KYUNGPOOK Math. J. 57(2017), 701-709
https://doi.org/10.5666/KMJ.2017.57.4.701
pISSN 1225-6951 eISSN 0454-8124

Minimum Covering Randic Energy of a Graph

Kunkunadu Nanjundappa Prakasha
Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru-570002, India
e-mail : prakashamaths@gmail.com

Siva Kota Reddy Polaepalli
Department of Mathematics, Siddaganga Institute of Technology, Tumkur-572 103, India
e-mail : pskreddy@sit.ac.in

Ismail Naci Cangul*
Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059 Bursa, Turkey
e-mail : ncangul@gmail.com

Abstract. In this paper, we introduce the minimum covering Randic energy of a graph. We compute minimum covering Randic energy of some standard graphs and establish upper and lower bounds for this energy. Also we disprove a conjecture on Randic energy which is proposed by S. Alikhani and N. Ghanbari, [2].

1. Introduction

Let G be a simple, finite, undirected graph. The energy of G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G and plays a very central role in mathematical chemistry. For more details on the energy of a graph, see [4, 5].

The Randic matrix $R(G) = (R_{ij})_{n\times n}$ is defined and used in [3] as follows:

* Corresponding Author.
Received December 5, 2016; revised October 12, 2017; accepted November 17, 2017.
2010 Mathematics Subject Classification: 05C50.
Key words and phrases: minimum covering set, minimum covering Randic eigenvalues, minimum covering Randic energy.
The third author is supported by the research fund of Uludag University project nos: F-2015/17, 2015/23 and 2016/9.

$$R_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 0 & otherwise. \end{cases}$$

Some lower and upper bounds on Randic energy were given in [3, 6].

2. The Minimum Covering Randic Energy of Graph

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$ and edge set E. A subset C of V is called a covering set of the graph G if every edge of G is incident to at least one vertex of C. A covering set with minimum cardinality is called a minimum covering set. Let C be a minimum covering set of a graph G. The minimum covering Randic matrix $R^C(G) = (R^C_{ij})_{n\times n}$ with respect to C is given by

$$R^C_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_i}} & \text{if } v_i \sim v_j, \\ 1 & \text{if } i = j \text{ and } v_i \in C, \\ 0 & otherwise. \end{cases}$$

The characteristic polynomial of $R^C(G)$ is denoted by

$$\phi^C_R(G, \lambda) = det(\lambda I - R^C(G)).$$

Since the minimum covering Randic matrix is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum covering Randic energy was given in [1] by

$$RE_C(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{2.1}$$

Recall that the spectrum of a graph G is the list of distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ with multiplicities $m_1, m_2, \ldots, m_r$ and in this paper, we shall denote it by

$$\text{Spec(G)} = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}.$$

This paper is organized as follows. In Section 3, we get some basic properties of minimum covering Randic energy of a graph, and in Section 4, minimum covering Randic energy of some well-known graph types are obtained. In Section 5, we give a counterexample to a conjecture recently proposed on density of Randic energy.

3. Some Basic Properties of Minimum Covering Randic Energy of a Graph

Let us consider the number

$$P = \sum_{i<j} \frac{1}{d_i d_j}$$

where $d_i d_j$ is the product of the degrees of two adjacent vertices.

Proposition 3.1. *The first three coefficients of $\phi_R^C(G,\lambda)$ are as follows:*

(i) $a_0 = 1$,

(ii) $a_1 = -|C|$,

(iii) $a_2 = |C|C_2 - P$.

Proof. (i) From the definition $\Phi_R^C(G,\lambda)$, we easily get $a_0 = 1$.

(ii) The sum of the determinants of all 1×1 principal submatrices of $R^C(G)$ is equal to the trace of $R^C(G)$ which implies that

$$a_1 = (-1)^1 \times \text{the trace of } [R^C(G)] = -|C|.$$

(iii) In a similar way, we obtain that

$$\begin{aligned} (-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ij}a_{ji} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ij}a_{ji} \\ &= |C|C_2 - P. \end{aligned}$$

□

Proposition 3.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the minimum covering Randic eigenvalues of $R^C(G)$, then*

$$\sum_{i=1}^{n} {\lambda_i}^2 = |C| + 2P.$$

Proof. We know that

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2 \\ &= 2\sum_{i<j} a_{ij}^2 + |C| \\ &= |C| + 2P. \end{aligned}$$

□

Now, we give an upper bound for $RE^C(G)$:

Theorem 3.3. *Let G be a graph with n vertices. Then*

$$RE^C(G) \leq \sqrt{n(|C| + 2[P])}.$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $R_C(G)$. Now by the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$ and $b_i =| \lambda_i |$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

implying

$$[RE^C(G)]^2 \leq n(|C| + 2P)$$

and hence we obtain

$$[RE^C(G)] \leq \sqrt{n(|C| + 2P)}$$

as an upper bound. □

The next result gives a lower bound for $RE^C(G)$:

Theorem 3.4. *Let G be a graph with n vertices. If R= det $R^C(G)$, then*

$$RE^C(G) \geq \sqrt{(|C| + 2P) + n(n-1)R^{\frac{2}{n}}}.$$

Proof. By definition, we have

$$\begin{aligned} \left(RE^C(G)\right)^2 &= \left(\sum_{i=1}^{n} | \lambda_i |\right)^2 \\ &= \sum_{i=1}^{n} | \lambda_i | \sum_{j=1}^{n} | \lambda_j | \\ &= \left(\sum_{i=1}^{n} | \lambda_i |^2\right) + \sum_{i \neq j} | \lambda_i || \lambda_j | . \end{aligned}$$

Using arithmetic and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i \neq j} | \lambda_i || \lambda_j | \quad \geq \quad \left(\prod_{i \neq j} | \lambda_i || \lambda_j |\right)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\begin{aligned}
[RE^C(G)]^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}} \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)R^{\frac{2}{n}} \\
&= |C| + 2P + n(n-1)R^{\frac{2}{n}}.
\end{aligned}$$

Thus,

$$RE^C(G) \geq \sqrt{|C| + 2P + n(n-1)R^{\frac{2}{n}}}.$$

□

4. Minimum Covering Randic Energy of Some Graph Types

In this section, we calculate the minimum covering Randic energy of some well-known graph types including complete graphs, star graphs, crown graphs, complete bipartite graphs and coctail party graphs.

Theorem 4.1. *The minimum covering Randic energy of the complete graph K_n is*

$$RE^C(K_n) = \frac{n^2 - 4n + 4 + \sqrt{4n^2 - 8n + 5}}{n-1}.$$

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. The minimum covering set is then $C = \{v_1, v_2, \cdots, v_{n-1}\}$. The minimum covering Randic matrix is

$$R^C(K_n) = \begin{bmatrix}
1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\
\frac{1}{n-1} & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\
\frac{1}{n-1} & \frac{1}{n-1} & 1 & \cdots & \frac{1}{n-1} & \frac{1}{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 1 & \frac{1}{n-1} \\
\frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & \frac{1}{n-1} & 0
\end{bmatrix}.$$

Then the characteristic equation is

$$\left(\lambda - \frac{n-2}{n-1}\right)^{n-2}\left(\lambda^2 - \frac{2n-3}{n-1}\lambda - \frac{1}{n-1}\right) = 0$$

and therefore the spectrum becomes

$$Spec_R^C(K_n) = \begin{pmatrix} \frac{2n-3+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{n-2}{n-1} & \frac{2n-3-\sqrt{4n^2-8n+5}}{2(n-1)} \\ 1 & n-2 & 1 \end{pmatrix}.$$

Therefore we obtain

$$RE^C(K_n) = \frac{n^2-4n+4+\sqrt{4n^2-8n+5}}{n-1}.$$

□

Theorem 4.2. *The minimum covering Randic energy of the star graph $K_{1,n-1}$ is*

$$RE^C(K_{1,n-1}) = \sqrt{5}.$$

Proof. Let $K_{1,n-1}$ be the star graph with vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ with the assumption that v_0 is the central vertex. The minimum covering set can be chosen as $C = \{v_0\}$. The minimum covering Randic matrix becomes

$$R^C(K_{1,n-1}) = \begin{bmatrix} 1 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \cdots & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{\sqrt{n-1}} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The characteristic equation will become

$$\lambda^{n-2}(\lambda^2 - \lambda - 1) = 0$$

and the spectrum will be

$$Spec_R^C(K_{1,n-1}) = \begin{pmatrix} 0 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Therefore,

$$RE^C(K_{1,n-1}) = \sqrt{5}.$$

□

Theorem 4.3. *The minimum covering Randic energy of the Crown graph S_n^0 is*

$$RE^C(S_n^0) = \sqrt{5} + \sqrt{n^2-2n+5}.$$

Proof. Let S_n^0 be the crown graph of order $2n$ with vertex set

$$\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$$

and a minimum covering set would be $C = \{u_1, u_2, \cdots, u_n\}$. Then the minimum covering Randic matrix is

$$R^C(S_n^0) = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 1 & 0 & \ldots & 0 & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & \frac{1}{n-1} \\ 0 & 0 & 1 & \ldots & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \\ 0 & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 1 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & 0 & \ldots & \frac{1}{n-1} & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

Then the characteristic equation is

$$(\lambda^2 - \lambda - 1)(\lambda^2 - \lambda - \frac{1}{(n-1)^2})^{n-1} = 0$$

and hence the spectrum would be

$$Spec_R^C(S_n^0) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{n-1+\sqrt{n^2-2n+5}}{2(n-1)} & \frac{n-1-\sqrt{n^2-2n+5}}{2(n-1)} & \frac{1-\sqrt{5}}{2} \\ 1 & n-1 & n-1 & 1 \end{pmatrix}.$$

Therefore,

$$RE^C(S_n^0) = \sqrt{5} + \sqrt{n^2 - 2n + 5}.$$

□

Theorem 4.4. *The minimum covering Randic energy of the complete bipartite graph $K_{n,n}$ is*

$$RE_{P_1}^C(K_{n,n}) = n - 1 + \sqrt{5}.$$

Proof. Let $K_{n,n}$ be the complete bipartite graph of order $2n$ with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. In that case, the minimum covering set would be found as $C = \{u_1, u_2, \cdots, u_n\}$ and the minimum covering Randic matrix would be

$$R^C(K_{n,n}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 1 & 0 & 0 & \ldots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 1 & 0 & \ldots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & 1 & \ldots & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & 0 & 0 & 0 & 0 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In that case, the characteristic equation would be

$$\lambda^{n-1}(\lambda - 1)^{n-1}(\lambda^2 - \lambda - 1) = 0$$

and hence, the spectrum would become

$$Spec^C_{P_1}(K_{n,n}) = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 & 0 & \frac{1-\sqrt{5}}{2} \\ 1 & n-1 & n-1 & 1 \end{pmatrix}.$$

Therefore,

$$RE^C(K_{n,n}) = n - 1 + \sqrt{5}$$

as required. □

Theorem 4.5. *The minimum covering Randic energy of the cocktail party graph* $K_{n\times 2}$ *is*

$$RE^C(K_{n\times 2}) = 2 + \sqrt{n^2 - 2n + 2}.$$

Proof. Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having the vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then for the cocktail party graph, the minimum covering set is $C = \{u_1, u_2, \cdots, u_n\}$ and the minimum covering Randic matrix is

$$R^C(K_{n\times 2}) = \begin{bmatrix} 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} \\ \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \cdots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 \end{bmatrix}.$$

Then the characteristic equation becomes

$$\left(\lambda^2 - 2\lambda + \frac{1}{2}\right)\left(\lambda^2 - \frac{n-2}{n-1}\lambda - \frac{1}{2n-2}\right) = 0$$

and hence the spectrum would be

$$Spec_R^C(K_{n\times 2}) = \begin{pmatrix} \frac{2+\sqrt{2}}{2} & \frac{n-2+\sqrt{n^2-2n+2}}{2(n-1)} & \frac{2-\sqrt{2}}{2} & \frac{n-2-\sqrt{n^2-2n+2}}{2(n-1)} \\ 1 & n-1 & 1 & n-1 \end{pmatrix}.$$

Therefore, we obtain

$$RE^C(K_{n\times 2}) = 2 + \sqrt{n^2-2n+2}.$$

□

References

[1] C. Adiga, A. Bayad, I. Gutman and A. S. Shrikanth, *The minimum covering energy of a graph*, Kragujevac J. Sci., **34**(2012), 39–56.

[2] S. Alikhani and N. Ghanbari, *More on Energy and Randic Energy of Specific Graphs*, Journal of Mathematical Extension, **9(3)**(2015), 73–85.

[3] S. B. Bozkurt, A. D. Gungor, I. Gutman and A. S. Cevik, *Randic matrix and Randic energy*, MATCH Commun. Math. Comput. Chem., **64**(2010), 239–250.

[4] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz, **103**(1978), 1–22.

[5] I. Gutman, *The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue and A. Wassermann (Eds.)*, Algebraic Combinatorics and Applications, Springer-Verlag, Berlin, 2001, 196–211.

[6] I. Gutman, B. Furtula and S. B. Bozkurt, *On Randic energy*, Linear Algebra Appl., **442**(2014), 50–57.