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Generalised Ricci Solitons on Sasakian Manifolds

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Abstract. In this paper, we show that a Sasakian manifold which also satisfies the generalised gradient Ricci soliton equation, satisfying some conditions, is necessarily Einstein.

1. Introduction and Main Results

By R and Ric we denote respectively the Riemannian curvature tensor and the Ricci tensor of a Riemannian manifold (M, g). Then R and Ric are defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\mathrm{Ric}(X,Y) = g(R(X,e_i)e_i, Y), \tag{1.2}$$

where ∇ is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. Given a smooth function f on M, the gradient of f is defined by:

$$g(\operatorname{grad} f, X) = X(f), \tag{1.3}$$

the Hessian of f is defined by:

$$(\operatorname{Hess} f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y), \tag{1.4}$$

where $X, Y \in \Gamma(TM)$. For $X \in \Gamma(TM)$, we define $X^\flat \in \Gamma(T^*M)$ by:

$$X^\flat(Y) = g(X,Y). \tag{1.5}$$

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(For more details, see for example [11]).
- The generalised Ricci soliton equation in Riemannian manifold (M, g) is defined by (see [10]):

$$\mathcal{L}_X g = -2c_1 X^\flat \odot X^\flat + 2c_2 \operatorname{Ric} + 2\lambda g, \tag{1.6}$$

where $X \in \Gamma(TM)$, $\mathcal{L}_X g$ is the Lie-derivative of g along X given by:

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y), \tag{1.7}$$

for all $Y, Z \in \Gamma(TM)$, and $c_1, c_2, \lambda \in \mathbb{R}$. Equation (1.6), is a generalization of:

- Killing's equation ($c_1 = c_2 = \lambda = 0$);
- Equation for homotheties ($c_1 = c_2 = 0$);
- Ricci soliton ($c_1 = 0$, $c_2 = -1$);
- Cases of Einstein-Weyl ($c_1 = 1$, $c_2 = \frac{-1}{n-2}$);
- Metric projective structures with skew-symmetric Ricci tensor in projective class ($c_1 = 1$, $c_2 = \frac{-1}{n-1}$, $\lambda = 0$);
- Vacuum near-horzion geometry equation ($c_1 = 1$, $c_2 = \frac{1}{2}$).

(For more details, see [5], [8], [9], [10]). Equation (1.6), is also a generalization of Einstein manifolds (see [1]). Note that, if $X = \operatorname{grad} f$, where $f \in C^\infty(M)$, the generalised Ricci soliton equation is given by:

$$\operatorname{Hess} f = -c_1 df \odot df + c_2 \operatorname{Ric} + \lambda g. \tag{1.8}$$

- An $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist a $(1,1)$-tensor field φ on M, a vector field $\xi \in \Gamma(TM)$ and a 1-form $\eta \in \Gamma(T^*M)$, such that:

$$\eta(\xi) = 1, \quad \varphi^2 X = -X + \eta(X)\xi \tag{1.9}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.10}$$

for all $X, Y \in \Gamma(TM)$. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and $\eta(X) = g(\xi, X)$. It can be proved that an almost contact metric manifold is Sasakian if and only if (see [2], [3], [12]):

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.11}$$

for any $X, Y \in \Gamma(TM)$. For a Sasakian manifold the following equations hold:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{1.12}$$

$$\nabla_X \xi = -\varphi X, \quad (\nabla_X \eta)Y = -g(\varphi X, Y). \tag{1.13}$$

The main result of this paper is the following:

Theorem 1.1. *Suppose $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold of dimension $(2n+1)$, and satisfies the generalised Ricci soliton equation (1.8) with $c_1(\lambda + 2c_2 n) \neq -1$, then f is a constant function. Furthermore, if $c_2 \neq 0$, then (M, g) is an Einstein manifold.*

From Theorem 1.1, we get the following:

- Suppose (M, g) is a Sasakian manifold and satisfies the gradient Ricci soliton equation (i.e. $\operatorname{Hess} f = -\operatorname{Ric} + \lambda g$), then f is a constant function and (M, g) is an Einstein manifold (this result is obtained by P. Nurowski and M. Randall [10]).
- In a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, there is no non-constant smooth function f, such that $\operatorname{Hess} f = \lambda g$, for some constant λ.

2. Proof of the Result

For the proof of Theorem 1.1, we need the following lemmas.

Lemma 2.1. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then:*

$$\big(\mathcal{L}_\xi(\mathcal{L}_X g)\big)(Y, \xi) = g(X, Y) + g(\nabla_\xi \nabla_\xi X, Y) + Y g(\nabla_\xi X, \xi),$$

where $X, Y \in \Gamma(TM)$, with Y is orthogonal to ξ.

Proof. First note that:

$$\begin{aligned} \big(\mathcal{L}_\xi(\mathcal{L}_X g)\big)(Y, \xi) &= \xi\big((\mathcal{L}_X g)(Y, \xi)\big) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y, \xi) \\ &\quad -(\mathcal{L}_X g)(Y, \mathcal{L}_\xi \xi), \end{aligned} \tag{2.1}$$

since $\mathcal{L}_\xi Y = [\xi, Y]$, $\mathcal{L}_\xi \xi = [\xi, \xi] = 0$, by equations (1.7) and (2.1), we have:

$$\begin{aligned} \big(\mathcal{L}_\xi(\mathcal{L}_X g)\big)(Y, \xi) &= \xi g(\nabla_Y X, \xi) + \xi g(\nabla_\xi X, Y) - g(\nabla_{[\xi, Y]} X, \xi) \\ &\quad - g(\nabla_\xi X, [\xi, Y]) \\ &= g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_Y X, \nabla_\xi \xi) + g(\nabla_\xi \nabla_\xi X, Y) \\ &\quad + g(\nabla_\xi X, \nabla_\xi Y) - g(\nabla_{[\xi, Y]} X, \xi) - g(\nabla_\xi X, \nabla_\xi Y) \\ &\quad + g(\nabla_\xi X, \nabla_Y \xi), \end{aligned}$$

from equation (1.13), we get $\nabla_\xi \xi = \varphi \xi = 0$, so that:

$$\begin{aligned} \big(\mathcal{L}_\xi(\mathcal{L}_X g)\big)(Y, \xi) &= g(\nabla_\xi \nabla_Y X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) - g(\nabla_{[\xi, Y]} X, \xi) \\ &\quad + Y g(\nabla_\xi X, \xi) - g(\nabla_Y \nabla_\xi X, \xi), \end{aligned} \tag{2.2}$$

by the definition of the Riemannian curvature tensor (1.1), and (2.2), we obtain:

$$\big(\mathcal{L}_\xi(\mathcal{L}_X g)\big)(Y, \xi) = g(R(\xi, Y)X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) + Y g(\nabla_\xi X, \xi), \tag{2.3}$$

from equation (1.12), with $g(Y,\xi)=0$, we have:

$$g(R(\xi,Y)X,\xi)=g(R(Y,\xi)\xi,X)=g(X,Y). \tag{2.4}$$

the Lemma follows from equations (2.3) and (2.4). □

Lemma 2.2. *Let (M,g) be a Riemannian manifold, and let $f\in C^\infty(M)$. Then:*

$$\big(\mathcal{L}_\xi(df\odot df)\big)(Y,\xi) \;=\; Y(\xi(f))\xi(f)+Y(f)\xi(\xi(f)),$$

where $\xi,Y\in\Gamma(TM)$.

Proof. We compute:

$$\begin{aligned}\big(\mathcal{L}_\xi(df\odot df)\big)(Y,\xi) &= \xi\big(Y(f)\xi(f)\big)-[\xi,Y](f)\xi(f)-Y(f)[\xi,\xi](f)\\ &= \xi(Y(f))\xi(f)+Y(f)\xi(\xi(f))-[\xi,Y](f)\xi(f),\end{aligned}$$

since $[\xi,Y](f)=\xi(Y(f))-Y(\xi(f))$, we get:

$$\begin{aligned}\big(\mathcal{L}_\xi(df\odot df)\big)(Y,\xi) &= [\xi,Y](f)\xi(f)+Y(\xi(f))\xi(f)+Y(f)\xi(\xi(f))\\ &\quad -[\xi,Y](f)\xi(f)\\ &= Y(\xi(f))\xi(f)+Y(f)\xi(\xi(f)).\end{aligned}$$

The proof is completed. □

Lemma 2.3. *Let (M,φ,ξ,η,g) be a Sasakian manifold of dimension $(2n+1)$, and satisfies the generalised Ricci soliton equation* (1.8)*. Then:*

$$\nabla_\xi \operatorname{grad} f=(\lambda+2c_2n)\xi-c_1\xi(f)\operatorname{grad} f.$$

Proof. Let $Y\in\Gamma(TM)$, from the definition of Ricci curvature (1.2), and the curvature condition (1.12), we have:

$$\begin{aligned}\operatorname{Ric}(\xi,Y) &= g(R(\xi,e_i)e_i,Y)\\ &= g(R(e_i,Y)\xi,e_i)\\ &= \eta(Y)g(e_i,e_i)-\eta(e_i)g(X,e_i)\\ &= (2n+1)\eta(Y)-\eta(Y)\\ &= 2n\eta(Y)=2ng(\xi,Y),\end{aligned}$$

where $\{e_i\}$ is an orthonormal frame on M, implies that:

$$\begin{aligned}\lambda g(\xi,Y)+c_2\operatorname{Ric}(\xi,Y) &= \lambda g(\xi,Y)+2c_2ng(\xi,Y)\\ &= (\lambda+2c_2n)g(\xi,Y),\end{aligned} \tag{2.5}$$

from equations (1.8) and (2.5), we obtain:

$$\begin{aligned}(\operatorname{Hess} f)(\xi,Y) &= -c_1\xi(f)Y(f)+(\lambda+2c_2n)g(\xi,Y)\\ &= -c_1\xi(f)g(\operatorname{grad} f,Y)+(\lambda+2c_2n)g(\xi,Y),\end{aligned} \tag{2.6}$$

the Lemma follows from equation (2.6) and the definition of the Hessian (1.3). □

Proof of Theorem 1.1. Let $Y \in \Gamma(TM)$, such that $g(\xi, Y) = 0$, from Lemma 2.1, with $X = \operatorname{grad} f$, we have:

$$2\big(\mathcal{L}_\xi(\operatorname{Hess} f)\big)(Y,\xi) = Y(f) + g(\nabla_\xi\nabla_\xi \operatorname{grad} f, Y) + Yg(\nabla_\xi \operatorname{grad} f, \xi), \tag{2.7}$$

by Lemma 2.3, and equation (2.7), we get:

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f)\big)(Y,\xi) &= Y(f) + (\lambda + 2c_2 n)g(\nabla_\xi\xi, Y) - c_1 g(\nabla_\xi(\xi(f)\operatorname{grad} f), Y) \\ &\quad + (\lambda + 2c_2 n)Yg(\xi,\xi) - c_1 Y(\xi(f)^2), \end{aligned} \tag{2.8}$$

since $\nabla_\xi\xi = 0$ and $g(\xi,\xi) = 1$, from equation (2.8), we obtain:

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f)\big)(Y,\xi) &= Y(f) - c_1\xi(\xi(f))Y(f) - c_1\xi(f)g(\nabla_\xi \operatorname{grad} f, Y) \\ &\quad - 2c_1\xi(f)Y(\xi(f)), \end{aligned} \tag{2.9}$$

from Lemma 2.3, equation (2.9), and since $g(\xi, Y) = 0$, we have:

$$\begin{aligned} 2\big(\mathcal{L}_\xi(\operatorname{Hess} f)\big)(Y,\xi) &= Y(f) - c_1\xi(\xi(f))Y(f) + c_1^2\xi(f)^2 Y(f) \\ &\quad - 2c_1\xi(f)Y(\xi(f)). \end{aligned} \tag{2.10}$$

Note that, from equation (1.13), we have $\mathcal{L}_\xi g = 0$ (i.e. ξ is a Killing vector field), implies that $\mathcal{L}_\xi \operatorname{Ric} = 0$, taking the Lie derivative to the generalised Ricci soliton equation (1.8) yields:

$$2\big(\mathcal{L}_\xi(\operatorname{Hess} f)\big)(Y,\xi) = -2c_1\big(\mathcal{L}_\xi(df \odot df)\big)(Y,\xi), \tag{2.11}$$

thus, from equations (2.10), (2.11) and Lemma 2.2, we have:

$$\begin{aligned} &Y(f) - c_1\xi(\xi(f))Y(f) + c_1^2\xi(f)^2Y(f) - 2c_1\xi(f)Y(\xi(f)) \\ &= -2c_1 Y(\xi(f))\xi(f) - 2c_1 Y(f)\xi(\xi(f)), \end{aligned} \tag{2.12}$$

is equivalent to:

$$Y(f)\big[1 + c_1\xi(\xi(f)) + c_1^2\xi(f)^2\big] = 0, \tag{2.13}$$

according to Lemma 2.3, we have:

$$\begin{aligned} c_1\xi(\xi(f)) &= c_1\xi g(\xi, \operatorname{grad} f) \\ &= c_1 g(\xi, \nabla_\xi \operatorname{grad} f) \\ &= c_1(\lambda + 2c_2 n) - c_1^2\xi(f)^2, \end{aligned} \tag{2.14}$$

by equations (2.13) and (2.14), we obtain:

$$Y(f)\big[1 + c_1(\lambda + 2c_2 n)\big] = 0, \tag{2.15}$$

since $c_1(\lambda + 2c_2 n) \neq -1$, we fined that $Y(f) = 0$, i.e., grad f is parallel to ξ. Hence grad $f = 0$ as $D = \ker \eta$ is nowhere integrable, i.e., f is a constant function. □

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References

[1] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Nots in Mathematics **509**, Springer, 1976, pp 17–35.

[3] C. P. Boyer, K. Galicki, *Sasakian geometry*, Oxford Univ. Press, Oxford, 2007.

[4] H. D. Cao, *Recent progress on Ricci solitons*, Adv. Lect. Math. (ALM), **11**(2010), 1–38.

[5] P. T. Chrusciel, H. S. Reall and P. Tod, *On non-existence of static vacuum black holes with degenerate components of the event horizon*, Classical Quantum Gravity, **23**(2006), 549–554.

[6] C. He and M. Zhu, *Ricci solitons on Sasakian manifolds*, http://arXiv:1109.4407v2 [math.DG], 26 Sep., 2011.

[7] T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl., **3**(1993), 301–307.

[8] J. Jezierski, *On the existence of Kundt's metrics and degenerate (or extremal) Killing horizons*, Classical Quantum Gravity, **26**(2009), 035011, 11 pp.

[9] H. K. Kunduri and J. Lucietti, *Classification of near-horizon geometries of extremal black holes*, Living Rev. Relativity, **16(8)**(2013).

[10] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J Geom. Anal., **26**(2016), 1280–1345.

[11] O'Neil, *Semi-Riemannian geometry*, Academic Press, New York, 1983.

[12] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics **3**, World Scientific Publishing Co., 1984.