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Numerical Solutions of Third-Order Boundary Value Problems associated with Draining and Coating Flows

Jishan Ahmed
Department of Mathematics, Faculty of Science and Engineering, University of Barisal, Bangladesh
e-mail : jishan2004_du@yahoo.com

Abstract. Some computational fluid dynamics problems concerning the thin films flow of viscous fluid with a free surface and draining or coating fluid-flow problems can be delineated by third-order ordinary differential equations. In this paper, the aim is to introduce the numerical solutions of the boundary value problems of such equations by variational iteration method. In this paper, it is shown that the third-order boundary value problems can be written as a system of integral equations, which can be solved by using the variational iteration method. These solutions are gleaned in terms of convergent series. Numerical examples are given to depict the method and their convergence.

1. Introduction

The numerical solution of third-order boundary value problems (BVPs) is of great importance due to its wide application in scientific research. The third-order differential equations arise in many physical problems such as electromagnetic waves, thin film flow, and gravity-driven flows [6, 11, 25, 26]. In this paper, variational iteration method (VIM) is used to obtain a numerical solution to the third-order boundary value problems associated with draining and coating flows of the following form:

$$y^{(3)}(x) = f(x, y, y', y'') \tag{1.1}$$

with boundary conditions

$$y(a) = A_1, \quad y^{(1)}(a) = A_2, \quad y^{(1)}(b) = A_3,$$

where $A_i (i = 1, 2, 3)$ are finite real constants. Many researchers have attempted for the numerical solutions of this type of boundary value problems to obtain high

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accuracy rapidly by using numerous method such as, finite difference method [19] and also some other methods using polynomial and nonpolynomial spline functions [23]. The above problem was solved by El-Danaf [10] using quartic nonpolynomial spline functions. Al-Said and Noor [5, 20] demonstrated a second-order method based on cubic and quartic polynomial spline functions respectively for the solution of a system of third-order boundary value problems. Al-Said and Noor [4] have generated a second-order finite difference scheme at midpoints. Khan and Aziz [18] established and discussed convergent fourth-order method for this problem with the change in the boundary conditions

$$y(a) = A_1, \quad y^{(1)}(a) = A_2, \quad y(b) = A_3$$

using quintic polynomial spline functions respectively. Caglar et al. [8] introduced fourth-degree B-splines for solving third-order BVPs. All these techniques have their inbuilt deficiencies. So we may reasonably infer that a large number of authors have solved third-order BVPs using spline functions which can be exploited easily but the numerical results converge slowly. Recently, Noor and Mohyud-Din [21] have employed homotopy perturbation for solving higher-order boundary value problems. He [12–15] developed the variational iteration method for solving nonlinear initial and boundary value problems. It is worth mentioning that the method was first considered by Inokuti et al. [17]. The main objective of this paper is to apply the variational iteration method to solve a system of integral equations. This technique provides a sequence of functions which converges to the exact solution of the problem. This technique solves the problem without any need to discretization of the variables. Therefore, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. The idea outlined in this paper can be applied to computational fluid dynamics (CFD) problems as well. For example, the two dimensional steady state laminar viscous flow over a semi-infinite flat plate is modeled by the nonlinear two-point boundary value Blasius problem [7]

$$f'''(x) = \frac{1}{2} f(x) f''(x), x \geq 0, \tag{1.2}$$

with boundary conditions

$$f(0) = f'(0) = 0, \tag{1.3}$$

$$f'(+\infty) = 1. \tag{1.4}$$

where a *prime* denotes differentiation with respect to x. Also, x and $f(x)$ are, respectively, the dimensionless coordinate and the dimensionless stream function. In addition to the unknown function f, the solution of (1.3)-(1.4) is characterized by the value of $\alpha = f''(0)$.The condition (1.4) can be replaced by the condition

$$f'(M) = 1. \tag{1.5}$$

for some sufficiently large M, which must be determined as a part of solution. The addition of the new unknown M to the problem warrants the asymptotic condition

$$f''(M) = 0. \tag{1.6}$$

By using VIM method, it would be possible to obtain a solution of Blasius equation in the form of a power series for small x. The two-dimensional flow of a fluid near a stagnation point is a classical problem in fluid mechanics. VIM can also be used to solve the two dimensional flow of fluid near a stagnation point named Hiemenz flow [16]. The Navier-Stokes equations governing the flow can be reduced to an ordinary differential equation of third order using similarity transformation. Due to its wide range of applications in cooling of electronic devices by fans, cooling of nuclear reactors during emergency shutdown, solar central receivers exposed to wind currents, and many hydrodynamic processes in engineering application, the flow near the stagnation-point has attracted the attention of many investigators for more than a century. Another important advantage is that the method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials [2], hence the iteration is direct and straightforward. The method has been successfully implemented for solving various linear and nonlinear problems with approximations converging rapidly to exact solutions [12–15]. This method is based on the use of Lagrange multipliers for identification of optimal value of a parameter in a functional. Generally finding the Lagrange multiplier is not easy for higher order differential equations. Moreover, the use of Lagrange multiplier avoids the successive application of integral operator, reduces the huge computational work and can be considered as an additional edge of this method over the decomposition method [2]. In this paper, the variational iteration method is used for solving the third- order boundary value problems. The boundary value problems are reformulated as a system of integral equations by introducing a suitable transformation, so that the Lagrange multiplier can be effectively identified. This equivalent system is useful in applying the variational iteration method. Therefore, this transformation [17] plays an important and fundamental part in solving the boundary value problems. This clearly indicates that the variational iteration technique may be considered as an alternative method for solving linear and nonlinear problems.

2. Differential Equations Relevent to Draining and Coating Flows

Coating flows involve covering a surface with one or more thin layers of fluid. They range from rain running down a window to manufacturing processes, such as the production of videotapes. Difficulties in modelling coating flows arise for a number of reasons. For example, operating conditions may require a running speed which leads to instabilities, such as air entrainment and ribbing. Some draining or coating fluid-flow problems, in which surface tension forces are important, can be described by third-order ordinary differential equations. Tuck and Schwartz [26] discussed a series of third-order ordinary differential equations (ODEs) arising in the study of the flow of a thin film of viscous fluid over a solid surface. When such a film drains down a vertical wall and the effects of surface tension and gravity as well as viscosity are taken into account, one is led to an equation of the form

$$\frac{d^3y}{dx^3} = f(y) \tag{2.1}$$

for the film profile $y(x)$ in a coordinate frame moving with the fluid. For drainage down a dry surface with the $x-axis$ pointing downwards, this function become

$$f(y) = -1 + y^{-2}.$$

If the surface is prewetted by a very thin film of thickness $\delta > 0$, the function f becomes

$$f(y) = -1 + (1 + \delta + \delta^2)y^{-2} - (\delta + \delta^2)y^{-3}.$$

In [26] the authors formulate a series of well-posed mathematical problems arising from the study of these draining flows.

Despite the seeming simplicity of equation (2.1), there are some fundamental difficulties in its solution, because it becomes singular at the contact line, $y = 0$. The genesis of this singularity is entirely physical and lies in the so-called contact line paradox; the classical no-slip boundary condition being in direct conflict with the requirement of contact line movement. Because of singularity, it is not possible to move a contact line over a no-slip surface. To overcome this difficulty, the boundary condition at an initial point can be modified as $y(0) = \epsilon$ where $\epsilon \to 0$. To illustrate this idea, consider the equation (2.1) with boundary conditions

$$y(0) = 0, y'(0) = 0, \lim_{x\to\infty} y'(x) = 0. \tag{2.2}$$

where $f(y) = -1 + y^{-2}$ arises in the study of draining and coating flows on a dry surface. $f(y)$ is singular at $y(0) = 0$ and we can modify boundary conditions (2.2) as

$$y(0) = \epsilon, y'(0) = 0, \lim_{x\to\infty} y''(x) = \beta. \tag{2.3}$$

In this case, it becomes convenient to apply the VIM for solving the equation (2.1) with the modified boundary conditions (2.2). In this paper, we are concerned with general third-order nonlinear boundary value problems, such problems arise in the study of draining and coating flows.

3. Physical Descriptions of the Third-Order Differential Equations for Draining and Coating Flows

Fluid dynamic problems involving surface tension forces are described in general by partial differential equations in space and time, with rather high, typically fourth-order, spatial differentiations. For example, the thickness y of a thin film of viscous fluid draining over a solid surface in an unsteady manner satisfies such an equation. In some cases, as for example at the front edge of a large drop of fluid moving on a plane surface, the flow can be treated as steady in a frame of reference moving with the front. If, in addition, there is only one spatial coordinate of interest, namely, that in the direction of motion along the plane, the problem has reduced to an ordinary differential equation in that variable, say x. Further, the original fourth-order system then permits one explicit integration, effectively due to conservation of mass, and the result is an autonomous third-order ordinary differential equation of the form (2.1) for some given function $f(y)$. In [26] different possible choices of the function f are given. We consider here some simple rational-function forms, namely,

$$f(y) = -1 + y^{-2}, \tag{3.1}$$

$$f(y) = -1 + (1+\delta+\delta^2)y^{-2} - (\delta+\delta^2)y^{-3}, \tag{3.2}$$

$$f(y) = y^{-2} - y^{-3}, \tag{3.3}$$

$$f(y) = y^{-2}. \tag{3.4}$$

Typically, Equation (3.1) is very important, and is relevant to fluid draining problems on a dry wall, involving the forces of viscosity, gravity and surface tension, subject to a lubrication approximation. This equation is singular at $y = 0$, that is, at the tip of the film. Equation (3.1) also occurs in different film flows, such as spin coating and spray coating [11]. The equation (3.1) describes the thickness $y(x)$ of a layer of fluid that is draining down a vertical wall, the third-derivative term representing surface tension effects, the constant term on the right representing gravity, and the term in y^{-2} the viscous shearing forces. The special draining flow of interest is assumed steady in a frame of reference that is falling with the layer; hence there is an apparent upward movement of the wall in this frame. Tuck and Schwartz [26] used a boundary condition of the form

$$y \to 1 \qquad as \qquad x \to -\infty. \tag{3.5}$$

Numerical solution of (3.1) subject to (3.5) is straight forward. If we choose any starting point $x = x_0$, then we could call upon VIM to solve this problem numerically.

Equation (3.2) is a generalization to a coating problem, or to draining over a wet wall, that sidesteps these difficulties when δ is a small positive parameter measuring wall wetness. Since y may now be expected to be bounded away from zero, the singularity at $y = 0$ is no longer relevant. Equation (3.2) can be used to describe draining over a wet wall, i.e. a case in which the draining layer is doubly infinite, extending forever down the wall as well as forever up up it. Hence it is appropriate to solve (3.2) subject to the boundary conditions

$$y \to 1 \qquad as \qquad x \to -\infty \tag{3.6}$$

and

$$y \to \delta \qquad as \qquad x \to +\infty. \tag{3.7}$$

When the surface is dry, insight into the shape of the film close to the tip may be obtained by studying the limit of solutions of equation (3.2) as $\delta \to 0$. In suitably scaled coordinates this leads to equations involving the functions [26] (3.3) and (3.4). In addition to the asymptotic context given above, equation (3.4) is interesting in its own right in that it describes the spreading of certain oil drops on horizontal surfaces [11]. Equation (3.3) is the so-called "small limit" of (3.2), or inner expansion of (3.2) valid for $x \approx x_0$, obtained by setting $x = x_0 + \delta X, y = \delta Y$, followed by the formal limit $\delta \to 0$. Equation (3.4) describes the dynamic balance between surface tension and viscous forces in a thin fluid layer in the absence (or neglect) of gravity. Tanner [24] and Tuck and Schwartz [26] obtained accurate numerical solutions of equation (3.4) that satisfy the conditions

$$y(0) = 1,\ y^{'}(0) = 0. \tag{3.8}$$

Tanner [24] conducted a series of experiments on the spreading of droplets of silicone oil, from which he obtained a relationship between the speed of the three-phase contact line and the maximum slope of the droplet, now usually referred to as Tanner's Law. Motivated by these experimental results, Tanner [24] solved equation (3.4) numerically subject to the boundary conditions (3.8) for values of $y^{''}(0) = 0$. The VIM can easily be applied to solve the third-order ordinary differential equations of the form (3.1) and (3.2) associated with draining and coating flows.

4. Variational Iteration Method

Consider the following differential equation

$$Lu(x) + Nu(x) = f(x), \tag{4.1}$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is a given continuous function. The primary objective of this method is to derive a correction functional for (4.1) in the following form [12–15]:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(t) + N\tilde{u}_n(t) - f(t))\, dt, \tag{4.2}$$

where λ is a Lagrange multiplier [12–15], which can be determined by imposing the stationary conditions. The subscripts n denotes nth approximation, $\tilde{u}_n$ is regarded as a restricted variation i.e. $\delta\tilde{u}_n = 0$. The solution of the linear problems can be achieved with a very small number of iterations by the exact identification of the Lagrange multiplier. The following system of differential equations will guide us in understanding the variational iteration method:

$$x_i' = g_i(t, x_i), \quad i = 1, 2, 3, \ldots, n \tag{4.3}$$

subject to the boundary conditions,

$$x_i(0) = c_i, \quad i = 1, 2, 3, \ldots, n. \tag{4.4}$$

The system (4.3) can be rewritten as

$$x_i'(t) = g_i(x_i) + f_i(t), \quad i = 1, 2, 3, \ldots, n \tag{4.5}$$

subject to the boundary conditions (4.4). The correct functional for the nonlinear system (4.5) can be represented as

$$x_1^{(k+1)}(t) = x_1^{(0)}(t) + \int_{x_0}^{t} \lambda_1 \Big(\tilde{x}_1'^{(k)}(T), g_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - f_1(T) \Big)\, dT,$$
$$x_2^{(k+1)}(t) = x_2^{(0)}(t) + \int_{x_0}^{t} \lambda_2 \Big(\tilde{x}_2'^{(k)}(T), g_2(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - f_2(T) \Big)\, dT,$$
$$x_3^{(k+1)}(t) = x_3^{(0)}(t) + \int_{x_0}^{t} \lambda_3 \Big(\tilde{x}_3'^{(k)}(T), g_3(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - f_3(T) \Big)\, dT,$$
$$\vdots$$
$$x_n^{(k+1)}(t) = x_n^{(0)}(t) + \int_{x_0}^{t} \lambda_n \Big(\tilde{x}_1'^{(k)}(T), g_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - f_n(T) \Big)\, dT,$$

where $\lambda_i = 1, i = 1, 2, 3, \ldots, n$ are Lagrange multipliers, while $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ define the restricted variations. If we start with the initial approximations

$$x_i(0) = c_i, \quad i = 1, 2, 3, \ldots, n,$$

then the approximations can be completely determined; finally we approximate the solution

$$x_i(t) = \lim_{k \to \infty} x_i^{(k)}(t)$$

by the nth term $x_i^{(n)}(t)\, for\, i = 1, 2, 3, \ldots, n$. To implement the method, third-order boundary value problems are considered in the following section.

5. Numerical Results

The third-order boundary value problems in an infinite interval has been widely used to describe the evolution of physical phenomena, for example some draining or coating fluid-flow problems, see [6,11,25,26]. We refer the reader to [4,5,8,18,20,25] for the study of the finite interval problems of third-order differential equations, and to [6, 11, 25, 26] for the study of the infinite interval problems. The third-order boundary value problems in an infinite interval is a model for a viscous fluid draining over a wet surface for a thin film flowing on an inclined plane with an opening at the bottom of the plane. In the study of draining and coating flows, the thin films flowing on an inclined plane with an opening (a gap) at the bottom of the plane, representing an outlet, can be modeled as a third-order ordinary differential equation. In the present paper, we have considered the problem of draining and coating flows, which can be modeled as the third-order ordinary differential equation of the type (1.1). We now give two numerical examples considered by El-Danaf [10] and Srivastava et al. [23]. They solved a couple of problems associated with draining and coating flows by using the nonpolynomial quartic spline and the polynomial quintic spline, respectively. In this section, the variational iteration method is applied for solving the third-order boundary value problems by converting the problems into a system of integral equations. The variational iteration method is then applied to the resultant system of integral equations. This procedure for solving third-order boundary value problems in finite interval can be implemented in infinite interval problems. For this reason, the third-order boundary value problems of the form (1.1) are considered in this paper. The analytical solutions are taken from [10]. All calculations are performed by MATHEMATICA 5.2.

Example 1. Consider the boundary value problem [10]:

$$y^{(3)} - xy = (x^3 - 2x^2 - 5x - 3)e^x \tag{5.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y^{(1)}(0) = 1, \quad y^{(1)}(1) = -e.$$

The analytical solution of this problem is

$$y(x) = x(1 - x)e^x. \tag{5.2}$$

Using the transformation, $\frac{dy}{dx} = q(x), \quad \frac{dq}{dx} = f(x)$, we can rewrite the third-order boundary value problem as a system of first-order differential equations as follows:

$$\begin{aligned}
\frac{dy}{dx} &= q(x),\\
\frac{dq}{dx} &= f(x),\\
\frac{df}{dx} &= xy + (x^3 - 2x^2 - 5x - 3)e^x,
\end{aligned}$$

with $y(0) = 0, \quad q(0) = 1, \quad f(0) = a$. Using the VIM, we can rewrite the above system of differential equations as a system of integral equations with Lagrange multipliers $\lambda_i = 1, \quad i = 1, 2, 3, \ldots, n$,

$$\begin{aligned}
y^{k+1}(x) &= 0 + \int_0^x q^{(k)}(t)\,dt,\\
q^{k+1}(x) &= 1 + \int_0^x f^{(k)}(t)\,dt,\\
f^{k+1}(x) &= a + \int_0^x (xf^{(k)}(t) + (t^3 - 2t^2 - 5t - 3)e^t)\,dt,
\end{aligned}$$

with $y(0) = 0, \quad q(0) = 1, \quad f(0) = a$. Consequently, the variational iteration method obtains the following approximations:

$$y^{(1)}(x) = x,$$

$$y^{(2)}(x) = x + \frac{ax^2}{2},$$

$$y^{(3)}(x) = 72 - 72e^x + 30x + 43e^x x + 4x^2 + \frac{ax^2}{2} - 11e^x x^2 + e^x x^3,$$

$$y^{(4)}(x) = 72 - 72e^x + 30x + 43e^x x + 4x^2 + \frac{ax^2}{2} - 11e^x x^2 + e^x x^3 + \frac{x^5}{60},$$

$$y^{(5)}(x) = 72 - 72e^x + 30x + 43e^x x + 4x^2 + \frac{ax^2}{2} - 11e^x x^2 + e^x x^3 + \frac{x^5}{60} + \frac{ax^6}{240},$$

$$y^{(6)}(x) = -1680 + 1680e^x - 756x - 923e^x x - 120x^2 + \frac{ax^2}{2} + 203e^x x^2 - 22e^x x^3 + 3x^4 + e^x x^4 + \frac{x^5}{2} + \frac{x^6}{30} + \frac{ax^6}{240},$$

$$\vdots$$

Thus, we get the resulting series solution as

$$y(x) = -997920 + 997920e^x - 428400x - 569519e^x x - 65520x^2 + \frac{ax^2}{2} + 136079e^x x^2 - 17640e^x x^3 + 1617x^4 + 1323e^x x^4 + 286x^5 - 55e^x x^5 + \frac{45x^6}{2} + \frac{ax^6}{240} + e^x x^6 - \frac{5x^8}{24} - \frac{x^9}{40} - \frac{x^{10}}{720} + \frac{ax^{10}}{172800} + \frac{x^{12}}{147840} + \frac{x^{13}}{1729728} + \frac{x^{14}}{47174400} + \frac{ax^{14}}{377395200} + \frac{x^{17}}{211718707200} + \frac{ax^{18}}{1847726899200}.$$

Imposing the boundary conditions at $x = 1$, we get $a = -3.75223 \times 10^{-13}$.
Finally, the series solution can be written as

$$y(x) = -997920 + 997920e^x - 428400x - 569519e^x x - 65520x^2 + 136079e^x x^2 - 17640e^x x^3 + 1617x^4 + 1323e^x x^4 + 286x^5 - 55e^x x^5 + \frac{45x^6}{2} + e^x x^6 - \frac{5x^8}{24} - \frac{x^9}{40} - 0.00138889x^10 + \frac{x^{12}}{147840} + \frac{x^{13}}{1729728} + 2.11979 \times 10^{-8}x^{14} + \frac{x^{17}}{211718707200} - 2.03073 \times 10^{-25}x^{18}.$$

Table 1 shows the comparison between exact solution and the numerical solution obtained using the proposed VIM. The maximum absolute error obtained by the proposed method is compared with that of obtained by [1,3,9,10,18,23,27] in Table 2.

Table 1: Absolute errors for the Example 1

x	Analytical solution $y(x) = x(1-x)e^x$	VIM	Error
0.0	0.0000000000000	0.0000000000000	0.00000000
0.1	0.0994653826268	0.0994653827630	1.362×10^{-10}
0.2	0.1954244413056	0.1954244413075	1.9×10^{-12}
0.3	0.2834703495909	0.2834703495898	1.1×10^{-12}
0.4	0.3580379274339	0.3580379274346	7.0×10^{-12}
0.5	0.4121803176750	0.4121803176650	1.0×10^{-11}
0.6	0.4373085120937	0.4373085120954	1.7×10^{-12}
0.7	0.4228880685688	0.4228880684808	8.80×10^{-11}
0.8	0.3560865485588	0.3560865486530	9.42×10^{-11}
0.9	0.2213642800041	0.2213642801409	1.368×10^{-10}
1.0	0.0000000000000	0.0000000000000	0.00000000

Table 2: Maximum absolute errors for Example 1

References	Results
Khan et al. [18]	1.84×10^{-6}
Abdullah et al. [1]	8.12×10^{-4}
El-Salam et al. [9]	5.30×10^{-7}
Akram et al. [3]	8.29×10^{-7}
Zhiyuan et al. [27]	2.37×10^{-7}
Srivastava et al. [23]	2.64×10^{-7}
El-Danaf [10]	1.64×10^{-2}
Present study	1.37×10^{-10}

Example 2. Consider the boundary value problem [10]:

(5.3) $$y^{(3)} + y = (7 - x^2)cosx + (x^2 - 6x - 1)sinx$$

subject to the boundary conditions

$$y(0) = 0, \quad y^{(1)}(0) = -1, \quad y^{(1)}(1) = 2sin1.$$

The analytical solution of this problem is

$$y(x) = (x^2 - 1)sinx.$$

Using the transformation, $\frac{dy}{dx} = q(x), \quad \frac{dq}{dx} = f(x)$, we can rewrite the third-order boundary value problem as a system of first-order differential equations as follows:

$$\frac{dy}{dx} = q(x),$$
$$\frac{dq}{dx} = f(x),$$
$$\frac{df}{dx} = -y + (7 - x^2)cosx + (x^2 - 6x - 1)sinx,$$

with $y(0) = 0, \quad q(0) = -1, \quad f(0) = a$. Using the VIM, we can rewrite the above system of differential equations as a system of integral equations with Lagrange multipliers $\lambda_i = 1, i = 1, 2, 3, \ldots, n$,

$$y^{k+1}(x) = 0 + \int_0^x q^{(k)}(t)\, dt,$$
$$q^{k+1}(x) = -1 + \int_0^x f^{(k)}(t)\, dt,$$
$$f^{k+1}(x) = a + \int_0^x (-f^{(k)}(t) + ((7 - t^2)cost + (t^2 - 6t - 1)sint)\, dt,$$

with $y(0) = 0, \quad q(0) = -1, \quad f(0) = a$. Consequently, the variational iteration method obtains the following approximations:

$$y^{(1)}(x) = x,$$
$$y^{(2)}(x) = -x + \frac{ax^2}{2},$$
$$y^{(3)}(x) = 13 - \frac{3x^2}{2} + \frac{ax^2}{2} - 13cosx + x^2cosx - sinx - 6xsinx + x^2sinx,$$
$$y^{(4)}(x) = 13 - \frac{3x^2}{2} + \frac{ax^2}{2} + \frac{x^4}{24} - 13cosx + x^2cosx - sinx - 6xsinx + x^2sinx,$$
$$y^{(5)}(x) = 13 - \frac{3x^2}{2} + \frac{ax^2}{2} + \frac{x^4}{24} - \frac{ax^5}{120} - 13cosx + x^2cosx - sinx - 6xsinx + x^2sinx,$$
$$y^{(6)}(x) = 31x + \frac{ax^2}{2} - \frac{13x^3}{6} + \frac{x^5}{40} - \frac{ax^5}{120} + 12cosx - 44sinx + 2x^2sinx,$$
$$\vdots$$

Thus, we get the resulting series solution as

$$\begin{aligned}y(x) = &-463 + \frac{381x^2}{2} + \frac{ax^2}{2} - \frac{307x^4}{24} - \frac{ax^5}{120} + \frac{241x^6}{720} - \frac{61x^8}{13440} + \frac{ax^8}{40320} + \frac{19x^{10}}{518400} \\ &- \frac{ax^{11}}{39916800} - \frac{13x^{12}}{68428800} + \frac{19x^{14}}{29059430400} + \frac{ax^{14}}{87178291200} - \frac{31x^{16}}{20922789888000} \\ &- \frac{ax^{17}}{55687428096000} + \frac{x^{18}}{492490285056000} - \frac{x^{20}}{810967336058880000} \\ &+ \frac{ax^{20}}{2432902008176640000} + \frac{x^{22}}{1124000727777607680000} + 463cosx - x^2cosx \\ &- sinx + 42xsinx + x^2sinx.\end{aligned}$$

Imposing the boundary conditions at $x = 1$, we obtain $a = -2.81078 \times 10^{-22}$. Finally, the series solution can be written as

$$\begin{aligned}y(x) = &-463 + \frac{381x^2}{2} - \frac{307x^4}{24} + 2.34232 \times 10^{-24}x^5 + \frac{241x^6}{720} - 0.00453869x^8 \\ &+ \frac{19x^{10}}{518400} + 7.04159 \times 10^{-30}x^{11} - \frac{13x^{12}}{68428800} + 6.53832 \times 10^{-10}x^{14} \\ &- \frac{31x^{16}}{20922789888000} + 7.90238 \times 10^{-37}x^{17} + \frac{x^{18}}{492490285056000} \\ &- 1.2331 \times 10^{-18}x^{20} + \frac{x^{22}}{1124000727777607680000} + 463cosx - x^2cosx \\ &- sinx + 42xsinx + x^2sinx.\end{aligned}$$

Table 3 shows the comparison between exact solution and the numerical solution obtained using the proposed VIM. The maximum absolute error obtained by the proposed method is compared with that of obtained by [1, 10, 23] in Table 4.

6. Conclusion

In this paper, the variational iteration method has been successfully implemented to get the numerical solutions of third-order BVPs associated with draining and coating flows. The given problems have been converted into a system of first-order differential equations, which leads to the system of integral equations. The mehod provides analytical results to a rather wide class of nonlinear equations without linearization, perturbation, or discretization, which can lead to complex numerical computations. The numerical results obtained by the present method are in good agreement with the exact solutions and is confirmation with great accuracy than the results obtained by the previous methods so far. The results obtained here can easily be extended to third-order ordinary differential equations of the form (2.1) modelling travelling waves on the free surface of a thin film flowing down a vertical wall where the effects of gravity and surface tension have been included. Hence, it is concluded that the method is easy to apply and can easily be applied to boundary value problems associated with draining and coating flows.

Table 3: Absolute errors for the Example 2

x	Analytical solution $y(x) = (x^2 - 1)sinx$	VIM	Error
0.0	0.00000000000	0.00000000000	0.00000000
0.1	-0.09883508248	-0.09883508248	3.03646×10^{-14}
0.2	-0.19072255756	-0.19072255756	2.14273×10^{-14}
0.3	-0.26892338806	-0.26892338806	3.21965×10^{-14}
0.4	-0.32711140754	-0.32711140754	2.33702×10^{-14}
0.5	-0.35956915395	-0.35956915395	2.57572×10^{-14}
0.6	-0.36137118297	-0.36137118297	2.24265×10^{-14}
0.7	-0.32855102049	-0.32855102049	2.40918×10^{-14}
0.8	-0.25824819272	-0.25824819272	9.2149×10^{-15}
0.9	-0.14883211282	-0.14883211282	3.18634×10^{-14}
1.0	0.000000000000	0.00000000000	0.00000000

Table 4: Maximum absolute errors for Example 2

References	Results
Abdullah et al. [1]	8.5594×10^{-5}
Srivastava et al. [23]	2.1572×10^{-8}
El-Danaf [10]	8.8839×10^{-3}
Present study	3.21965×10^{-14}

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