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Some Characterizations of Catenary Rotation Surfaces

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Abstract. We study the positive C^1 function $z = f(x, y)$ defined on the plane $\mathbb{R}^2$. For a rectangular domain $[a, b] \times [c, d] \subset \mathbb{R}^2$, we consider the volume V and the surface area S of the graph of $z = f(x, y)$ over the domain. We also denote by $(\bar{x}_V,\ \bar{y}_V,\ \bar{z}_V)$ and $(\bar{x}_S,\ \bar{y}_S,\ \bar{z}_S)$ the geometric centroid of the volume under the graph of $z = f(x,\ y)$ and the centroid of the graph itself defined on the rectangular domain, respectively.

In this paper, first we show that among nonconstant C^2 functions with isolated singularities, $S = kV,\ k \in \mathbb{R}$ characterizes the family of catenary rotation surfaces $f(x,\ y) = k\cosh(r/k), \quad r = |(x,\ y)|$. Next, we show that one of $(\bar{x}_S,\ \bar{y}_S) = (\bar{x}_V,\ \bar{y}_V)$, $(\bar{x}_S,\ \bar{z}_S) = (\bar{x}_V,\ 2\bar{z}_V)$ and $(\bar{y}_S,\ \bar{z}_S) = (\bar{y}_V,\ 2\bar{z}_V)$ characterizes the family of catenary rotation surfaces among nonconstant C^2 functions with isolated singularities.

1. Introduction

A well-known property of the catenary curve $y = k\cosh((x-c)/k),\ k > 0$

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is that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. That is, for every interval $[a,\ b]$, the area $A(a,b)$ under the curve always equals to $kL(a,\ b)$, where $L(a,\ b)$ denotes the arc length of the curve itself. This property characterizes the family of catenaries $y = k\cosh((x-c)/k)$ among nonconstant C^2 functions as follows ([17]):

Proposition 1.1. *For a nonconstant positive C^2 function $y = f(x)$ defined on an interval I, the following are equivalent.*

(1) *There exists a positive constant k such that for every interval $[a,b] \subset I$, $A(a,\ b) = kL(a,\ b)$.*

(2) *The function $f(x)$ satisfies $f(x) = k\sqrt{1+f'(x)^2}$, where k is a positive constant.*

(3) *For some $k > 0$ and $c \in \mathbb{R}$,*

$$f(x) = k\cosh\left(\frac{x-c}{k}\right).$$

For a positive C^1 function $y = f(x)$ defined on an interval I and an interval $[a,b] \subset I$, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a,b),\ \bar{y}_A(a,b))$ and $(\bar{x}_L,\ \bar{y}_L) = (\bar{x}_L(a,b),\ \bar{y}_L(a,b))$ the geometric centroid of the area under the graph of $y = f(x)$ and the centroid of the graph itself defined on this interval, respectively. Then, for a catenary curve $f(x) = k\cosh((x-c)/k)$ we always have $\bar{x}_L(a,\ b) = \bar{x}_A(a,\ b)$ and $\bar{y}_L(a,\ b) = 2\bar{y}_A(a,\ b)$ ([17]). Conversely, one of theses properties characterizes the catenary curves as follows ([13]).

Proposition 1.2. *For a nonconstant positive C^2 function $y = f(x)$ defined on an interval I, the following are equivalent.*

(1) *For every interval $[a,\ b] \subset I$, $\bar{x}_L(a,\ b) = \bar{x}_A(a,\ b)$.*

(2) *For every interval $[a,\ b] \subset I$, $\bar{y}_L(a,\ b) = 2\bar{y}_A(a,\ b)$.*

(3) *For some $k > 0$ and $c \in \mathbb{R}$,*

$$f(x) = k\cosh\left(\frac{x-c}{k}\right).$$

Some higher dimensional generalizations of Proposition 1.1, especially for rotation hypersurfaces in $\mathbb{R}^{n+1}$, were established in [1].

In this paper, we study a positive C^1 function $z = f(x,\ y)$ defined on $\mathbb{R}^2$. For a rectangular domain $[a,\ b] \times [c,\ d] \subset \mathbb{R}^2$, we consider the volume $V = V(a,\ b,\ c,\ d)$

and the surface area $S = S(a,\ b,\ c,\ d)$ of the graph of $z = f(x,\ y)$ over the domain. We also denote by

$$(\bar{x}_V,\ \bar{y}_V,\ \bar{z}_V) = (\bar{x}_V(a,\ b,\ c,\ d),\ \bar{y}_V(a,\ b,\ c,\ d),\ \bar{z}_V(a,\ b,\ c,\ d))$$

and

$$(\bar{x}_S,\ \bar{y}_S,\ \bar{z}_S) = (\bar{x}_S(a,\ b,\ c,\ d),\ \bar{y}_S(a,\ b,\ c,\ d),\ \bar{z}_S(a,\ b,\ c,\ d))$$

the geometric centroid of the volume under the graph of $z = f(x,\ y)$ and the centroid of the graph itself defined on the rectangular domain, respectively.

As a result, first of all we prove the following characterization theorem in Section 3.

Theorem 1.3. *Suppose that $z = f(x,\ y)$ denotes a nonconstant positive C^2 function defined on $\mathbb{R}^2$ with isolated singularities. Then, the following are equivalent.*

(1) *There exists a positive constant k such that for every rectangular domain, $V = kS$.*

(2) *By a Euclidean motion of $\mathbb{R}^2$ if necessary, we have*

$$f(x,\ y) = k \cosh\left(\frac{r}{k}\right), \quad r = \sqrt{x^2 + y^2}.$$

Next, in Section 4 we prove the following characterization theorem.

Theorem 1.4. *Suppose that $z = f(x,y)$ denotes a nonconstant positive C^2 function defined on $\mathbb{R}^2$ with isolated singularities. Then, the following are equivalent.*

(1) *For every rectangular domain, we have*

$$\bar{x}_S = \bar{x}_V, \quad \bar{y}_S = \bar{y}_V.$$

(2) *For every rectangular domain, we have*

$$\bar{x}_S = \bar{x}_V, \quad \bar{z}_S = 2\bar{z}_V.$$

(3) *For every rectangular domain, we have*

$$\bar{y}_S = \bar{y}_V, \quad \bar{z}_S = 2\bar{z}_V.$$

(4) *By a Euclidean motion of $\mathbb{R}^2$ if necessary, we have*

$$f(x,y) = k \cosh\left(\frac{r}{k}\right), \quad r = \sqrt{x^2 + y^2}.$$

In order to find the centroid of polygons, see [3]. For the perimeter centroid of a polygon, we refer to [2]. In [15], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [16]. The relationships between various centroids of a quadrangle were given in [6, 12]

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([18]). Some characterizations of parabolas using these properties were given in [5, 9, 11]. Furthermore, Archimedes also proved the volume properties of the region surrounded by a paraboloid of rotation and a plane ([18]). For characterizations of spheres, ellipsoids, elliptic paraboloid or elliptic hyperboloids with respect to these volume properties, we refer to [4, 7, 8, 14].

2. Some Lemmas

In this section, we prove some lemmas which are useful in the proof of our theorems.

We consider a positive C^1 function $z = f(x,y)$ defined on $\mathbb{R}^2$. For a rectangular domain $[a,\ b] \times [c,\ d] \subset \mathbb{R}^2$, the volume $V = V(a,\ b,\ c,\ d)$ and the surface area $S = S(a,\ b,\ c,\ d)$ of the graph of $z = f(x,\ y)$ over the domain are respectively given by

(2.1)
$$V(a,\ b,\ c,\ d) = \int_a^b \int_c^d f(x,\ y)dxdy, \quad S(a,\ b,\ c,\ d) = \int_a^b \int_c^d w(x,\ y)dxdy,$$

where $w(x,\ y)$ is a function defined by

$$w(x,\ y) = \sqrt{1 + |\nabla f|^2}, \quad \nabla f = (f_x,\ f_y). \tag{2.2}$$

The centroids over the rectangular domain $[a,\ b] \times [c,\ d]$ are also respectively given by

(2.3)
$$(\bar{x}_V,\ \bar{y}_V,\ \bar{z}_V) = \frac{1}{V}\left(\int_a^b \int_c^d xf(x,\ y)dxdy,\ \int_a^b \int_c^d yf(x,\ y)dxdy,\ \frac{1}{2}\int_a^b \int_c^d f(x,\ y)^2 dxdy\right)$$

and

(2.4)
$$(\bar{x}_S,\ \bar{y}_S,\ \bar{z}_S) = \frac{1}{S}\left(\int_a^b \int_c^d xw(x,y)dxdy,\ \int_a^b \int_c^d yw(x,y)dxdy,\ \int_a^b \int_c^d f(x,y)w(x,y)dxdy\right).$$

First, suppose that $\bar{x}_S = \bar{x}_V$. Then for all $a,\ b,\ c,\ d\ \in \mathbb{R}$ with $a < b$ and $c < d$, we have

(2.5)
$$\int_a^b \int_c^d f(x,y)dxdy \int_a^b \int_c^d xw(x,y)dxdy = \int_a^b \int_c^d xf(x,y)dxdy \int_a^b \int_c^d w(x,y)dxdy.$$

Note that (2.5) is valid for all $a, b, c, d \in \mathbb{R}$. We differentiate (2.5) with respect to b and a repeatedly. Then the fundamental theorem of calculus gives

$$\begin{aligned}&\int_c^d f(b,y)dy \int_c^d aw(a,y)dy + \int_c^d f(a,y)dy \int_c^d bw(b,y)dy \\ &= \int_c^d bf(b,y)dy \int_c^d w(a,y)dy + \int_c^d af(a,y)dy \int_c^d w(b,y)dy.\end{aligned} \tag{2.6}$$

We again differentiate (2.6) with respect to d and c repeatedly. Then, we get

$$\begin{aligned}&f(b,d)aw(a,c) + aw(a,d)f(b,c) + f(a,d)bw(b,c) + bw(b,d)f(a,c) \\ &= bf(b,d)w(a,c) + bw(a,d)f(b,c) + af(a,d)w(b,c) + aw(b,d)f(a,c),\end{aligned} \tag{2.7}$$

from which we obtain

$$f(b,d)w(a,c) + f(b,c)w(a,d) = f(a,d)w(b,c) + f(a,c)w(b,d). \tag{2.8}$$

Let us define $g(a,b,c,d)$ as follows:

$$g(a,b,c,d) = f(a,c)w(b,d) - f(b,d)w(a,c). \tag{2.9}$$

Then, it follows from (2.8) that

$$g(a,b,c,d) = g(b,a,c,d). \tag{2.10}$$

Next, suppose that $\bar{y}_S = \bar{y}_V$. Then for all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have

$$\int_a^b \int_c^d f(x,y)dxdy \int_a^b \int_c^d yw(x,y)dxdy = \int_a^b \int_c^d yf(x,y)dxdy \int_a^b \int_c^d w(x,y)dxdy. \tag{2.11}$$

Note that (2.11) is valid for all $a, b, c, d \in \mathbb{R}$. If we differentiate (2.11) with respect to b and a repeatedly, then we get

$$\begin{aligned}&\int_c^d f(b,y)dy \int_c^d yw(a,y)dy + \int_c^d f(a,y)dy \int_c^d yw(b,y)dy \\ &= \int_c^d yf(b,y)dy \int_c^d w(a,y)dy + \int_c^d yf(a,y)dy \int_c^d w(b,y)dy.\end{aligned} \tag{2.12}$$

By differentiating (2.12) with respect to d and then with respect to c, as before we get

$$f(b,d)w(a,c) + f(a,d)w(b,c) = f(b,c)w(a,d) + f(a,c)w(b,d), \tag{2.13}$$

which implies

$$g(a,b,c,d) = g(a,b,d,c). \tag{2.14}$$

Finally, suppose that $\bar{z}_S = 2\bar{z}_V$. Then for all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have

$$\int_a^b \int_c^d f(x,y)dxdy \int_a^b \int_c^d f(x,y)w(x,y)dxdy = \int_a^b \int_c^d f(x,y)^2 dxdy \int_a^b \int_c^d w(x,y)dxdy. \tag{2.15}$$

Note that (2.15) is valid for all $a, b, c, d \in \mathbb{R}$. We differentiate (2.15) with respect to b and a repeatedly. Then we get

$$\begin{aligned} &\int_c^d f(b,y)dy \int_c^d f(a,y)w(a,y)dy + \int_c^d f(a,y)dy \int_c^d f(b,y)w(b,y)dy \\ &= \int_c^d f(b,y)^2 dy \int_c^d w(a,y)dy + \int_c^d f(a,y)^2 dy \int_c^d w(b,y)dy. \end{aligned} \tag{2.16}$$

By differentiating (2.16) with respect to d and then with respect to c, we have

$$\begin{aligned} &f(b,d)f(a,c)w(a,c) + f(b,c)f(a,d)w(a,d) + f(a,d)f(b,c)w(b,c) + f(a,c)f(b,d)w(b,d) \\ &= f(b,d)^2 w(a,c) + f(b,c)^2 w(a,d) + f(a,d)^2 w(b,c) + f(a,c)^2 w(b,d). \end{aligned} \tag{2.17}$$

Together with (2.9), this shows that

$$\{f(b,d) - f(a,c)\}g(a,b,c,d) = \{f(a,d) - f(b,c)\}g(a,b,d,c). \tag{2.18}$$

Summarizing the above discussions, we obtain

Lemma 2.1. *Suppose that $z = f(x,y)$ denotes a positive C^2 function defined on $\mathbb{R}^{\not\models}$. Then, we have*

(1) *If the function $z = f(x,y)$ satisfies $\bar{x}_S = \bar{x}_V$ for every rectangular domain, then*
$$g(a,b,c,d) = g(b,a,c,d).$$

(2) *If the function $z = f(x,y)$ satisfies $\bar{y}_S = \bar{y}_V$ for every rectangular domain, then*
$$g(a,b,c,d) = g(a,b,d,c).$$

(3) *If the function $z = f(x,y)$ satisfies $\bar{z}_S = 2\bar{z}_V$ for every rectangular domain, then*
$$\{f(b,d) - f(a,c)\}g(a,b,c,d) = \{f(a,d) - f(b,c)\}g(a,b,d,c).$$

3. Theorem 1.3.

In this section, we prove Theorem 1.3 stated in Section 1.

Suppose that a positive C^2 function $z = f(x,y)$ defined on $\mathbb{R}^2$ satisfies $V = kS$ for some $k > 0$. Then for all $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, we have

$$\int_a^b \int_c^d f(x,y)dxdy = k \int_a^b \int_c^d w(x,y)dxdy, \tag{3.1}$$

where $w(x,y)$ is a function defined by

$$w(x,y) = \sqrt{1+|\nabla f|^2}. \tag{3.2}$$

Note that (2.1) is valid for all $a,b,c,d \in \mathbb{R}$. By differentiating (2.1) with respect to b and d repeatedly, the fundamental theorem of calculus gives $f(b,d) = kw(b,d)$ for all $a,b \in \mathbb{R}$. That is, we get a partial differential equation

$$f(x,y) = k\sqrt{1+|\nabla f|^2}. \tag{3.3}$$

This shows that the function $z = f(x,y)$ satisfies

$$|\nabla f(x,y)| = \phi(f(x,y)), \quad \phi(t) = \frac{\sqrt{t^2-k^2}}{k}. \tag{3.4}$$

Now, we need the following (the main theorem of [10]):

Proposition 3.1. *Suppose that a C^2 function $f : R^n \to R$ with isolated critical points satisfies $|\nabla f(x,y)| = \phi(f(x,y))$, where ϕ is a function. Then f is a function of either a distance function $r = ||p-o||$ from a fixed point o or a linear function. That is, the level sets are either concentric hyperspheres or parallel hyperplanes.*

It follows from Proposition 3.1 that by a Euclidean motion if necessary, the function $f(x,y)$ is either a radial function $f(x,y) = h(r), r = |(x,y)|$ or a function $f(x,y) = h(x)$ of only x.

We consider two cases as follows.

Case 1. $f(x,y) = h(r)$**.** In this case, we have from (3.4)

$$h'(r) = \pm\frac{\sqrt{h(r)^2-k^2}}{k}, \tag{3.5}$$

which shows that

$$f(x,y) = h(r) = k\cosh\left(\frac{r-c}{k}\right). \tag{3.6}$$

Since the function $f(x,y)$ has isolated singularities and $|\nabla f(x,y)| = |h'(r)| = |\sinh((r-c)/k)|$ vanishes where $r(x,y) = c$, the constant c must be nonpositive. But, if c is negative, the function $f(x,y)$ cannot be differentiable at the origin. This implies that $c = 0$, and hence

$$f(x,y) = k\cosh\left(\frac{r}{k}\right), \quad r = \sqrt{x^2+y^2}. \tag{3.7}$$

Case 2. $f(x,y) = h(x)$**.** In this case, we have from (3.4)

$$h'(x) = \pm\frac{\sqrt{h(x)^2-k^2}}{k}, \tag{3.8}$$

which shows that

$$f(x,y) = h(x) = k\cosh\left(\frac{x-c}{k}\right). \tag{3.9}$$

Since $|\nabla f(x,y)|$ vanishes on the line $x = c$, this case is impossible.

Summarizing the above two cases, we see that (1) $\Rightarrow$ (2).

Conversely, it is straightforward to show that (2) $\Rightarrow$ (1). This completes the proof of Theorem 1.3.

4. Theorem 1.4.

In this section, we prove Theorem 1.4 stated in Section 1.

First, suppose that a positive C^2 function $z = f(x,y)$ defined on $\mathbb{R}^2$ satisfies $\bar{x}_S = \bar{x}_V$ and $\bar{y}_S = \bar{y}_V$. Then for the function g defined by (2.9), it follows from Lemma 2.1 that

$$g(a,b,c,d) = g(b,a,c,d) \tag{4.1}$$

and

$$g(a,b,c,d) = g(a,b,d,c). \tag{4.2}$$

On the other hands, by the definition of the function g in (2.9) we have

$$g(a,b,c,d) = -g(b,a,d,c). \tag{4.3}$$

Hence, together with (4.1) and (4.2), this shows that for all $a,b,c,d \in \mathbb{R}$

$$g(a,b,c,d) = f(a,c)w(b,d) - f(b,d)w(a,c) = 0. \tag{4.4}$$

This shows that $f(x,y) = kw(x,y), k \in \mathbb{R}$, that is,

$$f(x,y) = k\sqrt{1 + |\nabla f(x,y)|^2}. \tag{4.5}$$

Thus, the proof of Theorem 1.3 implies that (1) $\Rightarrow$ (4).

Next, suppose that a positive C^2 function $z = f(x,y)$ defined on $\mathbb{R}^2$ satisfies $\bar{x}_S = \bar{x}_V$ and $\bar{z}_S = 2\bar{z}_V$. Then for the function g defined by (2.9), it follows from Lemma 2.1 that

$$g(a,b,c,d) = g(b,a,c,d) \tag{4.6}$$

and

$$\{f(b,d) - f(a,c)\}g(a,b,c,d) = \{f(a,d) - f(b,c)\}g(a,b,d,c). \tag{4.7}$$

Note that (4.3) implies

$$g(a,b,d,c) = -g(b,a,c,d) = -g(a,b,c,d), \tag{4.8}$$

where the second equality follows from (4.6). Together with (4.7), this yields

$$B(a,b,c,d)g(a,b,c,d) = 0, \tag{4.9}$$

where we put

$$B(a,b,c,d) = f(a,d) - f(a,c) + f(b,d) - f(b,c). \tag{4.10}$$

Now we claim that $g(a,b,c,d)$ vanishes. Otherwise, on an open set $I^2 \times J^2$ $g(a,b,c,d)$ does not vanish, and hence $B(a,b,c,d)$ vanishes on the open set. This implies the function $f_y(x,y) = 0$ on the open set $I \times J$, and hence $f(x,y) = h(x)$ is a function of one variable. The function $h(x)$ obviously satisfies (1) of Proposition 1.2. Thus, on the open set $I \times J$ the function $f(x,y)$ is either a constant function or a catenary

$$f(x,y) = k \cosh\left(\frac{x-c}{k}\right). \tag{4.11}$$

But in any cases, $g(a,b,c,d)$ vanishes on the open set $I^2 \times J^2$. This contradiction shows that $g(a,b,c,d)$ vanishes on $\mathbb{R}^{\nvDash}$, and hence $f(x,y) = kw(x,y)$ for some $k \in \mathbb{R}$. Therefore the proof of Theorem 1.3 implies that (2) $\Rightarrow$ (4).

Finally, suppose that a positive C^2 function $z = f(x,y)$ defined on $\mathbb{R}^{\nvDash}$ satisfies $\bar{y}_S = \bar{y}_V$ and $\bar{z}_S = 2\bar{z}_V$. Then, just the same argument as in the proof of (2) $\Rightarrow$ (4) yields that (3) $\Rightarrow$ (4).

Conversely, it is straightforward to show that (4) $\Rightarrow$ (1), (2) and (3). This completes the proof of Theorem 1.4.

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