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Oppenheim and Schur Type Inequalities for Khatri-Rao Products of Positive Definite Matrices

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Abstract. For partitioned matrices, the Khatri-Rao product is viewed as a generalized Hadamard product. In this paper we present Oppenheim's and Schur's determinantal inequalities for the Khatri-Rao product of two positive semidefinite matrices.

1. Introduction

Oppenheim proved in [2, pp.509] that for $n \times n$ positive semidefinite matrices $A = (a_{ij})$ and $B = (b_{ij})$,

$$\det(A \circ B) \geq \det A \left(\prod_{i=1}^{n} b_{ii} \right), \tag{1.1}$$

where $A \circ B = (a_{ij} b_{ij})$ denotes the Hadamard product (alternatively the entry-wise product). Marcus called the following inequality (1.2) that improves Oppenheim

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inequality (1.1) as Schur inequality (see [10, Theorem 3.7]):

$$\det(A \circ B) + \det(AB) \geq \det(A) \left(\prod_{i=1}^{n} b_{ii} \right) + \left(\prod_{i=1}^{n} a_{ii} \right) \det(B). \tag{1.2}$$

The importance and applicability of the Hadamard multiplication are well known. In mathematics, for example, this multiplication is used (i) for constructing discrete equipments by means of integer orthogonal matrices that allow fast transformations, and (ii) for finding the maximum of a determinant. This product has also been used in combinatorial analysis, finite geometry, group theory, number theory, and regular graphs. Applications of the Hadamard product can also be found in other fields, for example, in (i) correcting codes in satellite transmissions and cryptography, (ii) communication and information theory, (iii) signal processing and pattern recognition, (iv) neural behavior, and (v) lossy compression algorithms as images in JPEG format. In statistics, some applications of the Hadamard product pertain to (i) interrelations between Hadamard matrices and different combinatorial configurations such as block-designs, Latin square, and orthogonal F-square, (ii) linear models, (iii) maximum likelihood estimation of the variances in a multivariate normal population, (iv) multivariate statistical analysis, and (v) multivariate Tchebycheff equalities. For more details and applications of the Hadamard product, interested readers may refer to Vijayan [12], Hedayat and Wallis [3], Styan [10], Agaian [1], Seberry and Yamada [11], and the references therein. Relevant to this matrix product, the Khatri-Rao product for partitioned matrices is claimed to be a generalized Hadamard product (see [4, 8]). Rao and Kleffe [9] and Liu [6] have compiled several matrix inequalities involving the Khatri-Rao product.

The set of all complex matrices partitioned into $m \times n$ blocks with each block $p \times q$ is denoted by $\mathbb{M}_{m,n}(\mathbb{M}_{p,q})$, and we simply denote as $\mathbb{M}_n(\mathbb{M}_k) := \mathbb{M}_{n,n}(\mathbb{M}_{k,k})$. In this article, we generalize the Oppenheim inequality and the Schur inequality for the Khatri-Rao product on $\mathbb{M}_n(\mathbb{M}_k)$.

2. Preliminaries

We introduce the definitions of three matrix products, namely the Hadamard, Kronecker and Khatri-Rao products. We then give several identities involving the Khatri-Rao products.

Let $A = (a_{ij})$ and $B = (b_{ij})$ of size $m \times n$ and $C = (c_{kl})$ of size $p \times q$.

(1) Hadamard product
$$A \circ B = (a_{ij} b_{ij}),$$
where the scalar $a_{ij} b_{ij}$ is the (i, j)th entry. Note that $A \circ B$ is of size $m \times n$.

(2) Kronecker product
$$A \otimes C = (a_{ij} C),$$
where $a_{ij} C$ is the (i, j)th block submatrix of size $p \times q$. Note that $A \otimes B$ is of size $mp \times nq$.

Let $\mathbf{A} = (A_{ij})$ be partitioned with A_{ij} of size $m_i \times n_j$ as the (i, j)th block submatrix and $\mathbf{B} = (B_{kl})$ be partitioned with B_{kl} of size $p_k \times q_l$ as the (k, l)th block submatrix.

(3) Khatri-Rao product

$$\mathbf{A} * \mathbf{B} = (A_{ij} \otimes B_{ij})$$

where $A_{ij} \otimes B_{ij}$ is of size $m_i p_i \times n_j q_j$. Note that $\mathbf{A} * \mathbf{B}$ is of size $(\sum m_i p_i) \times (\sum n_j q_j)$.

Theorem 2.1. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be comparible partitioned matrices. Then*

(a) $\mathbf{A} * \mathbf{B} \neq \mathbf{B} * \mathbf{A}$, *in general.*

(b) $(\mathbf{A} * \mathbf{B})^* = \mathbf{A}^* * \mathbf{B}^*$, *where $\mathbf{A}^*$ is the complex conjugate transpose of $\mathbf{A}$.*

(c) $(\mathbf{A} + \mathbf{B}) * (\mathbf{C} + \mathbf{D}) = \mathbf{A} * \mathbf{C} + \mathbf{A} * \mathbf{D} + \mathbf{B} * \mathbf{C} + \mathbf{B} * \mathbf{D}$.

(d) $(\mathbf{A} * \mathbf{B}) * \mathbf{C} = \mathbf{A} * (\mathbf{B} * \mathbf{C})$.

(e) $(\mathbf{A} * \mathbf{B}) \circ (\mathbf{C} * \mathbf{D}) = (\mathbf{A} \circ \mathbf{C}) * (\mathbf{B} \circ \mathbf{D})$.

Proof. Straightforward. □

For Hermitian matrices A and B, we write $A \geq B$ in the sense of Loewner partial ordering, which means that $A - B$ is positive semidefinite. Liu [7] proved that $\mathbf{A} * \mathbf{B}$ is positive semidefinite if $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite.

Lemma 2.2.([7]) *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ be compatibly partitioned matrices such that $\mathbf{A} \geq \mathbf{B} \geq 0$ and $\mathbf{C} \geq \mathbf{D} \geq 0$. Then*

$$\mathbf{A} * \mathbf{C} \geq \mathbf{B} * \mathbf{D} \geq 0.$$

If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then the Schur complement of A_{22} in A is given by

$$A/A_{22} := A_{11} - A_{12} A_{22}^{-1} A_{21}$$

and the Schur complement of A_{11} in A is given by

$$A/A_{11} := A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

In the case that A_{11} or A_{22} is singular, the inverses on A/A_{11} and A/A_{22} can be replaced by a generalized inverse.

Lemma 2.3.([2, Theorem 7.7.7]) *Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ be Hermitian with $A_{11} \in \mathbb{M}_p$ and $A_{22} \in \mathbb{M}_q$. The following are equivalent:*

(a) *A is positive definite.*

(b) A_{11} *is positive definite and* A/A_{11} *is positive definite.*

3. Main Results

We generalize Oppenheim's inequality for the Khatrio-Rao product.

Theorem 3.1. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{M}_k)$. *If* $\mathbf{A}, \mathbf{B}$ *are positive semidefinite, then*

$$\det(\mathbf{A} * \mathbf{B}) \geq \left(\det \mathbf{A} \cdot \prod_{i=1}^{n} \det B_{ii} \right)^k . \tag{3.1}$$

Proof. We use induction with respect to n. If $n = 1$,

$$\det(\mathbf{A} * \mathbf{B}) = \det(\mathbf{A} \otimes \mathbf{B}) = (\det \mathbf{A})^k (\det \mathbf{B})^k = (\det \mathbf{A})^k (\det B_{11})^k.$$

Let $n \geq 2$ and assume that (3.1) is true for matrices in $\mathbb{M}_{n-1}(\mathbb{M}_k)$. Consider the partition

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{a} \\ \mathbf{a}^* & A_{nn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{b} \\ \mathbf{b}^* & B_{nn} \end{pmatrix},$$

where $\mathbf{A}_1, \mathbf{B}_1 \in \mathbb{M}_{n-1}(\mathbb{M}_k)$, $\mathbf{a}, \mathbf{b} \in \mathbb{M}_{n-1,1}(\mathbb{M}_k)$. Assume first that $\mathbf{A}$ and $\mathbf{B}$ are positive definite. Then Schur's determinant formula for block matrices gives

$$\begin{aligned} \det(\mathbf{A} * \mathbf{B}) &= \det \begin{pmatrix} \mathbf{A}_1 * \mathbf{B}_1 & \mathbf{a} * \mathbf{b} \\ \mathbf{a}^* * \mathbf{b}^* & A_{nn} \otimes B_{nn} \end{pmatrix} \\ &= \det(\mathbf{A}_1 * \mathbf{B}_1) \cdot \det\left(A_{nn} \otimes B_{nn} - (\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b}) \right). \end{aligned} \tag{3.2}$$

According to Lemma 2.2, the matrix $\mathbf{A} * \mathbf{B}$ is positive definite. Therefore,

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_1 * \mathbf{B}_1) = A_{nn} \otimes B_{nn} - (\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b}) \geq 0 \tag{3.3}$$

by Lemma 2.3. Moreover,

$$\mathbf{C} := \begin{pmatrix} \mathbf{A}_1 & \mathbf{a} \\ \mathbf{a}^* & \mathbf{a}^* \mathbf{A}_1^{-1} \mathbf{a} \end{pmatrix}$$

is positive semidefinite. Hence

$$\mathbf{C} * \mathbf{B} = \begin{pmatrix} \mathbf{A}_1 * \mathbf{B}_1 & \mathbf{a} * \mathbf{b} \\ \mathbf{a}^* * \mathbf{b}^* & \mathbf{a}^* \mathbf{A}_1^{-1} \mathbf{a} \otimes B_{nn} \end{pmatrix} \geq 0$$

and

$$(\mathbf{C} * \mathbf{B})/(\mathbf{A}_1 * \mathbf{B}_1) = \mathbf{a}^* \mathbf{A}_1^{-1} \mathbf{a} \otimes B_{nn} - (\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b}) \geq 0. \tag{3.4}$$

Note that

$$\begin{aligned} & A_{nn} \otimes B_{nn} - (\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b}) \\ &= (A_{nn} - \mathbf{a}^* \mathbf{A}_1^{-1} \mathbf{a}) \otimes B_{nn} + \mathbf{a}^* \mathbf{A}_1^{-1} \mathbf{a} \otimes B_{nn} - (\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b}) \end{aligned}$$

Since $(A_{nn} - \mathbf{a}^*\mathbf{A}_1^{-1}\mathbf{a}) \otimes B_{nn}$ is positive definite, we obtain from (3.3) and (3.4) that

$$\det\left(A_{nn}\otimes B_{nn}-(\mathbf{a}^* * \mathbf{b}^*)(\mathbf{A}_1 * \mathbf{B}_1)^{-1}(\mathbf{a} * \mathbf{b})\right) \geq \det\left((A_{nn}-\mathbf{a}^*\mathbf{A}_1^{-1}\mathbf{a})\otimes B_{nn}\right). \tag{3.5}$$

From (3.2), (3.5), and the induction assumption it follows that

$$\begin{aligned} \det(\mathbf{A} * \mathbf{B}) &\geq \det(\mathbf{A}_1 * \mathbf{B}_1) \cdot \det\left((A_{nn} - \mathbf{a}^*\mathbf{A}_1^{-1}\mathbf{a}) \otimes B_{nn}\right) \\ &\geq \left(\det \mathbf{A}_1 \cdot \prod_{i=1}^{n-1} \det B_{ii}\right)^k \cdot \left(\det(A_{nn} - \mathbf{a}^*\mathbf{A}_1^{-1}\mathbf{a})\right)^k (\det B_{nn})^k \\ &= \left(\det \mathbf{A} \cdot \prod_{i=1}^{n} \det B_{ii}\right)^k . \end{aligned}$$

Thus (3.1) is proved for positive definite matrices and can be extended to positive semidefinite matrices by a continuity argument. □

Lemma 3.2. *Let* $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & \mathbf{A}_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{22} \end{pmatrix} \in \mathbb{M}_n(\mathbb{M}_k)$ *with* $\mathbf{A}_{11}, \mathbf{B}_{11} \in \mathbb{M}_m(\mathbb{M}_k)$, $\mathbf{A}_{12}, \mathbf{B}_{12} \in \mathbb{M}_{m,n-m}(\mathbb{M}_k)$ $(m < n)$. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive definite, then*

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) + (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}) \geq \mathbf{A}_{22} * (\mathbf{B}/\mathbf{B}_{11}) + (\mathbf{A}/\mathbf{A}_{11}) * \mathbf{B}_{22}.$$

In particular, for $m = 1$

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) + (\mathbf{A}/\mathbf{A}_{11}) \otimes (\mathbf{B}/\mathbf{B}_{11}) \geq A_{22} \otimes (\mathbf{B}/\mathbf{B}_{11}) + (\mathbf{A}/\mathbf{A}_{11}) \otimes B_{22}.$$

Proof. Let

$$\mathbf{C} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & \mathbf{A}_{12}^*\mathbf{A}_{11}^{-1}\mathbf{A}_{12} \end{pmatrix}$$

and

$$\mathbf{D} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{12}^*\mathbf{B}_{11}^{-1}\mathbf{B}_{12} \end{pmatrix}.$$

Then $\mathbf{C}$ and $\mathbf{D}$ are positive semidefinite. Therefore,

$$\mathbf{C} * \mathbf{D} = \begin{pmatrix} \mathbf{A}_{11} * \mathbf{B}_{11} & \mathbf{A}_{12} * \mathbf{B}_{12} \\ (\mathbf{A}_{12} * \mathbf{B}_{12})^* & (\mathbf{A}_{12}^*\mathbf{A}_{11}^{-1}\mathbf{A}_{12}) * (\mathbf{B}_{12}^*\mathbf{B}_{11}^{-1}\mathbf{B}_{12}) \end{pmatrix}$$

is positive semidefinite and so

$$(\mathbf{A}_{12}^*\mathbf{A}_{11}^{-1}\mathbf{A}_{12}) * (\mathbf{B}_{12}^*\mathbf{B}_{11}^{-1}\mathbf{B}_{12}) \geq (\mathbf{A}_{12} * \mathbf{B}_{12})^*(\mathbf{A}_{11} * \mathbf{B}_{11})^{-1}(\mathbf{A}_{12} * \mathbf{B}_{12}).$$

That is,

$$(\mathbf{A}_{22} - \mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}_{22} - \mathbf{B}/\mathbf{B}_{11}) \geq \mathbf{A}_{22} * \mathbf{B}_{22} - (\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}). \tag{3.6}$$

Expanding (3.6) gives the required result. □

Lemma 3.3. *Let* $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^* & \mathbf{A}_{22} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^* & \mathbf{B}_{22} \end{pmatrix} \in \mathbb{M}_n(\mathbb{M}_k)$ *with* $\mathbf{A}_{11}, \mathbf{B}_{11} \in \mathbb{M}_m(\mathbb{M}_k)$, $\mathbf{A}_{12}, \mathbf{B}_{12} \in \mathbb{M}_{m,n-m}(\mathbb{M}_k)$ $(m < n)$. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive definite, then*

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) \geq \mathbf{A}_{22} * (\mathbf{B}/\mathbf{B}_{11}) \geq (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}) \tag{3.7}$$

$$(\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) \geq (\mathbf{A}/\mathbf{A}_{11}) * \mathbf{B}_{22} \geq (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}). \tag{3.8}$$

Proof. Since $\mathbf{A}_{22} \geq \mathbf{A}/\mathbf{A}_{11}$, by Lemma 2.2, we get

$$\mathbf{A}_{22} * (\mathbf{B}/\mathbf{B}_{11}) \geq (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}).$$

This proves the second inequality in (3.7). Since $\mathbf{B}_{22} \geq \mathbf{B}/\mathbf{B}_{11}$, we get similarly

$$(\mathbf{A}/\mathbf{A}_{11}) * \mathbf{B}_{22} \geq (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11}). \tag{3.9}$$

By Lemma 3.2,

$$\begin{aligned} (\mathbf{A} * \mathbf{B})/(\mathbf{A}_{11} * \mathbf{B}_{11}) &\geq \mathbf{A}_{22} * (\mathbf{B}/\mathbf{B}_{11}) + \big((\mathbf{A}/\mathbf{A}_{11}) * \mathbf{B}_{22} - (\mathbf{A}/\mathbf{A}_{11}) * (\mathbf{B}/\mathbf{B}_{11})\big) \\ &\geq \mathbf{A}_{22} * (\mathbf{B}/\mathbf{B}_{11}). \end{aligned}$$

The second inequality follows from (3.9). Similarly, we can prove (3.8). □

Lemma 3.4.([5]) *Let* A, B, C, D *be positive semidefinte matrices. If* $A+B \geq C+D$, $A \geq C \geq B$ *and* $A \geq D \geq B$, *then*

$$\det A + \det B \geq \det C + \det D.$$

For $\mathbf{A} = (A_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$, we denote as $\mathbf{A}_p = (A_{ij})_{i,j=1}^p \in \mathbb{M}_p(\mathbb{M}_k)$ the $p \times p$ leading principal block submatrix of $\mathbf{A}$.

Proposition 3.5. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{M}_k)$. *If* $\mathbf{A}, \mathbf{B}$ *are positive definite, then*

$$\det(\mathbf{A}*\mathbf{B}) \geq (\det \mathbf{AB})^k \times \prod_{p=2}^n \left(\left(\frac{\det A_{pp} \det \mathbf{A}_{p-1}}{\det \mathbf{A}_p} \right)^k + \left(\frac{\det B_{pp} \det \mathbf{B}_{p-1}}{\det \mathbf{B}_p} \right)^k - 1 \right). \tag{3.10}$$

Proof. Let $2 \leq p \leq n$. By Lemma 3.2, we have

$$(\mathbf{A}_p*\mathbf{B}_p)/(\mathbf{A}_{p-1}*\mathbf{B}_{p-1})+(\mathbf{A}_p/\mathbf{A}_{p-1})\otimes(\mathbf{B}_p/\mathbf{B}_{p-1}) \geq A_{pp}\otimes(\mathbf{B}_p/\mathbf{B}_{p-1})+(\mathbf{A}_p/\mathbf{A}_{p-1})\otimes B_{pp}.$$

Also, by Lemma 3.3, we have

$$(\mathbf{A}_p * \mathbf{B}_p)/(\mathbf{A}_{p-1} * \mathbf{B}_{p-1}) \geq A_{pp} \otimes (\mathbf{B}_p/\mathbf{B}_{p-1}) \geq (\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes (\mathbf{B}_p/\mathbf{B}_{p-1})$$

and

$$(\mathbf{A}_p * \mathbf{B}_p)/(\mathbf{A}_{p-1} * \mathbf{B}_{p-1}) \geq (\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes B_{pp} \geq (\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes (\mathbf{B}_p/\mathbf{B}_{p-1}).$$

Thus, by Lemma 3.4, we have

$$\begin{aligned} &\det\big((\mathbf{A}_p * \mathbf{B}_p)/(\mathbf{A}_{p-1} * \mathbf{B}_{p-1})\big) + \det\big((\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes (\mathbf{B}_p/\mathbf{B}_{p-1})\big) \\ &\quad \geq \det\big(A_{pp} \otimes (\mathbf{B}_p/\mathbf{B}_{p-1})\big) + \det\big((\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes B_{pp}\big). \end{aligned}$$

That is,

$$\begin{aligned} \frac{\det(\mathbf{A}_p * \mathbf{B}_p)}{\det\big(\mathbf{A}_{p-1} * \mathbf{B}_{p-1}\big)} &= \det\big((\mathbf{A}_p * \mathbf{B}_p)/(\mathbf{A}_{p-1} * \mathbf{B}_{p-1})\big) \\ &\geq \det\big(A_{pp} \otimes (\mathbf{B}_p/\mathbf{B}_{p-1})\big) + \det\big((\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes B_{pp}\big) \\ &\qquad - \det\big((\mathbf{A}_p/\mathbf{A}_{p-1}) \otimes (\mathbf{B}_p/\mathbf{B}_{p-1})\big) \\ &= \left(\frac{\det A_{pp} \det \mathbf{B}_p}{\det \mathbf{B}_{p-1}}\right)^k + \left(\frac{\det \mathbf{A}_p \det B_{pp}}{\det \mathbf{A}_{p-1}}\right)^k - \left(\frac{\det(\mathbf{A}_p\mathbf{B}_p)}{\det(\mathbf{A}_{p-1}\mathbf{B}_{p-1})}\right)^k \\ &= \frac{\det(\mathbf{A}_p\mathbf{B}_p)^k}{\det(\mathbf{A}_{p-1}\mathbf{B}_{p-1})^k} \\ &\qquad \times \left(\left(\frac{\det A_{pp} \det \mathbf{A}_{p-1}}{\det \mathbf{A}_p}\right)^k + \left(\frac{\det \mathbf{B}_{p-1} \det B_{pp}}{\det \mathbf{B}_p}\right)^k - 1\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\prod_{p=2}^{n} \frac{\det(\mathbf{A}_p * \mathbf{B}_p)}{\det\big(\mathbf{A}_{p-1} * \mathbf{B}_{p-1}\big)} \\ \geq\ &\prod_{p=2}^{n} \frac{\det(\mathbf{A}_p\mathbf{B}_p)^k}{\det(\mathbf{A}_{p-1}\mathbf{B}_{p-1})^k} \left(\left(\frac{\det A_{pp} \det \mathbf{A}_{p-1}}{\det \mathbf{A}_p}\right)^k + \left(\frac{\det \mathbf{B}_{p-1} \det B_{pp}}{\det \mathbf{B}_p}\right)^k - 1\right) \end{aligned}$$

which coincides with (3.10). □

Lemma 3.6.([Fisher's inequality]) *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is a positive definite matrix that is partitioned so that A_{11} and A_{22} are square and nonempty. Then

$$\det A \leq (\det A_{11})(\det A_{22}).$$

Lemma 3.7.([5]) *Let $a_k, b_k \geq 1$ for all $k = 1, \ldots, n$. Then*

$$\prod_{k=1}^{n}(a_k + b_k - 1) \geq \prod_{k=1}^{n} a_k + \prod_{k=1}^{n} b_k - 1.$$

We generalize Schur's inequality.

Theorem 3.8. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n(\mathbb{M}_k)$*. If* $\mathbf{A}, \mathbf{B}$ *are positive semidefinite, then*

$$\det(\mathbf{A} * \mathbf{B}) + \det(\mathbf{AB})^k \geq \left(\det \mathbf{A} \prod_{p=1}^{n} \det B_{pp}\right)^k + \left(\det \mathbf{B} \prod_{p=1}^{n} \det A_{pp}\right)^k. \tag{3.11}$$

Proof. If any of A_{pp}, B_{pp} in (3.11) is singular, then so is $\mathbf{A}$ or $\mathbf{B}$. In this case the right hand side of (3.11) vanishes. So without loss of generality, we can indeed assume that $\mathbf{A}$ and $\mathbf{B}$ are positive definite by a continuous argument.

We may rewrite (3.11) as

$$\det(\mathbf{A} * \mathbf{B}) \geq \det(\mathbf{AB})^k \left(\left(\frac{\prod_{p=1}^{n} \det A_{pp}}{\det \mathbf{A}}\right)^k + \left(\frac{\prod_{p=1}^{n} \det B_{pp}}{\det \mathbf{B}}\right)^k - 1 \right). \tag{3.12}$$

By Lemma , we have $\dfrac{\det A_{pp} \det \mathbf{A}_{p-1}}{\det \mathbf{A}_p} \geq 1$ and $\dfrac{\det B_{pp} \det \mathbf{B}_{p-1}}{\det \mathbf{B}_p} \geq 1$ in (3.10) for $p = 2, \ldots, n$. Then

$$\begin{aligned}
\prod_{p=2}^{n} \frac{\det A_{pp} \det \mathbf{A}_{p-1}}{\det \mathbf{A}_p} &= \left(\prod_{p=2}^{n} \det A_{pp}\right) \left(\frac{\det \mathbf{A}_1}{\det \mathbf{A}_2} \frac{\det \mathbf{A}_2}{\det \mathbf{A}_3} \cdots \frac{\det \mathbf{A}_{n-1}}{\det \mathbf{A}_n}\right) \\
&= \left(\prod_{p=2}^{n} \det A_{pp}\right) \frac{\det \mathbf{A}_1}{\det \mathbf{A}_n} \\
&= \frac{\prod_{p=1}^{n} \det A_{pp}}{\det \mathbf{A}}.
\end{aligned}$$

Similarly, we have

$$\prod_{p=2}^{n} \frac{\det B_{pp} \det \mathbf{B}_{p-1}}{\det \mathbf{B}_p} = \frac{\prod_{p=1}^{n} \det B_{pp}}{\det \mathbf{B}}.$$

Thus, (3.12) follows from Proposition 3.5 and Lemma 3.7. □

4. Concluding Remarks

The Hadamard product plays an important role in matrix methods for statistics and econometrics. Relevant to this matrix product, the Khatri-Rao product for partitioned matrices is claimed to be a generalized Hadamard product. In this paper, we have established the generalized Oppenheims and Schurs inequalities for the Khatrio-Rao product. We can get the lower bound of the determinant of Khatri-Rao product of partitioned positive semidefinite matrices.

References

[1] S. S. Agaian, *Hadamard Matrices and Their Applications*, Springer-Verlag, Berlin, 1985.

[2] R. Horn and C. Johnson, *Matrix analysis*, 2nd edition, Cambridge University Press, Cambridge, 2013.

[3] A. Hedayat, W. D. Wallis, *Hadamard matrices and their applications*, Annals of Statistics, **6**(1978), 11841238.

[4] C. G. Khatri and C. R. Rao, *Solutions to some functional equations and their applications to charaterization of probability distributions*, Sankhya, **30**(1968), 167–180.

[5] M. Lin, *An Oppenheim type inequality for a block Hadamard product*, Linear Algebra Appl., **452**(2014), 1–6.

[6] S. Liu, *Contributions to Matrix Calculus and Applications in Econometrics*, Thesis Publishers, Amsterdam, The Netherlands, 1995.

[7] S. Liu, *Matrix results on the Khatri-Rao and Tracy-Singh products*, Linear Algebra Appl., **289**(1999), 267–277.

[8] C. R. Rao, *Estimation of heteroscedastic variances in linear models*, J. Amer. Statist. Assoc., **65**(1970), 161–172.

[9] C. R. Rao and J. Kleffe, *Estimation of variance components and applications*, North-Holland, Amsterdam, The Netherlands, 1988.

[10] G. P. H. Styan, *Hadamard products and multivariate statistical analysis*, Linear Algebra Appl., **6**(1973), 217–240.

[11] J. Seberry, M. Yamada, *Hadamard matrices, sequences, and block designs*, in : J. H. Dinitz, D. R. Stinson(Eds.), Contemporary Design Theory : A Collection of Surveys, John Wiley Sons, NewYork, 1992 (Chapter11).

[12] K. Vijayan, *Hadamard matrices and submatrices*, J. Austr. Math. Soc. Ser. A, **22**(1976), 469–475.