East Asian Math. J.
Vol. 33 (2017), No. 1, pp. 011–022
http://dx.doi.org/10.7858/eamj.2017.002

EQUIVALENCE BETWEEN SOME ITERATIVE SCHEMES FOR GENERALIZED φ-WEAK CONTRACTION MAPPING IN $CAT(0)$ SPACES

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Abstract. The aim of this paper is to obtain equivalence of convergence between some iterative schemes for generalized φ-weak contraction mapping in $CAT(0)$ spaces.

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

A mapping $T : X \to X$ is a φ-*weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ and $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X. \tag{1}$$

If X is bounded, then the infinity condition can be omitted.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [20] in 2001, who extended the results of [1] to metric spaces.

Theorem 1.1. ([20]) *Let (X, d) be a complete metric space, $T : X \to X$ be a φ-weak contractive self-map on X. The T has a unique fixed point p in X.*

Remark 1. Theorem 1.1 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1-\alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

Received October 24, 2016; Accepted November 29, 2016.
2010 *Mathematics Subject Classification.* 47H09, 47H10, 41A65, 54E35.
Key words and phrases. $CAT(0)$ space, generalized φ-weak contraction, fixed point, iterative scheme.
This work was supported by Kyungnam University Foundation Grant, 2016.

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(pISSN 1226-6973, eISSN 2287-2833)

In 2016, Xue [23] introduced a new contraction type mapping as follows.

Definition 1. ([23]) A mapping $T : X \to X$ is a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \le d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{2}$$

holds.

We notice immediately that if $T : X \to X$ is φ-weak contraction, then T satisfies the following inequality

$$d(Tx, Ty) \le d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X.$$

However, the converse is not true in general.

Example 1. Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$. Define $\varphi(t) : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (2), but T does not satisfy inequality (1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\le |x - y| - \frac{4}{3} \cdot \frac{2}{5}|x - y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\ge |x - y| - \frac{4}{3}|x - y| \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

for all $x, y \in X$.

Example 2. ([23]) Let $X = [0, +\infty)$ be endowed by $d(x, y) = |x - y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x}{1 + x} - \frac{y}{1 + y}\right| = \frac{|x - y|}{(1 + x)(1 + y)} \\ &\le \frac{|x - y|}{1 + |x - y|} = |x - y| - \frac{|x - y|^2}{1 + |x - y|} \\ &= d(x, y) - \varphi(d(x, y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However T is not a contraction.

Remark 2. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions.

One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [22] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0,1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0,1]$. A metric space with a convex structure W is known as a convex metric space which denoted by (X, d, W). A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0,1]$. In fact, every normed linear space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [22]).

Example 3. ([13]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ and $\lambda \in [0,1]$. We define a mapping $W : X \times X \times [0,1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1-\lambda)y_1, \frac{\lambda x_1 x_2 + (1-\lambda)y_1 y_2}{\lambda x_1 + (1-\lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space but not a normed linear space.

A metric space X is a $CAT(0)$ space. This term is due to M. Gromov [9] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov. If X is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [4], p.159). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [4] or Burago *et al.* [5].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [4]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces*(see, [15]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [4], p.163). The above inequality has been extended by [7] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(see, [4], p.163). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha)d(z, y) \tag{3}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure

$$W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y.$$

It is easy to see that for any $x, y \in X$ and $\lambda \in [0,1]$,

$$d(x,(1-\lambda)x \oplus \lambda y) = \lambda d(x,y),$$
$$d(y,(1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x,y).$$

As a consequence,

$$1 \cdot x \oplus 0 \cdot y = x,$$
$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x,y] \subset K$.

The aim of this paper is to obtain equivalence of convergence between some iterative schemes for generalized φ-weak contraction mapping in $CAT(0)$ spaces.

2. Preliminaries

Definition 2. Let K be a nonempty convex subset of a $CAT(0)$ space X, $T : K \to K$ be a self mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1)$ satisfying some conditions.
(1) The Picard iterative scheme (*cf.*, [19]) is defined by $w_0 \in K$,

$$w_{n+1} = Tw_n, \quad n \geq 0. \tag{$\mathbb{P}$}$$

(2) The Mann iterative scheme (*cf.*, [18]) is defined by $u_0 \in K$,

$$u_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n Tu_n, \quad n \geq 0. \tag{$\mathbb{M}$}$$

(3) The Ishikawa iterative scheme (*cf.*, [10]) is defined by $r_0 \in K$,

$$\begin{cases} r_{n+1} = (1-\alpha_n)r_n \oplus \alpha_n Ts_n, \\ \quad s_n = (1-\beta_n)r_n \oplus \beta_n Tr_n, \quad n \geq 0. \end{cases} \tag{$\mathbb{I}$}$$

(4) The three-step iterative scheme (*cf.*, [11], [12]) is defined by $x_0 \in K$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n Ty_n, \\ \quad y_n = (1-\beta_n)x_n \oplus \beta_n Tz_n, \\ \quad z_n = (1-\gamma_n)x_n \oplus \gamma_n Tx_n, \quad n \geq 0. \end{cases} \tag{$\mathbb{TH}$}$$

Another iterative schemes and other some results in $CAT(0)$ space have been studied extensively by various authors(see *e.g.* [6], [8], [14], [16], [17], [21]).

Xue [23] proved the following very intersting fixed point theorem in complete metric space.

Theorem 2.1. ([23]) *Let (X,d) be a complete metric space and let $T : X \to X$ be a generalized φ-weak contraction. Then the Picard iterative scheme* ([19])

$$x_{n+1} = Tx_n$$

converges to the unique fixed point.

Theorem 2.2. *Let T be a generalized φ-weak contractive self mapping of a closed convex subset K of a Banach space X. Then the Picard iterative scheme*

$$x_{n+1} = Tx_n$$

converges strongly to the fixed point p with the following error estimate:

$$\|x_{n+1} - p\| \leq \Phi^{-1}(\Phi(\|x_1 - p\| - n)),$$

where Φ is defined by the antiderivative

$$\Phi(t) = \int \frac{1}{\varphi(t)}\, dt, \quad \Phi(0) = 0$$

and Φ^{-1} is the inverse of Φ.

Proof. The proof is similar as [20](Theorem 2). However, for completeness, we give a sketch of the proof. We can obtain convergence follows from Theorem 2.1. To establish the error estimete, from (2) with $\lambda_n = \|x_n - p\|$,

$$\begin{aligned} \lambda_{n+1} &= \|x_{n+1} - p\| = \|Tx_n - p\| \\ &\leq \|x_n - p\| - \varphi(\|x_{n+1} - p\|) \\ &= \lambda_n - \varphi(\lambda_{n+1}), \end{aligned}$$

so, we have

$$\varphi(\lambda_{n+1}) \leq \lambda_n - \lambda_{n+1}. \tag{4}$$

Thus

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \int_{\lambda_{n+1}}^{\lambda_n} \frac{1}{\varphi(t)}\, dt = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)},$$

for some $\lambda_{n+1} < \mu_n < \lambda_n$. Since φ is nondecreasing, from (4),

$$\Phi(\lambda_n) - \Phi(\lambda_{n+1}) = \frac{\lambda_n - \lambda_{n+1}}{\varphi(\mu_n)} \geq \frac{\lambda_n - \lambda_{n+1}}{\varphi(\lambda_n)} \geq 1.$$

Thus

$$\Phi(\lambda_{n+1}) \leq \Phi(\lambda_n) - 1 \leq \cdots \leq \Phi(\lambda_1) - n.$$

This completes the proof of Theorem 2.2. □

Lemma 2.3. ([2]) *Let $\{a_n\}$ and $\{b_n\}$ be sequence of nonnegative numbers and $0 \leq q < 1$ such that for all $n \geq 0$,*

$$a_{n+1} = qa_n + b_n.$$

If $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Theorem 3.1. *Let (X, d) be a complete $CAT(0)$ space and K be a nonempty bounded convex subset of X. Let $T : K \to K$ be a generalized φ-weak contraction mapping. Let $\{w_n\}$ and $\{x_n\}$ be the Picard and three step iterative scheme defined by $(\mathbb{P})$ and $(\mathbb{TH})$ respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n, \gamma_n \in [0, 1), \quad \forall\, n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 1$, $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$.

If $w_0 = x_0$, then the following statements are equivalent:

(1) *the Picard iterative scheme $\{w_n\}$ converegs to $p \in F(T)$;*
(2) *the three step iterative scheme $\{x_n\}$ converegs to $p \in F(T)$.*

Furthermore, p is the unique fixed point of T.

Proof. From Theorem 2.1 and Theorem 2.2, T has a fixed point. Take it p. From (3) and the generalized φ-weak contraction of T, we have

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n T x_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n[d(x_n, p) - \varphi(d(Tx_n, p))] \end{aligned} \tag{5}$$

and

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n[d(z_n, p) - \varphi(d(Tz_n, p))]. \end{aligned} \tag{6}$$

From (5) and (6), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)x_n \oplus \alpha_n T y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n[d(y_n, p) - \varphi(d(Ty_n, p))] \\ &\leq (1-\alpha_n)d(x_n, p) \\ &\quad + \alpha_n[(1-\beta_n)d(x_n, p) + \beta_n\{d(z_n, p) - \varphi(d(Tz_n, p))\}] \\ &\quad - \alpha_n\varphi(d(Ty_n, p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n(1-\beta_n)d(x_n, p) \\ &\quad + \alpha_n\beta_n[(1-\gamma_n)d(x_n, p) + \gamma_n\{d(x_n, p) - \varphi(d(Tx_n, p))\}] \\ &\quad - \alpha_n\beta_n\varphi(d(Tz_n, p)) - \alpha_n\varphi(d(Ty_n, p)) \end{aligned}$$

$$
\begin{aligned}
&= d(x_n,p) - \alpha_n\beta_n\gamma_n\varphi(d(Tx_n,p)) - \alpha_n\beta_n\varphi(d(Tz_n,p)) \\
&\quad - \alpha_n\varphi(d(Ty_n,p)) \\
&= d(x_n,p) - \alpha_n\beta_n\gamma_n\varphi(d(Tx_n,p)) \qquad (7)\\
&\le d(x_n,p).
\end{aligned}
$$

Therefore $\{d(x_n,p)\}$ is a nonnegative nonincreasing sequence, which converges to a limit $L \ge 0$. Suppose that $L > 0$. For notational convenience, let $\lambda_n = d(x_n,p)$. Since $\{d(x_n,p)\}$ is a nonincreasing sequence, we have $\lambda_n \ge L$, *i.e.*,

$$
d(x_n,p) \ge d(x_{n+1},p) \ge \cdots \ge L, \quad \forall\, n \in \mathbb{N}. \tag{8}
$$

Most of all, we want to show that

$$
d(Tx_n,p) \ge L, \quad \forall\, n \in \mathbb{N}.
$$

It is sufficient to show that there exists $n_1 \in \mathbb{N}$ such that

$$
d(x_{n_1},p) \le d(Tx_n,p), \quad n \ge 1.
$$

Suppose that $d(Tx_n,p) < L$. Then

$$
d(x_{n_1},p) > d(Tx_n,p), \quad \forall\, n_1 \in \mathbb{N}. \tag{9}
$$

Since $\lim_{n\to\infty} d(x_n,p) = L$ and (9), for $\frac{\varepsilon}{2} = L - d(Tx_n,p) > 0$, there exists $N \in \mathbb{N}$ with $d(x_N,p) < d(Tx_n,p) + \frac{\varepsilon}{4}$ such that

$$
\begin{aligned}
|d(x_n,p) - L| &\le |L - d(Tx_n,p)| + |d(Tx_n,p) - d(x_n,p)| \\
&= L - d(Tx_n,p) + d(x_n,p) - d(Tx_n,p) \\
&\le \frac{\varepsilon}{2} + d(x_N,p) - d(Tx_n,p) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}
$$

for $n \ge N$. On the other hand, from (9), we obtain

$$
\begin{aligned}
d(x_N,p) &< d(Tx_n,p) + \frac{\varepsilon}{4} = d(Tx_n,p) + \frac{1}{2}(L - d(Tx_n,p)) \\
&= \frac{1}{2}(L + d(Tx_n,p)) \\
&< \frac{1}{2}(L + d(x_N,p)),
\end{aligned}
$$

i.e.,

$$
d(x_N,p) < L.
$$

This is a contradiction to (8). Therefore

$$
d(Tx_n,p) \ge L. \tag{10}
$$

From (7), (8) and (10), it follows that, for any fixed integer $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=N}^{\infty} \alpha_n \beta_n \gamma_n \varphi(L) &\leq \sum_{n=N}^{\infty} \alpha_n \beta_n \gamma_n \varphi(d(Tx_n, p)) \\ &\leq \sum_{n=N}^{\infty} (d(x_n, p) - d(x_{n+1}, p)) \\ &\leq d(x_N, p).\end{aligned}$$

This is a contradiction to the condition (iii). Therefore

$$\lim_{n\to\infty} d(x_n, p) = L = 0.$$

For each $n \geq 0$,

$$\begin{aligned}d(z_n, w_n) &= d((1-\gamma_n)x_n \oplus \gamma_n T x_n, w_n) \\ &\leq (1-\gamma_n)d(x_n, w_n) + \gamma_n d(Tx_n, w_n) \\ &\leq (1-\gamma_n)d(x_n, w_n) + \gamma_n[d(Tx_n, Tw_n) + d(w_{n+1}, w_n)] \\ &\leq (1-\gamma_n)d(x_n, w_n) + \gamma_n[d(x_n, w_n) - \varphi(d(Tx_n, w_{n+1}))] \\ &\quad + \gamma_n d(w_{n+1}, w_n) \\ &= d(x_n, w_n) - \gamma_n \varphi(d(Tx_n, w_{n+1})) + \gamma_n d(w_{n+1}, w_n) \qquad (11)\end{aligned}$$

and

$$\begin{aligned}d(y_n, w_n) &= d((1-\beta_n)x_n \oplus \beta_n T z_n, w_n) \\ &\leq (1-\beta_n)d(x_n, w_n) + \beta_n d(Tz_n, w_n) \\ &\leq (1-\beta_n)d(x_n, w_n) + \beta_n[d(Tz_n, Tw_n) + d(w_{n+1}, w_n)] \\ &\leq (1-\beta_n)d(x_n, w_n) + \beta_n[d(z_n, w_n) - \varphi(d(Tz_n, w_{n+1}))] \\ &\quad + \beta_n d(w_{n+1}, w_n). \qquad (12)\end{aligned}$$

Substitute (11) to (12), we have

$$\begin{aligned}d(y_n, w_n) &\leq (1-\beta_n)d(x_n, w_n) \\ &\quad + \beta_n[d(x_n, w_n) - \gamma_n \varphi(d(Tx_n, w_{n+1})) + \gamma_n d(w_{n+1}, w_n)] \\ &\quad - \beta_n \varphi(d(Tz_n, w_{n+1})) + \beta_n d(w_{n+1}, w_n) \\ &= d(x_n, w_n) - \beta_n[\varphi(d(Tz_n, w_{n+1})) + \gamma_n \varphi(d(Tx_n, w_{n+1}))] \\ &\quad + \beta_n(1+\gamma_n)d(w_{n+1}, w_n). \qquad (13)\end{aligned}$$

From (13), we obtain

$$
\begin{aligned}
d(x_{n+1}, w_{n+1}) &= d((1-\alpha_n)x_n \oplus \alpha_n Ty_n, Tw_n) \\
&\le (1-\alpha_n)d(x_n, Tw_n) + \alpha_n d(Ty_n, Tw_n) \\
&\le (1-\alpha_n)d(x_n, Tw_n) + \alpha_n[d(y_n, w_n) - \varphi(d(Ty_n, w_{n+1}))] \\
&\le (1-\alpha_n)d(x_n, Tw_n) + \alpha_n[d(x_n, w_n) - \beta_n\{\varphi(d(Tz_n, w_{n+1})) \\
&\quad + \gamma_n\varphi(d(Tx_n, w_{n+1}))\} + \beta_n(1+\gamma_n)d(w_{n+1}, w_n)] \\
&\quad - \alpha_n\varphi(d(Ty_n, w_{n+1})) \\
&= \alpha_n d(x_n, w_n) + (1-\alpha_n)d(x_n, Tw_n) - \alpha_n[\beta_n\varphi(d(Tz_n, w_{n+1})) \\
&\quad + \beta_n\gamma_n\varphi(d(Tx_n, w_{n+1})) + \varphi(d(Ty_n, w_{n+1}))] \\
&\quad + \alpha_n\beta_n(1+\gamma_n)d(w_{n+1}, w_n) \\
&\le q\, d(x_n, w_n) + (1-\alpha_n)d(x_n, Tw_n) \\
&\quad + \alpha_n\beta_n(1+\gamma_n)d(w_{n+1}, w_n), \qquad (14)
\end{aligned}
$$

where $q = \max\{\alpha_n : n \ge 1\}$. By Lemma 2.3 and conditions (i),(ii), we know that

$$\lim_{n\to\infty} d(x_n, w_n) = 0.$$

If $w_n \to p \in F(T)$ as $n \to \infty$, we have

$$d(x_n, p) \le d(x_n, w_n) + d(w_n, p) \to 0$$

as $n \to \infty$. If $x_n \to p \in F(T)$ as $n \to \infty$, we have

$$d(w_n, p) \le d(w_n, x_n) + d(x_n, p) \to 0$$

as $n \to \infty$. Therefore, the equivalence between the statement (1) and (2) was proved. Finally, we show that $p \in K$ is the unique fixed point of T. In fact, let $p, q \in K$ be two fixed point of T. Since T is a generalized φ-weak contraction mapping, we have

$$
\begin{aligned}
d(p, q) &= d(Tp, Tq) \\
&\le d(p, q) - \varphi(d(Tp, Tq)) \\
&= d(p, q) - \varphi(d(p, q)).
\end{aligned}
$$

This implies

$$\varphi(d(p, q)) = 0.$$

From the property of φ, $\varphi^{-1}(0) = \{0\}$, we have

$$d(p, q) = 0,$$

i.e., $p = q$. This completes the proof. □

Corollary 3.2. *Let (X, d) be a complete $CAT(0)$ space and K be a nonempty bounded convex subset of X. Let $T : K \to K$ be a generalized φ-weak contraction* *mapping. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Ishikawa iterative scheme defined by* ($\mathbb{P}$) *and* ($\mathbb{I}$) *respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n \in [0,1), \quad \forall\, n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 1$, $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

If $w_0 = r_0$, then the following statements are equivalent:

(1) *the Picard iterative scheme $\{w_n\}$ converegs to $p \in F(T)$;*
(2) *the Ishikawa iterative scheme $\{r_n\}$ converegs to $p \in F(T)$.*

Furthermore, p is the unique fixed point of T.

Corollary 3.3. *Let (X, d) be a complete $CAT(0)$ space and K be a nonempty bounded convex subset of X. Let $T : K \to K$ be a generalized φ-weak contraction mapping. Let $\{w_n\}$ and $\{u_n\}$ be the Picard and Mann iterative scheme defined by* ($\mathbb{P}$) *and* ($\mathbb{M}$) *respectively and satisfying the following conditions:*

(i) $\alpha_n \in [0,1), \quad \forall\, n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 1$.

If $w_0 = u_0$, then the following statements are equivalent:

(1) *the Picard iterative scheme $\{w_n\}$ converegs to $p \in F(T)$;*
(2) *the Mann iterative scheme $\{u_n\}$ converegs to $p \in F(T)$.*

Furthermore, p is the unique fixed point of T.

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