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Real Hypersurfaces in the Complex Hyperbolic Quadric with Killing Shape Operator

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Abstract. We introduce the notion of Killing shape operator for real hypersurfaces in the complex hyperbolic quadric $Q^{m*} = SO_{m,2}/SO_mSO_2$. The Killing shape operator implies that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of real hypersurfaces in $Q^{m*} = SO_{m,2}/SO_mSO_2$ with Killing shape operator.

1. Introduction

As examples of some Hermitian symmetric spaces of rank 2, usually we can consider Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1], [2], [3], [6], [7], [8], [16], [17] and [23]). Those are said to be Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J} = \text{Span}\{J_1, J_2, J_3\}$ on $SU_{2,m}/S(U_2U_m)$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another example of Hermitian symmetric space with rank 2 of compact type different from the above ones, we could give a complex quadric $Q^m =$

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SO_{m+2}/SO_2SO_m, which is a complex hypersurface in complex projective space CP^m (see Reckziegel [14], Suh [19], [20], [22] and Smyth [15]). The complex quadric also can be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [10]). Accordingly, the complex quadric admits two important geometric structures that a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [9] and Reckziegel [14]).

About the latter part of twentieth century, many geometers have investigated some real hypersurfaces in Hermitian symmetric spaces of rank 1 like the complex projective space $\mathbb{C}P^m$ or the complex hyperbolic space $\mathbb{C}H^m$. For the complex projective space $\mathbb{C}P^m$ and the quaternionic projective space $\mathbb{Q}P^m$ a characterization with isometric Reeb flow was obtained by Okumura [11], $\mathcal{D}$-parallel shape operator $\nabla_{\mathcal{D}} A = 0$ by Pérez [12], and $\mathcal{D}$-parallel curvature tensor $\nabla_{\mathcal{D}} R = 0$ by Pérez and Suh [13], respectively, where $\mathcal{D} = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, $\xi_i = -J_i N$, $i = 1, 2, 3$.

Now let us introduce a complex hyperbolic quadric $Q^{m*} = SO^o_{m,2}/SO_2SO_m$, which can be regarded as a Hermitian symmetric space with rank 2 of noncompact type. Here we consider a real hypersurface M in Q^{m*} with shape operator of Codazzi type, that is, $(\nabla_X S)Y = (\nabla_Y S)X$ for the shape operator S and any vector fields X and Y on M in Q^{m*}. In Suh [18] we gave a non-existence property for real hypersurfaces of Codazzi type in the complex quadric Q^{m*} as follows:

Theorem A. *There do not exist any Hopf real hypersurfaces in complex quadric Q^{m*}, $m \geq 3$, with shape operator of Codazzi type.*

Next we have considered the notion of parallel shape operator for real hypersurfaces M in Q^{m*}. Usually, parallelism is naturally included in the notion of shape operator of Codazzi type. So by Theorem A we can assert the following

Theorem B. *There do not exist any Hopf real hypersurfaces in complex quadric Q^{m*}, $m \geq 3$, with parallel shape operator.*

Apart from the complex structure J there is another distinguished geometric structure on Q^{m*}. Namely, a vector subbundle $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset C\}$, $[z] \in Q^{m*}$, which consists of all complex conjugations defined on the complex quadric Q^{m*}. The vector bundle $\mathfrak{A}$ contains a S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^{m*} and becomes a parallel rank 2-subbundle of of End TQ^{m*}. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^{m*}.

Recall that a nonzero tangent vector $W \in T_z Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. Here maximal flat means a 2-dimensional curvature flat maximal totally geodesic submanifold in Q^{m*}. Such a maximal flat always exists, because the rank of Q^{m*} is 2. There are two types of singular tangent vectors for the complex quadric Q^{m*} as follows:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Here we note that the unit normal N is said to be $\mathfrak{A}$-principal if N is invariant under the complex conjugation A, that is, $AN = N$.

Moreover, the derivative of the complex conjugation A on Q^{m*} is defined by

$$(\bar{\nabla}_X A)Y = q(X)JAY$$

for any vector fields X and Y on M and q denotes a certain 1-form defined on M.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. There are two types of singular tangent vectors for the complex quadric Q^{m*}:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

The shape operator S of M in Q^{m*} is said to be *Killing* if the operator S satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0$$

for any $X, Y \in T_zM$, $z \in M$. The equation is equivalent to $(\nabla_X S)X = 0$ for any $X \in T_zM$, $z \in M$, because of linearization. Moreover, we can give the geometric meaning of Killing Jacobi tensor as follows:

When we consider a geodesic γ with initial conditions such that $\gamma(0) = z$ and $\dot{\gamma}(0) = X$. Then the transformed vector field $S\dot{\gamma}$ is Levi-Civita *parallel* along the geodesic γ of the vector field X (see Blair [5] and Tachibana [24]).

In the study of real hypersurfaces in the complex quadric Q^m we considered the notion of parallel Ricci tensor, that is, $\nabla \mathrm{Ric} = 0$ (see Suh [19]). But from the assumption of Ricci being parallel, it was difficult for us to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [19] we gave a classification with the further assumption of $\mathfrak{A}$-isotropic. But fortunately, when we consider Killing shape operator for real hypersurfaces in the complex hyperbolic quadric Q^{m*}, first we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal as follows:

Main Theorem 1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$, with Killing shape operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Then motivated by such a result, next we give a complete classification for real hypersurfaces in the complex hyperbolic quadric Q^{m*} with Killing shape operator as follows:

Main Theorem 2. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 4$, with Killing shape operator. Then M is locally congruent to a horosphere or has* 6 *distinct constant principal curvatures given by*

$$\alpha \neq 0,\ \beta = \gamma = 0,\ \lambda_1 = \frac{(\alpha^2-1)+\sqrt{(\alpha^2-1)^2-2\alpha^2}}{2\alpha},$$

and

$$\lambda_2 = \frac{(\alpha^2-1)-\sqrt{(\alpha^2-1)^2-2\alpha^2}}{2\alpha}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi],\ T_\beta = [AN],\ T_\gamma = [A\xi],\ \phi(T_{\lambda_i}) = T_{\mu_i},$$

and

$$\dim\ (T_{\lambda_1} + T_{\lambda_2}) = \dim\ (T_{\mu_1} + T_{\mu_2}) = m-2.$$

2. The Complex Hyperbolic Quadric

Let us denote by C_1^{m+2} an indefinite complex Euclidean space $\mathbb{C}^{m+2}$, on which the indefinite Hermitian product

$$H(z,w) = -z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_{n+2}\bar{w}_{n+2}$$

is negative definite.

The homogeneous quadratic equation $z_1^2 + \ldots + z_m^2 - z_{m+1}^2 - z_{m+2}^2 = 0$ consists of the points in $\mathbb{C}_1^{m+2}$ defines a noncompact complex hyperbolic quadric $Q^{*m} = SO^o_{2,m}/SO_2SO_m$ which can be immersed in the $(m+1)$-dimensional in complex hyperbolic space $\mathbb{C}H^{m+1} = SU_{1,m+1}/S(U_{m+1}U_1)$. The complex hypersurface Q^{m*} in $\mathbb{C}H^{m+1}$ is known as the m-dimensional complex hyperbolic quadric. The complex structure J on $\mathbb{C}H^{m+1}$ naturally induces a complex structure on Q^{*m} which we will denote by J as well. We equip Q^{m*} with the Riemannian metric g which is induced from the Begerman metric on $\mathbb{C}H^{m+1}$ with constant holomorphic sectional curvature 4. For $m \geq 2$ the triple (Q^{m*}, J, g) is a Hermitian symmetric space of rank two and its minimal sectional curvature is equal to -4. The 1-dimensional quadric Q^{1^*} is isometric to the 2-dimensional real hyperbolic space $\mathbb{R}H^2 = SO^o_{1,2}/SO_1SO_2$. The 2-dimensional complex quadric Q^{2^*} is isometric to the Riemannian product of complex hyperbolic spaces $\mathbb{C}H^1 \times \mathbb{C}H^1$. We will assume $m \geq 3$ for the main part of this paper.

For a nonzero vector $z \in \mathbb{C}_1^{m+2}$ we denote by $[z]$ the complex span of z, that is, $[z] = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}H^{m+1}$. As usual, for each $[z] \in \mathbb{C}H^{m+1}$ we identify $T_{[z]}\mathbb{C}H^{m+1}$ with the orthogonal complement $\mathbb{C}_1^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}_1^{m+2}$. For $[z] \in Q^{m*}$ the tangent space $T_{[z]}Q^{m*}$ can then be identified canonically with the orthogonal complement $\mathbb{C}_1^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}_1^{m+2}$. Note that $\bar{z} \in \nu_{[z]}Q^{m*}$ is a unit normal vector of Q^{m*} in $\mathbb{C}H^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^{m*} in $\mathbb{C}H^{m+1}$ with respect to $\bar{z}$. Then we have $A_{\bar{z}}w = \overline{w}$ for all $w \in T_{[z]}Q^{m*}$, that is, $A_{\bar{z}}$ is just complex conjugation restricted to $T_{[z]}Q^{m*}$. The shape operator $A_{\bar{z}}$ is an antilinear involution on the complex vector space $T_{[z]}Q^{*m}$ and

$$T_{[z]}Q^{m*} = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}_1^{m+2} \cap T_{[z]}Q^{m*}$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}_1^{m+2} \cap T_{[z]}Q^{m*}$ is the (-1)-eigenspace of $A_{\bar{z}}$. Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^{m*}$. Recall that a real structure on a complex vector space V is by definition an antilinear involution $A : V \to V$. Since the normal space $\nu_{[z]}Q^{m*}$ of Q^{m*} in $\mathbb{C}H_1^{m+1}$ at $[z]$ is a complex subspace of $T_{[z]}\mathbb{C}H^{m+1}$ of complex dimension one, every normal vector in $\nu_{[z]}Q^{m*}$ can be written as $\lambda\bar{z}$ with some $\lambda \in \mathbb{C}$. The shape operators $A_{\lambda\bar{z}}$ of Q^{m*} define a rank two vector subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^{m*})$. Since the second fundamental form of the embedding $Q^{m*} \subset \mathbb{C}H^{m+1}$ is parallel (see e.g. [15]), $\mathfrak{A}$ is a parallel subbundle of $\mathrm{End}(TQ^{m*})$. For $\lambda \in S^1 \subset \mathbb{C}$ we again get a real structure $A_{\lambda\bar{z}}$ on $T_{[z]}Q^{m*}$ and we have $V(A_{\lambda\bar{z}}) = \lambda V(A_{\bar{z}})$. We thus have an S^1-subbundle of $\mathfrak{A}$ consisting of real structures on the tangent spaces of Q^{m*}.

The Gauss equation for the complex hypersurface $Q^{m*} \subset \mathbb{C}H^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^{m*} can be expressed in terms of the Riemannian metric g, the complex structure J and a generic real structure A in $\mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = &-g(Y,Z)X + g(X,Z)Y \\ &- g(JY,Z)JX + g(JX,Z)JY + 2g(JX,Y)JZ \\ &- g(AY,Z)AX + g(AX,Z)AY \\ &- g(JAY,Z)JAX + g(JAX,Z)JAY.\end{aligned}$$

Note that the complex structure J anti-commutes with each endomorphism $A \in \mathfrak{A}$, that is, $AJ = -JA$.

A nonzero tangent vector $W \in T_{[z]}Q^{m*}$ is called singular if it is tangent to more than one maximal flat in Q^{m*}. There are two types of singular tangent vectors for the complex quadric Q^{m*}:

(i) If there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

(ii) If there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_{[z]}Q^{m*}$ there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$.

Let M be a real hypersurface in Q^{m*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathcal{F}$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$. We denote by νM the normal bundle of M.

We first introduce some notations. For a fixed real structure $A \in \mathfrak{A}_{[z]}$ and $X \in T_{[z]}M$ we decompose AX into its tangential and normal component, that is,

$$AX = BX + \rho(X)N$$

where BX is the tangential component of AX and

$$\rho(X) = g(AX, N) = g(X, AN) = g(X, AJ\xi) = g(JX, A\xi).$$

Since $JX = \phi X + \eta(X)N$ and $A\xi = B\xi + \rho(\xi)N$ we also have

$$\rho(X) = g(\phi X, B\xi) + \eta(X)\rho(\xi) = \eta(B\phi X) + \eta(X)\rho(\xi).$$

We also define

$$\delta = g(N, AN) = g(JN, JAN) = -g(JN, AJN) = -g(\xi, A\xi).$$

At each point $[z] \in M$ we define

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M \mid AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\},$$

which is the maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$. Then by using the same method for real hypersurfaces in Q^{*m} as in Berndt and Suh [4] we get the following

Lemma 2.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*}. Then the following statements are equivalent:*

(i) *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal,*

(ii) $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$,

(iii) *There exists a real structure* $A \in \mathfrak{A}_{[z]}$ *such that* $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$.

Assume now that the normal vector $N_{[z]}$ of M is not $\mathfrak{A}$-principal. Then there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$N_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 < t \leq \frac{\pi}{4}$. This implies

$$\begin{aligned} N_{[z]} &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_{[z]} &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi_{[z]} &= \sin(t)Z_2 + \cos(t)JZ_1, \end{aligned}$$

and therefore $\mathcal{Q}_{[z]} = T_{[z]}Q^m \ominus ([Z_1] \oplus [Z_2])$ is strictly contained in $\mathcal{C}_{[z]}$. Moreover, we have

$$A\xi_{[z]} = B\xi_{[z]} \text{ and } \rho(\xi_{[z]}) = 0.$$

We have

$$\begin{aligned} g(B\xi_{[z]} + \delta\xi_{[z]}, N_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, \xi_{[z]}) &= 0, \\ g(B\xi_{[z]} + \delta\xi_{[z]}, B\xi_{[z]} + \delta\xi_{[z]}) &= \sin^2(2t), \end{aligned}$$

where the function δ denotes $\delta = -g(\xi, A\xi) = -(\sin^2 t - \cos^2 t) = \cos 2t$. Therefore

$$U_{[z]} = \frac{1}{\sin(2t)}(B\xi_{[z]} + \delta\xi_{[z]})$$

is a unit vector in $\mathcal{C}_{[z]}$ and

$$\mathcal{C}_{[z]} = \mathcal{Q}_{[z]} \oplus [U_{[z]}] \text{ (orthogonal sum).}$$

If $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$, then N is not $\mathfrak{A}$-principal in an open neighborhood of $[z]$, and therefore U is a well-defined unit vector field on that open neighborhood. We summarize this in the following

Lemma 2.2. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} whose unit normal $N_{[z]}$ is not $\mathfrak{A}$-principal at $[z]$. Then there exists an open neighborhood of $[z]$ in M and a section A in $\mathfrak{A}$ on that neighborhood consisting of real structures such that*

(i) $A\xi = B\xi$ *and* $\rho(\xi) = 0$,

(ii) $U = (B\xi + \delta\xi)/||B\xi + \delta\xi||$ *is a unit vector field tangent to* $\mathcal{C}$,

(iii) $\mathcal{C} = \mathcal{Q} \oplus [U]$.

3. The Codazzi Equation and Some Consequences

From the explicit expression of the Riemannian curvature tensor of the complex hyperbolic quadric Q^{m*} we can easily derive the Codazzi equation for a real hypersurface $M \subset Q^{*m}$:

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, Z)\\
&\quad = -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y)\\
&\quad\quad - \rho(X)g(BY, Z) + \rho(Y)g(BX, Z)\\
&\quad\quad + \eta(BX)g(BY, \phi Z) + \eta(BX)\rho(Y)\eta(Z)\\
&\quad\quad - \eta(BY)g(BX, \phi Z) - \eta(BY)\rho(X)\eta(Z).
\end{aligned}$$

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^{m*} satisfies

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M. Inserting $Z = \xi$ into the Codazzi equation leads to

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = 2g(\phi X, Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX).$$

On the other hand, we have

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&\quad = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X)\\
&\quad = d\alpha(X)\eta(Y) - d\alpha(Y)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$d\alpha(Y) = d\alpha(\xi)\eta(Y) + 2\delta\rho(Y),$$

where δ denotes $\delta = -g(A\xi, \xi)$. Reinserting this into the previous equation yields

$$\begin{aligned}
&g((\nabla_X S)Y - (\nabla_Y S)X, \xi)\\
&\quad = -2\delta\eta(X)\rho(Y) + 2\delta\rho(X)\eta(Y) + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y).
\end{aligned}$$

Altogether this implies

$$\begin{aligned}
0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) + 2g(\phi X, Y)\\
&\quad - 2\delta\rho(X)\eta(Y) - 2\rho(X)\eta(BY) + 2\rho(Y)\eta(BX) + 2\delta\eta(X)\rho(Y)\\
&= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y)\\
&\quad - 2\rho(X)\eta(BY + \delta Y) + 2\rho(Y)\eta(BX + \delta X)\\
&= g((2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X, Y)\\
&\quad - 2\rho(X)g(Y, B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\rho(Y).
\end{aligned}$$

If $AN = N$ we have $\rho = 0$, otherwise we can use Lemma 2.2 to calculate $\rho(Y) = g(Y, AN) = g(Y, AJ\xi) = -g(Y, JA\xi) = -g(Y, JB\xi) = -g(Y, \phi B\xi)$. Thus we have proved

Lemma 3.1. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then we have*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\phi B\xi.$$

If the unit normal vector field N is $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ such that $AN = N$. Then we have $\rho = 0$ and $\phi B\xi = -\phi\xi = 0$, and therefore

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi.$$

If N is not $\mathfrak{A}$-principal, we can choose a real structure $A \in \mathfrak{A}$ as in Lemma 2.2 and get

$$\begin{aligned}
&\rho(X)(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi B\xi \\
&\quad = -g(X, \phi(B\xi + \delta\xi))(B\xi + \delta\xi) + g(X, B\xi + \delta\xi)\phi(B\xi + \delta\xi) \\
&\quad = ||B\xi + \delta\xi||^2(g(X, U)\phi U - g(X, \phi U)U) \\
&\quad = \sin^2(2t)(g(X, U)\phi U - g(X, \phi U)U),
\end{aligned}$$

which is equal to 0 on $\mathcal{Q}$ and equal to $\sin^2(2t)\phi X$ on $\mathcal{C} \ominus \mathcal{Q}$. Altogether we have proved:

Lemma 3.2. *Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m*}, $m \geq 3$. Then the tensor field*

$$2S\phi S - \alpha(\phi S + S\phi)$$

leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have

$$2S\phi S - \alpha(\phi S + S\phi) = -2\phi \text{ on } \mathcal{Q}$$

and

$$2S\phi S - \alpha(\phi S + S\phi) = -2\delta^2\phi \text{ on } \mathcal{C} \ominus \mathcal{Q},$$

where $\delta = -g(A\xi, \xi) = \cos 2t$ as in Section 3.

4. Killing Shape Operator and a Key Lemma

Let us put $AX = BX + \rho(X)N$ for any vector field $X \in T_zQ^{m*}$, $z \in M$, $\rho(X) = g(AX, N)$, where BX and $\rho(X)N$ respectively denote the tangential and normal component of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi, N) = 0$. Then it follows that

$$\begin{aligned}
AN &= AJ\xi = -JA\xi = -J(B\xi + \rho(\xi)N) \\
&= -(\phi B\xi + \eta(B\xi)N).
\end{aligned}$$

The shape operator S of M in the complex hyperbolic quadric Q^{m*} is said to be *Killing* if the operator S satisfies

$$(\nabla_X S)Y + (\nabla_Y S)X = 0 \tag{4.1}$$

for any $X, Y \in T_z M$, $z \in M$.
From (4.1), together with the equation of Codazzi, it follows that

$$\begin{aligned} 2g((\nabla_X S)Y, Z) = & -\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z) + 2\eta(Z)g(\phi X, Y) \\ & - g(X, AN)g(AY, Z) + g(Y, AN)g(AX, Z) \\ & - g(X, A\xi)g(JAY, Z) + g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{4.2}$$

Since we have assumed the real hypersurface M in Q^{m*} is *Hopf*, then $S\xi = \alpha\xi$. This gives

$$(\nabla_X S)\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX.$$

From this, let us put $Y = \xi$ in (4.2) and use $g(A\xi, N) = 0$, we see that

$$\begin{aligned} 2g((X\alpha)\xi + \alpha\phi SX - S\phi SX, Z) = & g(\phi X, Z) - g(X, AN)g(A\xi, Z) \\ & - g(X, A\xi)g(JA\xi, Z) + g(\xi, A\xi)g(JAX, Z). \end{aligned} \tag{4.3}$$

Here, let us put $X = \xi$ in (4.3) and also use $g(\xi, AN) = 0$, we have

$$2(\xi\alpha)\eta(Z) = -g(\xi, A\xi)g(JA\xi, Z) + g(\xi, A\xi)g(JA\xi, Z) = 0.$$

From this we get $\xi\alpha = 0$. Then the derivative $Y\alpha$ in Section 3 becomes

$$Y\alpha = -2g(Y, AN)g(\xi, A\xi). \tag{4.4}$$

From this, together with (4.3), it follows that

$$\begin{aligned} & 2g(-2g(X, AN)g(\xi, A\xi)\xi + \alpha\phi SX - S\phi SX, Z) \\ & = g(\phi X, Z) - g(X, AN)g(A\xi, Z) \\ & \quad - g(X, A\xi)g(JA\xi, Z) + g(\xi, A\xi)g(JAX, Z). \end{aligned} \tag{4.5}$$

Then by putting $Z = \xi$ into (4.3), we have

$$\begin{aligned} -4g(X, AN)g(\xi, A\xi) = & - g(X, AN)g(A\xi, \xi) - g(X, A\xi)g(JA\xi, \xi) \\ & + g(\xi, A\xi)g(JAX, \xi) \\ = & - 2g(X, AN)g(A\xi, \xi). \end{aligned} \tag{4.6}$$

Since $g(A\xi, N) = 0$, (4.6) gives that

$$g(A\xi, \xi)g(AN, X) = 0.$$

Then we have $g(A\xi, \xi) = 0$ or $(AN)^T = 0$, where $(AN)^T$ denotes the tangential part of the vector AN.

Summing up above discussions, we conclude the following

Lemma 4.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m{\geq}3$, with Killing shape operator. Then the unit normal vector field N is singular, that is, N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. In above discussion, let us consider the first case $g(A\xi,\xi) = 0$. Then it implies that

$$0 = g(A\xi,\xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N).$$

If we insert $N = \cos tZ_1 + \sin tJZ_2$ for $Z_1, Z_2 \in V(A)$ into the above equation, we have $\cos^2 t - \sin^2 t = 0$. Then by Section 2, we have $t = \frac{\pi}{4}$, that is, $N = \frac{1}{\sqrt{2}}(X + JY)$ for some $X, Y \in V(A)$. So the unit normal N is $\mathfrak{A}$-isotropic.

Next we consider the case that $(AN)^T = 0$. Then $AN = (AN)^T + g(AN, N)N = g(AN, N)N$. So it follows that

$$N = A^2N = g(AN, N)AN = g^2(AN, N)N.$$

So $g(AN, N) = \pm 1$ gives that $AN = \pm N$.
That is, the unit normal N is $\mathfrak{A}$-principal. □

Then we are able to consider the classification of Killing shape operator S of M in Q^{m*} into two cases that the unit normal N is $\mathfrak{A}$-principal or N is $\mathfrak{A}$-isotropic. In Section 5 we will discuss a classification of real hypersurfaces in Q^{m*} with Killing shape operator and $\mathfrak{A}$-isotropic unit normal and in Section 6 a non-existence of Killing shape operator for hypersurfaces in Q^m when N is $\mathfrak{A}$-principal will be explained in detail.

5. Proof of Main Theorem with $\mathfrak{A}$-isotropic Unit Normal

In this section let us assume that the unit normal vector field N is $\mathfrak{A}$-isotropic. Then the normal vector field N can be written

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes the $(+1)$-eigenspace of the complex conjugation $A \in \mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2), AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2), \text{and } JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

From this, together with anti-commuting $AJ = -JA$, it follows that

$$g(\xi, A\xi) = g(JN, AJN) = 0,\ g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

Then (4.3) gives the following for any $X, Z \in T_zM$, $z \in M$

$$\begin{aligned}(5.1)\qquad & 2g(\alpha\phi SX - S\phi SX, Z)\\ &= g(\phi X, Z) - g(X, AN)g(A\xi, Z) - g(X, A\xi)g(JA\xi, Z)\\ &= g(\phi X, Z) - g(X, AN)g(A\xi, Z) + g(X, A\xi)g(AN, Z).\end{aligned}$$

Since $A\xi, AN \in T_zM$, $z \in M$, it implies

$$(5.2)\qquad 2(\alpha\phi SX - S\phi SX) = \phi X - g(X, AN)A\xi + g(X, A\xi)AN.$$

On the other hand, we have the following from Lemma 3.1 in Section 3 a Hopf real hypersurface M with $\mathfrak{A}$-isotropic unit normal N

$$(5.3)\qquad 2S\phi SX = \alpha(S\phi + \phi S)X - 2\phi X + 2g(X, AN)A\xi - 2g(X, A\xi)AN.$$

Then by virtue of (5.2) and (5.3), first we have

$$-S\phi SX = -2\alpha\phi SX + \frac{\alpha}{2}(\phi S + S\phi)X.$$

Then naturally it follows that

$$(5.4)\qquad -2S\phi SX = \alpha S\phi X - 3\alpha\phi SX.$$

We know that the tangent space T_zM, $z \in M$ is decomposed as follows:

$$T_zM = [\xi] \oplus [A\xi, AN] \oplus \mathcal{Q},$$

where $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^{\perp} = \mathrm{Span}[A\xi, AN]$.

Lemma 5.1. *Let M be a Hopf real hypersurface in the complex hyperbolic quadric Q^{m*}, $m\geq 3$, with $\mathfrak{A}$-isotropic unit normal vector field. Then*

$$SAN = 0, \quad \text{and} \quad SA\xi = 0.$$

Proof. Let us denote by $\mathcal{C} \ominus \mathcal{Q} = \mathcal{Q}^{\perp} = \mathrm{Span}[A\xi, AN]$. Since N is isotropic, $g(AN, N) = 0$ and $g(A\xi, \xi) = 0$. By differentiating $g(AN, N) = 0$ and using $(\bar\nabla_X A)Y = q(X)JAY$ and the equation of Weingarten, we know that

$$\begin{aligned}0 &= g(\bar\nabla_X(AN), N) + g(AN, \bar\nabla_X N)\\ &= g(q(X)JAN - ASX, N) - g(AN, SX)\\ &= -2g(ASX, N).\end{aligned}$$

Then $SAN = 0$. Moreover, by differentiating $g(A\xi, N) = 0$ and using $g(AN, N) = 0$, we have the following formula

$$\begin{aligned}0 &= g(\bar\nabla_X(A\xi), N) + g(A\xi, \bar\nabla_X N)\\ &= g(q(X)JA\xi + A(\phi SX + g(SX, \xi)N), N) - g(SA\xi, X)\\ &= -2g(SA\xi, X)\end{aligned}$$

for any $X \in T_zM$, $z \in M$, where in the third equality we have used $\phi AN = JAN = -AJN = A\xi$. Then it follows that

$$SA\xi = 0.$$

It completes the proof of our assertion. □

By Lemma 5.1 we know that the distribution $\mathcal{Q}^\perp$ for a Hopf real hypersurface M in Q^{m*} is invariant by the shape operator S, so the distribution $\mathcal{Q}$ is also S invariant. From this fact we may consider a principal curvature vector $X \in \mathcal{Q}$ such that $SX = \lambda X$, because the distribution $\mathcal{Q}$ can be diagonalized. Then (5.4) gives

$$S\phi X = \frac{3\alpha\lambda}{2\lambda + \alpha}\phi X. \tag{5.5}$$

In fact, if $2\lambda + \alpha = 0$, then $0 = \alpha\lambda = -2\lambda^2$. This implies $\alpha = \lambda = 0$. Then (5.3) gives $\phi X = 0$ for $X \in \mathcal{Q}$. This leads a contradiction.

For $X \in \mathcal{Q}$, we know that $g(X, AN) = g(X, A\xi) = 0$. So (5.3) gives the following

$$2S\phi SX = \alpha(S\phi + \phi S)X - 2\phi X. \tag{5.6}$$

Then we consider two cases for $X \in \mathcal{Q}$ or $X \in \mathcal{Q}^\perp$.

As a first, for $X \in \mathcal{Q}$ such that $SX = \lambda X$ the formula (5.6) gives

$$2\lambda S\phi X = \alpha S\phi X + (\alpha\lambda - 2)\phi X. \tag{5.7}$$

Then we can consider two cases as follows:

Case 1. $\alpha = 2\lambda$

Then $\alpha\lambda = 2$ gives $\alpha = 2$ and $\lambda = 1$ with multiplicities 1 and $2(m-2)$ respectively. This case implies that the shape operator commutes with the structure tensor, that is, $S\phi = \phi S$. Then by a result due to Suh [21], M is locally congruent to a horosphere.

Case 2. $\alpha \neq 2\lambda$

Then (5.7) gives that

$$S\phi X = \mu\phi X \quad \text{where} \quad \mu = \frac{\alpha\lambda - 2}{2\lambda - \alpha}. \tag{5.8}$$

Then (5.5) and (5.8) give

$$\frac{\alpha\lambda - 2}{2\lambda - \alpha}\phi X = \frac{3\alpha\lambda}{2\lambda + \alpha}\phi X.$$

From this, any principal curvatures λ and μ of the distribition $\mathcal{Q}$ satisfy the following quadratic equation

$$2\alpha\lambda^2 - 2(\alpha^2 - 1)\lambda + \alpha = 0. \tag{5.9}$$

The solutions become the following constant principal curvatures given by

$$\lambda_1, \lambda_2 = \frac{(\alpha^2-1)\pm\sqrt{(\alpha^2-1)^2-2\alpha^2}}{2\alpha}, \tag{5.10}$$

because the Reeb function α is constant for $\mathfrak{A}$-isotropic unit normal N. Here we note that the Reeb function α can not vanishing. If the function α identically vanishes, then (5.9) gives $\lambda = 0$. From this, together with (5.7), we have $\phi X = 0$, which implies a contradiction.

From this, together with Lemma 5.1, the expression of the shape operator becomes the following

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_i & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_i \end{bmatrix}.$$

Here the principal curvatures λ_1 and λ_2 are given by (5.10) with multiplicities $p + q = m - 2$ where $\dim T_{\lambda_1} = p$ and $\dim T_{\lambda_2} = q$, respectively. Moreover, the principal curvatures μ_i $(i = 1, 2)$ with respect to the eigenspaces $T_{\mu_i} = \phi(T_{\lambda_i})$ satisfy $\mu_i = \frac{\alpha\lambda_i - 2}{2\lambda_i - \alpha}$, respectively.

Summing up the above discussions, we give the following

Theorem 5.2. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{m*} with $\mathfrak{A}$-isotropic unit normal vector field. Then M is locally congruent to a horosphere or M has 4 distinct constant principal curvatures given by*

$$\alpha \neq 0,\ \beta = \gamma = 0,\ \lambda_1 = \frac{(\alpha^2-1)+\sqrt{(\alpha^2-1)^2-2\alpha^2}}{2\alpha},\ \textit{and}$$

$$\lambda_2 = \frac{(\alpha^2-1)-\sqrt{(\alpha^2-1)^2-2\alpha^2}}{2\alpha}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi],\ T_\beta = [AN],\ T_\gamma = [A\xi],\ \phi(T_{\lambda_i}) = T_{\mu_i},$$

and

$$\dim\ (T_{\lambda_1} + T_{\lambda_2}) = \dim\ (T_{\mu_1} + T_{\mu_2}) = m - 2.$$

6. Proof of Main Theorem with $\mathfrak{A}$-principal

In this section let us consider a real hypersurface M in the complex hyperbolic quadric Q^{m*} with Killing shape operator for the case that the unit normal N is $\mathfrak{A}$-principal. In this case the Killing shape operator (4.3) gives that

$$2g(\{\alpha\phi SX - S\phi SX\}, Z) = g(\phi X, Z) - g(\phi AX, Z),$$

where we have used $g(\xi, A\xi) = -1$ and $JAX = \phi AX + \eta(AX)N$. Then it follows that

$$2(\alpha\phi SX - S\phi SX) = \phi X - \phi AX. \tag{6.1}$$

Since the unit normal vector field N is $\mathfrak{A}$-principal, $A\xi = -\xi$. Then differentiating this and using Gauss equation give

$$\nabla_X(A\xi) = \bar{\nabla}_X(A\xi) - g(SX, A\xi)N = -q(X)N + A\phi SX + 2\alpha\eta(X)N, \tag{6.2}$$

where q denotes a certain 1-form defined on M as in the introduction. From this, together with $\nabla_X(A\xi) = -\nabla_X\xi = -\phi SX$, it follows that

$$-\phi SX = -q(X)N + A\phi SX + 2\alpha\eta(X)N. \tag{6.3}$$

By taking the inner product of (6.3) with the unit normal N, we have $q(X) = 2\alpha\eta(X)$. From this, we know

$$A\phi SX = -\phi SX. \tag{6.4}$$

So when we consider $SX = \lambda X$ for $X \in \mathcal{C}$ and $\lambda \neq 0$, (6.4) becomes $A\phi X = -\phi X$, where the distribution $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ. When the principal curvature $\lambda = 0$, $SX = 0$ in (6.1) gives that

$$\phi AX = \phi X.$$

Accordingly, for any cases we know that

$$AX = X - 2\eta(X)\xi.$$

Then we have

$$\begin{aligned} \mathrm{Tr}A &= g(AN, N) + \sum_{i=1}^{2m-1} g(Ae_i, e_i) \\ &= 1 + \sum_{i=1}^{2m-1} g(e_i - 2\eta(e_i)\xi, e_i) \\ &= 2(m-1). \end{aligned} \tag{6.5}$$

But $\mathrm{Tr}A = 0$, because $T_zQ^m = V(A) \oplus JV(A)$, where $V(A) = \{X \in T_zQ^{m*} | \, AX = X\}$ and $JV(A) = \{X \in T_zQ^m | \, AX = -X\}$. This leads to a contradiction, which implies another theorem as follows:

Theorem 6.1. *There do not exist any real hypersurface in the complex hyperbolic quadric Q^{m*} with Killing shape operator if the unit normal vector field is $\mathfrak{A}$-principal.*

Summing up all of discussions including Sections 4 and 5, by Lemma 4.1, Theorems 5.2 and 6.1, we give a complete proof of our Main Theorem 2 in the introduction.

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