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On Some Binomial Difference Sequence Spaces

Jian Meng* and Meimei Song
Department of Mathematics, Tianjin University of Technology, Tianjin 300384, China
e-mail : mengjian0710@163.com *and* ms153106305@126.com

Abstract. The aim of this paper is to introduce the binomial sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ by combining the binomial transformation and difference operator. We prove that these spaces are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively. Furthermore, we compute the Schauder bases and the α-, β- and γ-duals of these sequence spaces.

1. Introduction and Preliminaries

Let w denote the space of all sequences. By ℓ_p, ℓ_∞, c and c_0, we denote the spaces of p-absolutely summable, bounded, convergent and null sequences respectively. Let Z be a sequence space, then Kizmaz[12] introduced the following difference sequence spaces

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z \in \{\ell_\infty, c, c_0\}$, where $\Delta x_k = x_k - x_{k+1}$ for each $k \in \mathbb{N} = \{1, 2, 3...\}-$the set of positive integers. Since then, many authors have studied further generalization of the difference sequence spaces [5, 9, 15, 17]. Moreover, Altay and Polat [3], Polat and Başar [14] and many others have studied new sequence spaces from matrix point of view that represent difference operators.

For an infinite matrix $A = (a_{n,k})$ and $x = (x_k) \in w$, the A-transform of x is defined by $Ax = \{(Ax)_n\}$ and is supposed to be convergent for all $n \in \mathbb{N}$, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k$. For two sequence spaces X and Y and an infinite matrix $A = (a_{n,k})$, the sequence space X_A is defined by $X_A = \{x = (x_k) \in w : Ax \in X\}$, which is called the domain of matrix A in the space X. By $(X : Y)$, we denote the class of all matrices such that $X \subseteq Y_A$.

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The Euler means E^r of order r is defined by the matrix $E^r = (e^r_{n,k})$, where $0 < r < 1$ and

$$e^r_{n,k} = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

The Euler sequence spaces e^r_0, e^r_c and e^r_∞ were defined by Altay and Başar [1] and Altay, Başar and Mursaleen [2] as follows

$$e^r_0 = \{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k x_k = 0\},$$

$$e^r_c = \{x = (x_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k x_k \text{ exists}\},$$

and

$$e^r_\infty = \{x = (x_k) \in w : \sup_{n\in\mathbb{N}} | \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-r)^{n-k} r^k x_k |< \infty\}.$$

Altay and Polat [3] defined further generalization of the Euler sequence spaces $e^r_0(\nabla)$, $e^r_c(\nabla)$ and $e^r_\infty(\nabla)$ by

$$Z(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in Z\}$$

for $Z \in \{e^r_0, e^r_c, e^r_\infty\}$, where $\nabla x_k = x_k - x_{k-1}$ for each $k \in \mathbb{N}$. Here any term with negative subscript is equal to naught. Moreover, many authors have used especially the Euler matrix for defining new sequence spaces. For instance, Kara and Başarir [10], Karakaya and Polat [11] and Polat and Başar [14].

Recently Bişgin [6, 7] defined another type of generalization of the Euler sequence spaces and introduced the binomial sequence spaces $b^{r,s}_0$, $b^{r,s}_c$ and $b^{r,s}_\infty$. Let $r, s \in \mathbb{R}$ and $r + s \neq 0$. Then the binomial matrix $B^{r,s} = (b^{r,s}_{n,k})$ is defined by

$$b^{r,s}_{n,k} = \begin{cases} \frac{1}{(s+r)^n} \begin{pmatrix} n \\ k \end{pmatrix} s^{n-k} r^k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. For $sr > 0$ we have

(i) $\| B^{r,s} \| < \infty$,

(ii) $\lim_{n\to\infty} b^{r,s}_{n,k} = 0$ for each $k \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \sum_k b^{r,s}_{n,k} = 1$.

Thus, the binomial matrix $B^{r,s}$ is regular for $sr > 0$. Unless stated otherwise, we assume that $sr > 0$. If we take $s + r = 1$, we obtain the Euler matrix E^r. So, the binomial matrix generalizes the Euler matrix. Bişgin defined the following spaces of binomial sequences

$$b_0^{r,s} = \{x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k = 0\},$$

$$b_c^{r,s} = \{x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \text{ exists}\},$$

and

$$b_\infty^{r,s} = \{x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right| < \infty\}.$$

The main purpose of the present paper is to study the difference spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ of the binomial sequence whose $B^{r,s}(\nabla)$-transforms are in the spaces c_0, c and ℓ_∞, respectively. These new sequence spaces are the generalization of the sequence spaces defined in [3, 6, 7]. Also, we compute the bases and the α-, β- and γ-duals of these sequence spaces.

2. The Binomial Difference Sequence Spaces

In this section, we introduce the spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ and prove that these sequence spaces are linearly isomorphic to the spaces c_0, c and ℓ_∞, respectively.

We first define the binomial difference sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ by

$$b_0^{r,s}(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in b_0^{r,s}\},$$
$$b_c^{r,s}(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in b_c^{r,s}\},$$

and

$$b_\infty^{r,s}(\nabla) = \{x = (x_k) \in w : (\nabla x_k) \in b_\infty^{r,s}\}.$$

Let us define the sequence $y = (y_n)$ as the $B^{r,s}(\nabla)$-transform of a sequence $x = (x_k)$, that is

$$y_n = [B^{r,s}(\nabla x_k)]_n = \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\nabla x_k). \tag{2.1}$$

for each $n \in \mathbb{N}$. Then, the binomial difference sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ can be redefined by all sequences whose $B^{r,s}(\nabla)$-transforms are in the space c_0, c and ℓ_∞. Let X be the one of the spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$. It is obvious that these sequence spaces are linear spaces normed by

$$\| x \|_X = \| y \|_\infty = \sup_{n\in\mathbb{N}} | y_n | . \tag{2.2}$$

Theorem 2.1. *The sequence space X is a complete linear metric space with the norm defined by the equation (2.2).*

Proof. Let $(x_m)_{m=1}^{\infty}$ be a Cauchy sequence in X, where $x_m = (x_{m_k})_{k=1}^{\infty} \in X$ for each $m \in \mathbb{N}$. For every $\varepsilon > 0$, there is a positive integer m_0 such that $\| x_m - x_l \| < \varepsilon$ for $m, l \geq m_0$. Then we get

$$| B^{r,s}[\nabla(x_{m_k} - x_{l_k})] | < \varepsilon$$

for $m, l \geq m_0$ and each $k \in \mathbb{N}$. So $(B^{r,s}(\nabla x_{m_k}))_{m=1}^{\infty}$ is a Cauchy sequence in the set of complex numbers $\mathbb{C}$. Since $\mathbb{C}$ is complete, we have $\lim_{l\to\infty} B^{r,s}(\nabla x_{l_k}) = B^{r,s}(\nabla x_k)$ for each $k \in \mathbb{N}$. Hence

$$\lim_{l\to\infty} | B^{r,s}[\nabla(x_{m_k} - x_{l_k})] | = | B^{r,s}[\nabla(x_{m_k} - x_k)] | \leq \varepsilon \text{ for } m > m_0,$$

which implies that $\| x_m - x \| < \varepsilon$ for all $m > m_0$. Then we have $x_m \to x$ as $m \to \infty$.

Next, we shall prove that $x \in b_{\infty}^{r,s}(\nabla)$. And we have

$$\begin{aligned} | B^{r,s}(\nabla x_k) | &= | B^{r,s}(x_k - x_{k-1}) | \\ &= | B^{r,s}(x_k - x_{m_k} + x_{m_k} - x_{m_{k-1}} + x_{m_{k-1}} - x_{k-1}) | \\ &\leq | B^{r,s}(x_{m_k} - x_{m_{k-1}}) | + | B^{r,s}(x_k - x_{m_k} + x_{m_{k-1}} - x_{k-1}) | \\ &\leq \| x_m \| + \| x_m - x \| \\ &< \infty, \end{aligned}$$

which implies that $x \in b_{\infty}^{r,s}(\nabla)$. Thus, $b_{\infty}^{r,s}(\nabla)$ is a complete linear metric space. Obviously, $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ are closed subspaces of $b_{\infty}^{r,s}(\nabla)$, so $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ are also complete linear metric spaces. □

Theorem 2.2. *The sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_{\infty}^{r,s}(\nabla)$ are linearly isomorphic to the spaces c_0, c and ℓ_{∞}, respectively.*

Proof. Similarly, we only prove the theorem for the space $b_0^{r,s}(\nabla)$. To prove $b_0^{r,s}(\nabla) \cong c_0$, we must show the existence of a linear bijection between the spaces $b_0^{r,s}(\nabla)$ and c_0.

Consider $T : b_0^{r,s}(\nabla) \to c_0$ by $T(x) = B^{r,s}(\nabla x_k)$. The linearity of T is obvious and $x = 0$ whenever $T(x) = 0$. Therefore, T is injective.

Let $y = (y_n) \in c_0$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{i=0}^{k} (s+r)^i \sum_{j=i}^{k} \binom{j}{i} r^{-j} (-s)^{j-i} y_i \tag{2.3}$$

for each $k \in \mathbb{N}$. Then we have

$$\lim_{n\to\infty} [B^{r,s}(\nabla x_k)]_n = \lim_{n\to\infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k (\nabla x_k) = \lim_{n\to\infty} y_n = 0,$$

which implies that $x \in b_0^{r,s}(\nabla)$ and $T(x) = y$. Consequently, T is surjective and is norm preserving. Thus, $b_0^{r,s}(\nabla) \cong c_0$. □

Theorem 2.3. *The inclusions* $c_0(\nabla) \subseteq e_0^r(\nabla) \subseteq b_0^{r,s}(\nabla)$, $c(\nabla) \subseteq e_c^r(\nabla) \subseteq b_c^{r,s}(\nabla)$ *and* $\ell_\infty(\nabla) \subseteq e_\infty^r(\nabla) \subseteq b_\infty^{r,s}(\nabla)$ *strictly hold.*

Proof. Similarly, we only prove the inclusion $c_0(\nabla) \subseteq e_0^r(\nabla) \subseteq b_0^{r,s}(\nabla)$. By the Theorem 2.3 of Altay and Polat [3], we deduce that $c_0(\nabla) \subseteq e_0^r(\nabla)$ strictly holds. Now, we prove that $e_0^r(\nabla) \subseteq b_0^{r,s}(\nabla)$ holds. If $r + s = 1$, we have $E^r = B^{r,s}$. So $e_0^r(\nabla) \subseteq b_0^{r,s}(\nabla)$ holds. Let $0 < r < 1$ and $s = 4$. We define a sequence $x = (x_k)$ by $x_k = (-\frac{3}{r})^k$ for each $k \in \mathbb{N}$. It is clearly that $[E^r(\nabla x_k)]_n = (\frac{r+3}{r}(-2-r)^n) \notin c_0$ and $[B^{r,s}(\nabla x_k)]_n = (\frac{r+3}{r}(\frac{1}{4+r})^n) \in c_0$. So, we have $x \in b_0^{r,s}(\nabla) \setminus e_0^r(\nabla)$. This shows that the inclusion $e_0^r(\nabla) \subseteq b_0^{r,s}(\nabla)$ strictly holds. □

3. The Schauder Basis and α-, β- and γ-duals

For a normed space $(X, \|\cdot\|)$, a sequence $\{x_k : x_k \in X\}_{k\in\mathbb{N}}$ is called a *Schauder basis* [8] if for every $x \in X$, there is a unique scalar sequence (λ_k) such that $\|x - \sum_{k=0}^n \lambda_k x_k\| \to 0$, as $n \to \infty$. Next, we shall give a Schauder basis for the sequence spaces $b_0^{r,s}(\nabla)$ and $b_c^{r,s}(\nabla)$.

We define the sequence $g^{(k)}(r,s) = \{g_i^{(k)}(r,s)\}_{i\in\mathbb{N}}$ by

$$g_i^{(k)}(r,s) = \begin{cases} 0 & \text{if } 0 \le i < k, \\ (s+r)^k \sum_{j=k}^i \begin{pmatrix} j \\ k \end{pmatrix} r^{-j}(-s)^{j-k} & \text{if } i \ge k, \end{cases}$$

for each $k \in \mathbb{N}$.

Theorem 3.1. *The sequence* $(g^{(k)}(r,s))_{k\in\mathbb{N}}$ *is a Schauder basis for the binomial sequence space* $b_0^{r,s}(\nabla)$ *and every* $x = (x_i) \in b_0^{r,s}(\nabla)$ *has a unique representation by*

$$x = \sum_k \lambda_k(r,s) g^{(k)}(r,s), \tag{3.1}$$

where $\lambda_k(r,s) = [B^{r,s}(\nabla x_i)]_k$ *for each* $k \in \mathbb{N}$.

Proof. Obviously, $B^{r,s}(\nabla g_i^{(k)}(r,s)) = e_k \in c_0$, where e_k is the sequence with 1 in the kth place and zeros elsewhere for each $k \in \mathbb{N}$. This implies that $g^{(k)}(r,s) \in b_0^{r,s}(\nabla)$ for each $k \in \mathbb{N}$.

For $x \in b_0^{r,s}(\nabla)$ and $m \in \mathbb{N}$, we put

$$x^{(m)} = \sum_{k=0}^m \lambda_k(r,s) g^{(k)}(r,s).$$

By the linearity of $B^{r,s}(\nabla)$, we have

$$B^{r,s}(\nabla x_i^{(m)}) = \sum_{k=0}^m \lambda_k(r,s) B^{r,s}(\nabla g_i^{(k)}(r,s)) = \sum_{k=0}^m \lambda_k(r,s) e_k$$

and

$$[B^{r,s}(\nabla(x_i - x_i^{(m)}))]_k = \begin{cases} 0 & \text{if } 0 \leq k < m, \\ [B^{r,s}(\nabla x_i)]_k & \text{if } k \geq m, \end{cases}$$

for each $k \in \mathbb{N}$.

For any given $\varepsilon > 0$, there is a positive integer m_0 such that

$$| [B^{r,s}(\nabla x_i)]_k | < \tfrac{\varepsilon}{2}$$

for all $k \geq m_0$. Then we have

$$\| x - x^{(m)} \| = \sup_{k \geq m} | [B^{r,s}(\nabla x_i)]_k | \leq \sup_{k \geq m_0} | [B^{r,s}(\nabla x_i)]_k | < \frac{\varepsilon}{2} < \varepsilon,$$

which implies that $x \in b_0^{r,s}(\nabla)$ is represented as (3.1).

To show the uniqueness of this representation, we assume that

$$x = \sum_k \mu_k(r,s) g^{(k)}(r,s).$$

Then we have

$$[B^{r,s}(\nabla x_i)]_k = \sum_k \mu_k(r,s)[B^{r,s}(\nabla g_i^{(k)}(r,s))]_k = \sum_k \mu_k(r,s)(e_k)_k = \mu_k(r,s),$$

which is a contradiction with the assumption that $\lambda_k(r,s) = [B^{r,s}(\nabla x_i)]_k$ for each $k \in \mathbb{N}$. This shows the uniqueness of this representation. □

Theorem 3.2. *Let $g = (1,2,3,4,...)$ and $\lim_{k\to\infty} \lambda_k(r,s) = l$. The set $\{g, g^{(0)}(r,s), g^{(1)}(r,s), ..., g^{(k)}(r,s), ...\}$ is a Schauder basis for the space $b_c^{r,s}(\nabla)$ and every $x \in b_c^{r,s}(\nabla)$ has a unique representation by*

$$x = lg + \sum_k [\lambda_k(r,s) - l] g^{(k)}(r,s). \tag{3.2}$$

Proof. Obviously, $B^{r,s}(\nabla g_i^k(r,s)) = e^k \in c_0 \subseteq c$ and $g \in c(\nabla) \subseteq b_c^{r,s}(\nabla)$. For $x \in b_c^{r,s}(\nabla)$, we put $y = x - lg$ and we have $y \in b_0^{r,s}(\nabla)$. Hence, we deduce that y has a unique representation by (3.1), which implies that x has a unique representation by (3.2). Thus, we complete the proof. □

Corollary 3.1. *The sequence spaces $b_0^{r,s}(\nabla)$ and $b_c^{r,s}(\nabla)$ are separable.*

For the duality theory, the study of sequence spaces is more useful when we investigate them equipped with linear topologies. Köthe and Toeplitz [13] first computed the duals whose elements can be represented as sequences and defined the α-dual (or Köthe-Toeplitz dual). Next, we compute the α-, β- and γ-duals of the binomial sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$.

For the sequence spaces X and Y, define multiplier space $M(X,Y)$ by

$$M(X,Y) = \{u = (u_k) \in w : ux = (u_k x_k) \in Y \text{ for all } x = (x_k) \in X\}.$$

Then the α-, β- and γ-duals of a sequence space X are defined by

$$X^{\alpha} = M(X, \ell_1),\ X^{\beta} = M(X, c) \text{ and } X^{\gamma} = M(X, \ell_\infty),$$

respectively. Let us give the following properties:

$$\sup_{K\in\Gamma} \sum_n \mid \sum_{k\in K} a_{n,k} \mid < \infty \tag{3.3}$$

$$\sup_{n\in\mathbb{N}} \sum_k \mid a_{n,k} \mid < \infty \tag{3.4}$$

$$\lim_{n\to\infty} a_{n,k} = a_k \text{ for each } k \in \mathbb{N} \tag{3.5}$$

$$\lim_{n\to\infty} \sum_k a_{n,k} = a \tag{3.6}$$

$$\lim_{n\to\infty} \sum_k \mid a_{n,k} \mid = \sum_k \mid \lim_{n\to\infty} a_{n,k} \mid \tag{3.7}$$

where Γ is the collection of all finite subsets of $\mathbb{N}$.

Lemma 3.1.([16]) *Let $A = (a_{n,k})$ be an infinite matrix. Then the following statements hold:*

(i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ *if and only if (3.3) holds.*

(ii) $A \in (c_0 : c)$ *if and only if (3.4) and (3.5) hold.*

(iii) $A \in (c : c)$ *if and only if (3.4), (3.5) and (3.6) hold.*

(iv) $A \in (\ell_\infty : c)$ *if and only if (3.5) and (3.7) hold.*

(v) $A \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ *if and only if (3.4) holds.*

Theorem 3.3. *The α-dual of the spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ is the set*

$$U_1^{r,s} = \{u = (u_k) \in w : \sup_{I\in\Gamma} \sum_k \mid \sum_{i\in I} (s+r)^i \sum_{j=i}^{k} \binom{j}{i} r^{-j} (-s)^{j-i} u_k \mid < \infty\}.$$

Proof. Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we have

$$u_k x_k = \sum_{i=0}^{k} (s+r)^i \sum_{j=i}^{k} \binom{j}{i} r^{-j} (-s)^{j-i} u_k y_i = (G^{r,s} y)_k$$

for each $k \in \mathbb{N}$, where $G^{r,s} = (g^{r,s}_{k,i})$ is defined by

$$g^{r,s}_{k,i} = \begin{cases} (s+r)^i \sum_{j=i}^{k} \begin{pmatrix} j \\ i \end{pmatrix} r^{-j}(-s)^{j-i}u_k & \text{if } 0 \leq i \leq k, \\ 0 & \text{if } i > k. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in \ell_1$ whenever $x \in b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ or $b_\infty^{r,s}(\nabla)$ if and only if $G^{r,s}y \in \ell_1$ whenever $y \in c_0, c$ or ℓ_∞, which implies that $u = (u_k) \in [b_0^{r,s}(\nabla)]^\alpha, [b_c^{r,s}(\nabla)]^\alpha$ or $[b_\infty^{r,s}(\nabla)]^\alpha$ if and only if $G^{r,s} \in (c_0 : \ell_1)$ $G^{r,s} \in (c : \ell_1)$ or $G^{r,s} \in (\ell_\infty : \ell_1)$ by Part (i) of Lemma 3.1. So we obtain that

$$u = (u_k) \in [b_0^{r,s}(\nabla)]^\alpha = [b_c^{r,s}(\nabla)]^\alpha = [b_\infty^{r,s}(\nabla)]^\alpha$$

if and only if

$$\sup_{I \in \Gamma} \sum_k \mid \sum_{i \in I} (s+r)^i \sum_{j=i}^{k} \begin{pmatrix} j \\ i \end{pmatrix} r^{-j}(-s)^{j-i}u_k \mid < \infty.$$

Thus, we have $[b_0^{r,s}(\nabla)]^\alpha = [b_c^{r,s}(\nabla)]^\alpha = [b_\infty^{r,s}(\nabla)]^\alpha = U_1^{r,s}$. □

Now, we define the sets $U_2^{r,s}$, $U_3^{r,s}$, $U_4^{r,s}$ and $U_5^{r,s}$ by

$$U_2^{r,s} = \{u = (u_k) \in w : \sup_{n \in \mathbb{N}} \sum_k \mid u_{n,k} \mid < \infty\},$$
$$U_3^{r,s} = \{u = (u_k) \in w : \lim_{n \to \infty} u_{n,k} \text{ exists for each } k \in \mathbb{N}\},$$
$$U_4^{r,s} = \{u = (u_k) \in w : \lim_{n \to \infty} \sum_k \mid u_{n,k} \mid = \sum_k \mid \lim_{n \to \infty} u_{n,k} \mid\},$$

and

$$U_5^{r,s} = \{u = (u_k) \in w : \lim_{n \to \infty} \sum_k u_{n,k} \text{ exists}\},$$

where

$$u_{n,k} = (s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \begin{pmatrix} j \\ k \end{pmatrix} r^{-j}(-s)^{j-k}u_i.$$

Theorem 3.4. *We have the following relations:*

(i) $[b_0^{r,s}(\nabla)]^\beta = U_2^{r,s} \bigcap U_3^{r,s}$,

(ii) $[b_c^{r,s}(\nabla)]^\beta = U_2^{r,s} \bigcap U_3^{r,s} \bigcap U_5^{r,s}$,

(iii) $[b_\infty^{r,s}(\nabla)]^\beta = U_3^{r,s} \bigcap U_4^{r,s}$.

Proof. Let $u = (u_k) \in w$ and $x = (x_k)$ be defined by (2.3), then we consider the following equation

$$\begin{aligned}\sum_{k=0}^{n} u_k x_k &= \sum_{k=0}^{n} u_k[\sum_{i=0}^{k}(s+r)^i \sum_{j=i}^{k} \binom{j}{i} r^{-j}(-s)^{j-i} y_i] \\ &= \sum_{k=0}^{n}[(s+r)^k \sum_{i=k}^{n}\sum_{j=k}^{i} \binom{j}{k} r^{-j}(-s)^{j-k} u_i] y_k \\ &= (U^{r,s} y)_n\end{aligned}$$

where $U^{r,s} = (u^{r,s}_{n,k})$ is defined by

$$u_{n,k} = \begin{cases} (s+r)^k \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{j}{k} r^{-j}(-s)^{j-k} u_i & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore, we deduce that $ux = (u_k x_k) \in c$ whenever $x \in b_0^{r,s}(\nabla)$ if and only if $U^{r,s} y \in c$ whenever $y \in c_0$, which implies that $u = (u_k) \in [b_0^{r,s}(\nabla)]^\beta$ if and only if $U^{r,s} \in (c_0 : c)$ by Part (ii) of Lemma 3.1. So we obtain that $[b_0^{r,s}(\nabla)]^\beta = U_2^{r,s} \bigcap U_3^{r,s}$. Using Parts (iii), (iv) instead of (ii) of Lemma 3.1, the proof can be proved in the similar way. So, we omit the detail. □

Similarly, we give the following theorem without proof.

Theorem 3.5. *The γ-dual of the spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$ is the set $U_2^{r,s}$.*

4. Conclusion

By considering the definitions of the binomial matrix $B^{r,s} = (b^{r,s}_{n,k})$ and difference operator, we introduce the sequence spaces $b_0^{r,s}(\nabla)$, $b_c^{r,s}(\nabla)$ and $b_\infty^{r,s}(\nabla)$. These spaces are the natural continuation of [3, 6, 7]. Our results are the generalization of the matrix domain of the Euler matrix of order r. In order to give full knowledge to the reader on related topics with applications and a possible line of further investigation, the e-book[4] is added to the list of references.

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