\chi e^{-2\varphi}dA)}=P_\varphi(f\chi)(z).$$

Let $u_f=f\chi-P_\varphi(f\chi)$, Then u_f is the canonical solution of

$$\bar\partial u=\bar\partial(f\chi)=f\bar\partial\chi$$

in $L^2(e^{-2\varphi}dA)$. Since $\chi(z)=0$, we have $|u_f(z)|=|P_\varphi(f\chi)(z)|$. Therefore,

$$|K(z,w)|^2e^{-2\varphi(w)}\lesssim\frac{1}{\tau(w)^2}\sup_f|u_f(z)|^2, \tag{4}$$

where f is holomorphic on $D^\beta(w)$ with $\|f\|_{L^2(\chi e^{-2\varphi}dA)}=1$. Since u_f is holomorphic in $D^\beta(z)$, we have the followings by Lemma 2.4:

$$\begin{aligned}|u_f(z)|^2e^{-2\varphi(z)} &\lesssim\frac{1}{\tau(z)^2}\int_{D^\beta(z)}|u_f(\zeta)|^2e^{-2\varphi(\zeta)}\,dA(\zeta)\\ &\lesssim\frac{1}{\tau(z)^2}\int_{D^\beta(z)}e^{-\epsilon\frac{|\zeta-z|}{\beta\tau(z)}}|u_f(\zeta)|^2e^{-2\varphi(\zeta)}\,dA(\zeta)\\ &\lesssim\frac{1}{\tau(z)^2}\int_{D^\beta(z)}e^{-C\epsilon d_\varphi(\zeta,z)}|u_f(\zeta)|^2e^{-2\varphi(\zeta)}\,dA(\zeta).\end{aligned}$$

The function $h_z(\zeta)=d_\varphi(\zeta,z)$ is locally Lipschitz. By the approximation theorem of the locally Lipschitz function [5], we can obtain a smooth function g_z such that

$$|g_z(\zeta)-d_\varphi(\zeta,z)|\le 1 \tag{5}$$

and

$$|dg_z(\zeta)|\lesssim\frac{1}{\tau(\zeta)}+1. \tag{6}$$

By using (5), we get the relation

$$e^{-C\epsilon d_\varphi(\zeta,z)} \sim e^{-C\epsilon g_z(\zeta)}.$$

Thus, we have

$$|u_f(z)|^2 e^{-2\varphi(z)} \lesssim \frac{1}{\tau(z)^2}\int_{\mathbb{C}} e^{-C\epsilon g_z(\zeta)}|u_f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta). \tag{7}$$

By using (6) and boundedness of τ, we have $\tau(\zeta)\,|dg_z(\zeta)| \le K$ for some constant $K > 0$. It implies $\tau(\zeta)\left|de^{-C\epsilon g_z(\zeta)}\right| \le \mu e^{-C\epsilon g_z(\zeta)}$, where $\mu = C\epsilon K$. We choose sufficiently small ϵ so that $0 < \mu < \sqrt{2}$. Then by Theorem 2.5, we have

$$\begin{aligned}
&\int_{\mathbb{C}} e^{-C\epsilon g_z(\zeta)}|u_f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta)\\
&\qquad\lesssim \int_{\mathbb{C}} e^{-C\epsilon g_z(\zeta)}\tau(\zeta)^2|\bar\partial\chi(\zeta)|^2|f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta)\\
&\qquad\lesssim \int_{\mathbb{C}} e^{-C\epsilon d_\varphi(\zeta,z)}\chi(\zeta)|f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta)\\
&\qquad= \int_{D^\beta(w)} e^{-C\epsilon d_\varphi(\zeta,z)}\chi(\zeta)|f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta).
\end{aligned}$$

Because $e^{-C\epsilon d_\varphi(\zeta,z)} \lesssim e^{-C\epsilon d_\varphi(z,w)}$, we get the followings from (7) and previous estimates:

$$\begin{aligned}
|u_f(z)|^2 e^{-2\varphi(z)} &\lesssim \frac{1}{\tau(z)^2}\int_{D^\beta(w)} e^{-C\epsilon d_\varphi(\zeta,z)}\chi(\zeta)|f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta)\\
&\lesssim \frac{1}{\tau(z)^2} e^{-C\epsilon d_\varphi(z,w)}\int_{D^\beta(w)} \chi(\zeta)|f(\zeta)|^2 e^{-2\varphi(\zeta)}\, dA(\zeta)\\
&\lesssim \frac{1}{\tau(z)^2} e^{-C\epsilon d_\varphi(z,w)}.
\end{aligned}$$

By using (4),

$$|K(z,w)| \lesssim \frac{e^{\varphi(z)+\varphi(w)}}{\tau(z)\tau(w)} \exp\left(-\sigma d_\varphi(z,w)\right), \quad \sigma > 0.$$

Thus we get the result for every $z, w \in \mathbb{C}$. □

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