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The Intrinsic Topology on a Quandle

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Abstract. Let $Inn(Q)$ denote the inner automorphism group on a quandle Q. For a subset M of Q, let $c(M)$ denote the orbit of M under the $Inn(Q)$-action on Q. Then c satisfies the axioms of the closure operator. In this paper, we study the topological space Q corresponding to the topology obtained from the closure operator c.

1. Introduction and Preliminaries

A quandle was introduced by Joyce[8] and Matveev[9] independently in 1980s. A quandle is an algebraic structure which is closely related to Reidemeister moves in knot theory. By using quandles, various knot invariants can be introduced, for example, quandle colorability, quandle cocycle invariant[1], [2], [3], etc.

Definition 1.1. A *quandle* is a set Q equipped with a binary operation $* : Q \times Q \longrightarrow Q$ satisfying the following axioms ;

(1) For all $a \in Q$, $a * a = a$.

(2) For all $a, b \in Q$, $\exists ! c \in Q$ such that $c * a = b$.

(3) For all $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation satisfying axioms above except the first condition.

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Example 1.2.

(1) Let X be a set. Define a binary operation $*$ by $x * y = x$ for $x, y \in X$. Then $(X, *)$ is a quandle, which is called the *trivial quandle*.

(2) Let G be a group. Define a binary operation $*$ by $g*h = h^{-n}gh^n$ for $g, h \in G$. Then $(G, *)$ is a quandle, which is called the *n-fold conjugation quandle*.

(3) Let $(M, +)$ be an abelian group. Let t be a group automorphism of M. Define a binary operation $*_t$ by $a *_t b = ta + (1-t)b$ for $a, b \in M$. Then $(M, *_t)$ is a quandle, which is called an *Alexander quandle*.

Example 1.3. ([7]) Let (,) be a symmetric bilinear form defined on Euclidean vector space $\mathbb{R}^n$. Then if S is the subset of $\mathbb{R}^n$ consisting of vectors $\mathbf{v}$ satisfying $(\mathbf{v}, \mathbf{v}) \neq 0$, there is a rack structure on S defined by the operation

$$\mathbf{u} *_0 \mathbf{v} := \mathbf{u} - \tfrac{2(\mathbf{u},\mathbf{v})}{(\mathbf{v},\mathbf{v})}\mathbf{v}.$$

Geometrically, this is the result of reflecting $\mathbf{u}$ in the hyperplane $H = \{\mathbf{w} | (\mathbf{w}, \mathbf{v}) = 0\}$. This rack structure is called the *coxeter rack*. Since $\mathbf{u} *_0 \mathbf{u} = -\mathbf{u}$, the operation $*_0$ is not a quandle operation. If we multiply the right-hand side of the above formula by -1, then the operation

$$\mathbf{u} * \mathbf{v} := \tfrac{2(\mathbf{u},\mathbf{v})}{(\mathbf{v},\mathbf{v})}\mathbf{v} - \mathbf{u}$$

defines a quandle structure on S. This quandle structure is called the *coxeter quandle on* $\mathbb{R}^n$.

Notice that for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, $\mathbf{u}*\mathbf{v}$ is the result of reflecting $\mathbf{u}$ in the line containing $\mathbf{v}$ as described Figure 1.

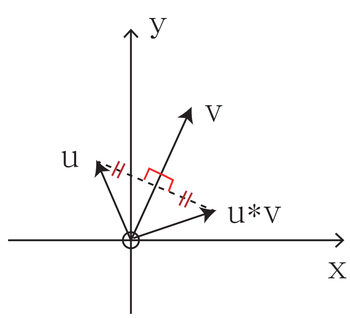

Figure 1: Coxeter quandle operation on $\mathbb{R}^2$

In the second axiom of Definition 1.1, one can see that the operation $\bar{*} : Q \times Q \longrightarrow Q$ defined by $c = a \mathbin{\bar{*}} b$ also satisfies all quandle axioms. We called $\bar{*}$ the *reverse operation* of $*$. For each $a \in Q$, one can obtain two functions $\sigma_a : Q \to Q$ and $\bar{\sigma}_a : Q \to Q$ defined by

$$\sigma_a(x) = x \ * \ a, \ \bar{\sigma}_a(x) = x \ \bar{*} \ a.$$

Definition 1.4. Let $(Q_1, *_1)$ and $(Q_2, *_2)$ be quandles and $f : Q_1 \longrightarrow Q_2$ a function.

(1) f is called a *quandle homomorphism* if $f(a *_1 b) = f(a) *_2 f(b)$ for all $a, b \in Q_1$.

(2) f is called a *quandle isomorphism* if f is a bijective quandle homomorphism.

(3) f is called a *quandle automorphism* if $(Q_1, *_1) = (Q_2, *_2)$ and f is a quandle isomorphism. That is, a quandle automorphism is a quandle isomorphism from a quandle to itself.

(4) A *quandle automorphism group* of Q_1, denoted by $Aut(Q_1)$, is the group of all quandle automorphisms of Q_1.

(5) A *quandle inner automorphism group* of Q_1, denoted by $Inn(Q_1)$, is the subgroup of $Aut(Q_1)$ generated by the set of $\{\sigma_a, \bar{\sigma}_a | \ a \in Q_1\}$.

Definition 1.5. Let $(Q, *)$ be a quandle and $a \in Q$. The *orbit* of Q corresponding to a, denoted by $orb_Q(a)$ or simply $orb(a)$, is the set of all $b \in Q$ such that

$$(\cdots((a *_1 x_1) *_2 x_2) \cdots) *_n x_n = b,$$

where $x_i \in Q$ and $*_i \in \{*, \bar{*}\}$. For any non-empty subset M of Q, the union of all orbits corresponding to elements in M is called the *orbit* of M. The *orbit set* of Q, denoted by $orb(Q)$, is the set of all orbits $orb_Q(a)$ of Q. A quandle Q is said to be *connected* if Q has only one orbit, Q itself. That is, its inner automorphism group $Inn(Q)$ acts on Q transitively.

We consider a function c which maps a subset M of Q to the orbit of M. Then c satisfies the axioms of the closure operator. In this paper, we study the topological space Q endowed with the topology obtained from the closure operator c.

2. Main Results

In this section, we study the construction of topological space on a quandle and its topological properties.

Definition 2.1. Let Q be a quandle with a binary operation $*$ and $M \subset Q$. M is called a *subquandle* of Q if M is a quandle under the binary operation $*$.

In [10], S. Nelson anc C.-Y. Wong showed the following lemma.

Lemma 2.2. *Let Q be a quandle and $M \subset Q$. Then M is a subquandle if and only if M is closed under the operation $*$.*

Now, we introduce a method to get a closure operator on a quandle Q.

Definition 2.3.([4]) Let X be a set. A *closure operator* on X is a function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ which associates with each subset A of X satisfying following properties;

(1) $c(\emptyset) = \emptyset$,

(2) $A \subset c(A)$,

(3) $c(c(A)) = c(A)$,

(4) $c(A \cup B) = c(A) \cup c(B)$,

for all subsets A, B of X. Here $\mathcal{P}(X)$ is the power set of X. A subset A of X is said to be *c-closed* provided that $c(A) = A$. A subset B of X is said to be *c-open* provided that $X \backslash B$ is c-closed.

It is known that for a closure operator c on X, the family $\mathfrak{T}$ of all c-open sets forms a topology for X.

Lemma 2.4. *Let Q be a quandle. Define a function $c : \mathcal{P}(Q) \to \mathcal{P}(Q)$ by, for any subset M of Q,*

$$c(M) = \{x \in Q \mid x \in orb_Q(a), a \in M\}.$$

Then c is a closure operator on Q.

Proof. We will check all conditions for the definition of a closure operator. Let M and N be subsets of Q.

(1) It is clear that $c(\emptyset) = \emptyset$.

(2) Let a be an element of M. Since $a * a = a$, $a \in orb_Q(a)$. It implies that $a \in c(M)$. Hence, $M \subset c(M)$.

(3) By (2), it is clear that $c(M) \subset c(c(M))$. Now we claim that $c(c(M)) \subset c(M)$. Let x be an element of $c(c(M))$. Then

$$\begin{aligned} x \in c(c(M)) &\Rightarrow x \in orb_Q(a) \text{ and } a \in c(M) \\ &\Rightarrow x \in orb_Q(a), a \in orb_Q(m) \text{ and } m \in M \\ &\Rightarrow x \in orb_Q(m) \text{ and } m \in M \\ &\Rightarrow x \in c(M). \end{aligned}$$

Hence, we have that $c(c(M)) = c(M)$.

(4) Let x be an element of $c(M \cup N)$. Then

$$\begin{aligned} x \in c(M \cup N) &\Leftrightarrow x \in orb_Q(a) \text{ and } a \in M \cup N \\ &\Leftrightarrow x \in orb_Q(a) \text{ and } (a \in M \text{ or } a \in N) \\ &\Leftrightarrow (x \in orb_Q(m) \text{ and } a \in M) \text{ or } (x \in orb_Q(m) \text{ and } a \in N) \\ &\Leftrightarrow x \in c(M) \text{ or } x \in c(N) \\ &\Leftrightarrow x \in c(M) \cup c(N). \end{aligned}$$

Hence, we have that $c(M \cup N) = c(M) \cup c(N)$. □

Remark 2.5.

(1) In [6], V. Even and M. Gran studied the category of quandles and also proved Lemma 2.4 by using the category theory.

(2) Note that $orb(a)$ for $a \in Q$ is the smallest set containing a which is closed under the operation of Q. Therefore, $orb(a)$ is the smallest subquandle of Q containing a by Lemma 2.2 and the $c(M)$ is the smallest subquandle of Q containing M.

Definition 2.6. Let Q be a quandle and c the closure operator defined in Lemma 2.4. The *intrinsic topology* $\mathfrak{T}_Q$ is the topology on Q defined from the closure operator c. Indeed, $\mathfrak{T}_Q = \{M \subset Q \mid M \text{ is } c\text{-open}\}$.

Example 2.7. Let Q_1 and Q_2 be two quandles whose operation tables are in Table 1. Then we have $orb_{Q_1}(1) = orb_{Q_1}(2) = \{1, 2\}$, $orb_{Q_1}(3) = orb_{Q_1}(4) = \{3, 4\}$, $orb_{Q_1}(5) = \{5\}$ and $orb_{Q_2}(1) = orb_{Q_2}(2) = orb_{Q_2}(3) = orb_{Q_2}(4) = orb_{Q_2}(5) = \{1, 2, 3, 4, 5\}$. Hence, the intrinsic topologies are $\mathfrak{T}_{Q_1} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{5\}, \{1, 2, 3, 4\}, \{1, 2, 5\}, \{3, 4, 5\}, Q_1\}$ and $\mathfrak{T}_{Q_2} = \{\emptyset, Q_2\}$.

Q_1	1	2	3	4	5
1	1	1	2	2	2
2	2	2	1	1	1
3	3	3	3	3	4
4	4	4	4	4	3
5	5	5	5	5	5

Q_2	1	2	3	4	5
1	1	4	5	3	2
2	3	2	4	5	1
3	2	5	3	1	4
4	5	1	2	4	3
5	4	3	1	2	5

Table 1: The operation tables of two quandles of order 5

Example 2.8.

(1) Let Q be a trivial quandle. Since $orb_Q(a) = \{a\}$ for any element $a \in Q$, $c(M) = M$ for any subset M of Q. Indeed, the intrinsic topology $\mathfrak{T}_Q$ on Q is the discrete topology on Q.

(2) Let Q be a connected quandle. Then $orb_Q(a) = Q$ for any element $a \in Q$ and its intrinsic topology $\mathfrak{T}_Q$ on Q consists of the empty set and Q itself. Indeed, $\mathfrak{T}_Q$ is the trivial topology on Q.

Example 2.9. Let $O(n)$ be the group of $n \times n$ orthogonal matrices. Then $det(A) = \pm 1$ for all $A \in O(n)$. Consider the 1-fold conjugation quandle operation, defined by $A * B = B^{-1}AB$, on $O(n)$. It is not hard to see that the orbit set of $O(n)$ is the set of all conjugacy classes. Hence, the intrinsic topology of $(O(n), *)$ is the topology generated by all conjugacy classes of $O(n)$.

Example 2.10. Consider the Alexander quandle $(\mathbb{Z}, *)$ on the integer group $\mathbb{Z}$ with the quandle operation defined by

$$x * y = 2y - x \text{ for any } x, y \in \mathbb{Z}.$$

Let $\mathbb{Z}_o$ be the set of all odd integers and $\mathbb{Z}_e$ the set of all even integers. We know that if $z \in \mathbb{Z}$, then

$$orb_{\mathbb{Z}}(a) = \begin{cases} \mathbb{Z}_o & \text{if } a \text{ is odd,} \\ \mathbb{Z}_e & \text{if } a \text{ is even.} \end{cases}$$

Hence, the intrinsic topology $\mathfrak{T}_{\mathbb{Z}}$ on $(\mathbb{Z}, *)$ is $\{\emptyset, \mathbb{Z}_o, \mathbb{Z}_e, \mathbb{Z}\}$.

Now, we study topological properties of the intrinsic topology on a quandle.

Theorem 2.11. *Let Q be a quandle and $\mathfrak{T}_Q$ the intrinsic topology. Let M be a subset of Q. Then*

(1) The interior of M is the union of all orbits which is contained in M.

(2) Every member of $\mathfrak{T}_Q$ is both open and closed.

(3) $\mathfrak{T}_Q$ is generated by all orbit sets of Q.

(4) $(Q, \mathfrak{T}_Q)$ is locally connected.

Proof. They are straightforward from the definition of the intrinsic topology. □

Theorem 2.12. *Let Q be a quandle and $\mathfrak{T}_Q$ the intrinsic topology. Then the following statements are equivalent.*

(1) Q is the trivial quandle.

(2) $\mathfrak{T}_Q$ is the discrete topology.

(3) $(Q, \mathfrak{T}_Q)$ is totally disconnected.

Proof. In Example 2.8. (1), it was introduced that the intrinsic topology on a trivial quandle Q was discrete. Indeed, the topology is totally disconnected. Conversely, if the intrinsic topology is totally disconnected, every two-point set is disconnected. Then every orbit of Q is singleton and Q is a trivial quandle. □

Theorem 2.13. *Let Q be a quandle and $\mathfrak{T}_Q$ the intrinsic topology. Then the following statements are equivalent.*

(1) Q is a connected quandle.

(2) $\mathfrak{T}_Q$ is the trivial topology.

(3) $(Q, \mathfrak{T}_Q)$ is connected.

Proof. In Example 2.8.(2), it was introduced that the intrinsic topology on a connected quandle Q was trivial. Indeed, the topology is connected. Conversely, if the intrinsic topology is connected, then there exist no pair of open sets which separate Q. Then there exist no proper orbits of Q and Q is a connected quandle. □

Remark 2.14. Note that the separation axioms of the intrinsic topology $\mathfrak{T}_Q$ depends on the quandle Q. Consider the quandle operation table Q_2 in Example 2.7. For any distinct two points in Q_2, there exists no open set containing one of the points but not the other. Moreover, one can see that $\mathfrak{T}_Q$ is a T_0-space if and only in Q is a trivial quandle.

Theorem 2.15. *Let Q be a quandle and M a subquandle of Q. Then $(M, \mathfrak{T}_M)$ is a subspace of $(Q, \mathfrak{T}_Q)$.*

Proof. It is true that $orb_M(a) = orb_Q(a) \cap M$ for any $a \in M$. □

Definition 2.16. Let $(Q_1, *_1)$ and $(Q_2, *_2)$ be two quandles. The *product quandle* of $(Q_1, *_1)$ and $(Q_2, *_2)$ is the set $Q_1 \times Q_2$ with the binary operation $*$ defined by, for any $(a, z), (b, y) \in Q_1 \times Q_2$,

$$(a, x) * (b, y) = (a *_1 b, x *_2 y).$$

Proposition 2.17.([5]) *Let Q_1 and Q_2 be two quandle. Then $Q_1 \times Q_2$ is connected if both Q_1 and Q_2 are connected.*

Theorem 2.18. *Let Q_1 and Q_2 be two quandles. Then $\mathfrak{T}_{Q_1 \times Q_2} = \mathfrak{T}_{Q_1} \times \mathfrak{T}_{Q_2}$.*

Proof. We will show that $\mathfrak{T}_{Q_1 \times Q_2}$ and $\mathfrak{T}_{Q_1} \times \mathfrak{T}_{Q_2}$ have the same basis. By Definition 2.6, $\mathfrak{T}_{Q_1 \times Q_2}$ is generated by all orbit sets of $Q_1 \times Q_2$. By Proposition 2.17, all the orbits of the quandle $\mathfrak{T}_{Q_1 \times Q_2}$ are obtained by the product of each orbit of $\mathfrak{T}_{Q_1}$ and $\mathfrak{T}_{Q_2}$. One can see that it is the definition of the product topology of $\mathfrak{T}_{Q_1} \times \mathfrak{T}_{Q_2}$. □

Example 2.19. Let Q_1 and Q_2 be two quandles with operation tables in Table 2. Then $\mathfrak{T}_{Q_1} = \{\emptyset, Q_1\}$ and $\mathfrak{T}_{Q_2} = \{\emptyset, \{1, 2\}, \{3\}, Q_2\}$ and the product quandle of Q_1 and Q_2 has the operation table in Table 3. Hence, the intrinsic topology $\mathfrak{T}_{Q_1 \times Q_2}$ on the product quandle is the equal to the product topology of $\mathfrak{T}_{Q_1}$ and $\mathfrak{T}_{Q_2}$

$$\mathfrak{T}_{Q_1} \times \mathfrak{T}_{Q_2} = \{\emptyset, \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}, \{(1,3), (2,3), (3,3)\}, Q_1 \times Q_2\}.$$

Q_1	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Q_2	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

Table 2: The operation tables of two quandle of order 3

$Q_1 \times Q_2$	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	(1,1)	(1,1)	(1,2)	(3,1)	(3,1)	(3,2)	(2,1)	(2,1)	(2,2)
(1,2)	(1,2)	(1,2)	(1,1)	(3,2)	(3,2)	(3,1)	(2,2)	(2,2)	(2,1)
(1,3)	(1,3)	(1,3)	(1,3)	(3,3)	(3,3)	(3,3)	(2,3)	(2,3)	(2,3)
(2,1)	(3,1)	(3,1)	(3,2)	(2,1)	(2,1)	(2,2)	(1,1)	(1,1)	(1,2)
(2,2)	(3,2)	(3,2)	(3,1)	(2,2)	(2,2)	(2,1)	(1,2)	(1,2)	(1,1)
(2,3)	(3,3)	(3,3)	(3,3)	(2,3)	(2,3)	(2,3)	(1,3)	(1,3)	(1,3)
(3,1)	(2,1)	(2,1)	(2,2)	(1,1)	(1,1)	(1,2)	(3,1)	(3,1)	(3,2)
(3,2)	(2,2)	(2,2)	(2,1)	(1,2)	(1,2)	(1,1)	(3,2)	(3,2)	(3,1)
(3,3)	(2,3)	(2,3)	(2,3)	(1,3)	(1,3)	(1,3)	(3,3)	(3,3)	(3,3)

Table 3: The operation table of the product quandle of Q_1 and Q_2

Example 2.20. Consider the coxeter quandle Q on $\mathbb{R}^2$. Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^2$. By the geometric meaning of the coxeter quandle operation, it is easy to show that $orb(\mathbf{u}) = \{\mathbf{v} \in \mathbb{R}^2 \backslash \{\mathbf{0}\} \mid \|\mathbf{v}\| = \|\mathbf{u}\|\}$. That is, $orb(\mathbf{u})$ is a circle centered at the origin of radius $\|\mathbf{u}\|$ as depicted in Figure 2. Hence, the intrinsic topology $\mathfrak{T}$ on Q is generated by all circles centered at origin.

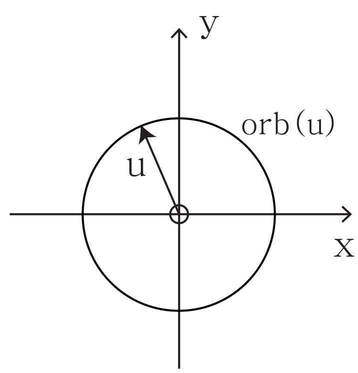

Figure 2: The orbit of the coxeter quandle on $\mathbb{R}^2$

Let $C(1)$ denote the circle centered at the origin of the radius 1. For any element $x, y \in C(1)$, we see that $\|x * y\| = \|x\| = \|y\|$. Then $x * y$ is also an element in $C(1)$. By Lemma. 2.2, $C(1)$ is a connected subquandle of the coxeter quandle on $\mathbb{R}^2$. We denote the intrinsic topology on $C(1)$ by $\mathfrak{T}_{C(1)}$.

Let $\mathbb{R}$ be the real line in $\mathbb{R}^2$. One can easily check that $\mathbb{R}$ is a subquandle of the coxeter quandle on $\mathbb{R}^2$. Since for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$, $\mathbf{u} * \mathbf{v} = \mathbf{u}$, $(\mathbb{R}, *)$ is the trivial quandle. We denote the intrinsic topology on $\mathbb{R}$ by $\mathfrak{T}_{\mathbb{R}}$.

Then the product topology $\mathfrak{T}_{C(1)} \times \mathfrak{T}_{\mathbb{R}}$of $\mathfrak{T}_{C(1)}$ and $\mathfrak{T}_{\mathbb{R}}$ is generated by $\{C(1) \times \{\mathbf{u}\} \mid \mathbf{u} \in \mathbb{R}\}$. Since the bases for the intrinsic topology $\mathfrak{T}$ of the coxeter quandle and $\mathfrak{T}_{C(1)} \times \mathfrak{T}_{\mathbb{R}}$ are the same, the intrinsic topology $\mathfrak{T}$ of the coxeter quandle on $\mathbb{R}^2$ is the product topology $\mathfrak{T}_{C(1)} \times \mathfrak{T}_{\mathbb{R}}$.

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