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Coefficient Estimates for Sãlãgean Type λ-bi-pseudo-starlike Functions

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Abstract. In this paper, we have constructed subclasses of bi-univalent functions associated with $\lambda-$bi-pseudo-starlike functions in the unit disc U. Furthermore we established bound on the coefficients for the subclasses $S_{\Sigma}^{\lambda}(k,\alpha)$ and $S_{\Sigma}^{\lambda}(k,\beta)$.

1. Introduction

Let A denote the class of functions f which are analytic in the open unit disc $U=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$, of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Let S be the subclass of A consisting of the form (1.1) which are univalent in U. It is well known that every function $f\in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z))=z,\ (z\in U)$ and $f\left(f^{-1}(w)\right)=w,\ \left(|w|<r_0(f)\ ,\ r_0(f)\geq\frac{1}{4}\right)$, where

$$f^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots. \tag{1.2}$$

A function $f\in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disc U. For a brief history and interesting examples of functions in the class Σ, see the

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pioneering work on this area by Srivastava *et al.* [15], which has apparently revived the study of bi-univalent functions in recent years. From the work of Srivastava *et al.* [15], we recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

However, the familiar Koebe function is not a member of the bi-univalent function class Σ. Such other common examples of functions in S as

$$z-\frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [15]).

Historically, Lewin [11] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [12] showed that $\max|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [5] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leqq \beta < 1)$ in $\mathbb{U}$, respectively (see [12]). The classes $\mathcal{S}^\star_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\beta)$ of bi-starlike functions of order β in $\mathbb{U}$ and bi-convex functions of order β in $\mathbb{U}$, corresponding to the function classes $\mathcal{S}^\star(\beta)$ and $\mathcal{K}(\beta)$, were also introduced analogously. For each of the function classes $\mathcal{S}^\star_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\beta)$, they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned work on this area Srivastava *et al.* [15], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, [2, 6, 7, 17, 18, 19, 20, 21]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions (see, for example, [1, 8, 9]). The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N}\setminus\{1,2\};\ \mathbb{N}=\{1,2,3,\cdots\})$ is still an open problem.

For f belongs to A, Sãlãgean (see [14]) defined differential operator D^k, $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, by

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= Df(z) = zf'(z); \\ &\vdots \\ D^k f(z) &= D\left(D^{k-1}f(z)\right). \end{aligned}$$

We note that

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n.$$

In this paper, motivated by the earlier work of Babalola [3] and Joshi et. al. [10], we aim at introducing two new subclasses of the function class Σ and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [15] (see also [7, 16, 22, 23, 24]).

We note the following lemma required for obtaining our results.

Lemma 1.1.([13]) *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{1.3}$$

2. Coefficient Bounds for the Function Class $S_\Sigma^\lambda(k, \alpha)$

Definition 2.1. A function $f \in \Sigma$ is said to be *in the class* $S_\Sigma^\lambda(k, \alpha)$ if the following conditions are satisfied:

$$\left| \arg \left(\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \lambda \geq 1,\ \ z \in U) \tag{2.1}$$

and

$$\left| \arg \left(\frac{w\left[\left(D^k g(w)\right)'\right]^\lambda}{D^k g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \lambda \geq 1,\ w \in U) \tag{2.2}$$

where the function $g = f^{-1}$.

Theorem 2.2. *Let f given by (1.1) be in the class $S_\Sigma^\lambda(k, \alpha)$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda + \alpha - 1)}}$$

and

$$|a_3| \leq \frac{2\alpha}{(3\lambda - 1)} + \frac{4\alpha^2}{(2\lambda - 1)^2}.$$

Proof. Let $f \in S_\Sigma^\lambda(k, \alpha)$. Then

$$\frac{z\left[\left(D^k f(z)\right)'\right]^\lambda}{D^k f(z)} = [p(z)]^\alpha \tag{2.3}$$

$$\frac{w\left[\left(D^k g(w)\right)'\right]^{\lambda}}{D^k g(w)} = [q(w)]^{\alpha} \tag{2.4}$$

where $g = f^{-1}$, p, q in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots .$$

Now, equating the coefficients in (2.3) and (2.4), we get

$$(2\lambda - 1)\, 2^k a_2 = \alpha p_1, \tag{2.5}$$

$$(3\lambda - 1)\, 3^k a_3 + \left(2\lambda^2 - 4\lambda + 1\right) 2^{2k} a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.6}$$

and

$$-(2\lambda - 1)\, 2^k a_2 = \alpha q_1, \tag{2.7}$$

$$\left(2\lambda^2 + 2\lambda - 1\right) 2^{2k} a_2^2 - (3\lambda - 1)\, 3^k a_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{2.8}$$

From (2.5) and (2.7) we obtain

$$p_1 = -q_1 \tag{2.9}$$

and

$$(2\lambda - 1)^2\, 2^{2k+1} a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{2.10}$$

Also from (2.6), (2.8) and (2.10) we have

$$\begin{aligned}(2\lambda^2 - \lambda) 2^{2k+1} a_2^2 &= \alpha\,(p_2 + q_2) + \tfrac{\alpha(\alpha-1)}{2}(p_1^2 + q_1^2)\\ &= \alpha\,(p_2 + q_2) + \tfrac{\alpha(\alpha-1)}{2}\,\tfrac{(2\lambda-1)^2 2^{2k+1}}{\alpha^2} a_2^2.\end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2\,(p_2 + q_2)}{(2\lambda - 1)(2\lambda + \alpha - 1) 2^{2k}}. \tag{2.11}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we obtain

$$|a_2| \leq \frac{2^{1-k}\alpha}{\sqrt{(2\lambda - 1)(2\lambda + \alpha - 1)}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.8) from (2.6), we obtain

$$2\left(3\lambda-1\right)3^{k}a_{3}-2\left(3\lambda-1\right)2^{2k}a_{2}^{2}=\alpha\left(p_{2}-q_{2}\right)+\frac{\alpha(\alpha-1)}{2}(p_{1}^{2}-q_{1}^{2}).$$

Then, in view of (1.3) and (2.10), we have

$$|a_3|\leq\frac{2\alpha}{(3\lambda-1)3^k}+\frac{4\alpha^2}{(2\lambda-1)^2 2^{2k}}.$$

This completes the proof of Theorem 2.2. □

Putting $k=0$ in Theorem 2.2 , we have

Remark 2.3.([10]) Let f given by (1.1) be in the class $S_{\Sigma}^{\lambda}(\alpha),\ 0<\alpha\leq 1$. Then

$$|a_2|\leq\frac{2\alpha}{\sqrt{(2\lambda-1)(2\lambda+\alpha-1)}}$$

and

$$|a_3|\leq\frac{2\alpha}{(3\lambda-1)}+\frac{4\alpha^2}{(2\lambda-1)^2}.$$

3. Coefficient Bounds for the Function Class $S_{\Sigma}^{\lambda}(k,\beta)$

Definition 3.1. A function $f\in\Sigma$ is said to be in the class $S_{\Sigma}^{\lambda}(k,\beta)$ if the following conditions are satisfied:

(3.1) $$\Re\left(\frac{z\left[\left(D^k f(z)\right)'\right]^{\lambda}}{D^k f(z)}\right)>\beta\quad(0\leq\beta<1,\ \lambda\geq 1,\ z\in U)$$

and

(3.2) $$\Re\left(\frac{w\left[\left(D^k g(w)\right)'\right]^{\lambda}}{D^k g(w)}\right)>\beta\quad(0\leq\beta<1,\ \lambda\geq 1,\ w\in U)$$

where the function $g=f^{-1}$.

Theorem 3.2. *Let f given by (1.1) be in the class $S_{\Sigma}^{\lambda}(k,\beta),\ 0\leq\beta<1$. Then*

$$|a_2|\leq\sqrt{\frac{2\left(1-\beta\right)}{\lambda(2\lambda-1)2^{2k}}}$$

and

$$|a_3|\leq\frac{2(1-\beta)}{(3\lambda-1)3^k}+\frac{4(1-\beta)^2}{(2\lambda-1)^2 2^{2k}}.$$

Proof. Let $f \in S_{\Sigma}^{\lambda}(k,\beta)$. Then

$$\frac{z\left[\left(D^k f(z)\right)'\right]^{\lambda}}{D^k f(z)} = \beta + (1-\beta)p(z) \tag{3.3}$$

$$\frac{w\left[\left(D^k g(w)\right)'\right]^{\lambda}}{D^k g(w)} = \beta + (1-\beta)q(w) \tag{3.4}$$

where $p, q \in P$ and $g = f^{-1}$.

It follows from (3.3) and (3.4) that

$$(2\lambda-1)\, 2^k a_2 = (1-\beta)p_1, \tag{3.5}$$

$$(3\lambda-1)\, 3^k a_3 + \left(2\lambda^2 - 4\lambda + 1\right) 2^{2k} a_2^2 = (1-\beta)p_2, \tag{3.6}$$

and

$$-(2\lambda-1)\, 2^k a_2 = (1-\beta)q_1, \tag{3.7}$$

$$\left(2\lambda^2 + 2\lambda - 1\right) 2^{2k} a_2^2 - (3\lambda-1)\, 3^k a_3 = (1-\beta)q_2. \tag{3.8}$$

From (3.6) and (3.8) we obtain

$$p_1 = -q_1 \tag{3.9}$$

and

$$(2\lambda-1)^2\, 2^{2k+1} a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{3.10}$$

Also from (3.6), (3.8) and (3.9) we have

$$(2\lambda^2 - \lambda)2^{2k+1} a_2^2 = (1-\beta)\,(p_2 + q_2)\,.$$

Therefore, we have

$$a_2^2 = \frac{(1-\beta)\,(p_2+q_2)}{(2\lambda^2-\lambda)2^{2k+1}}. \tag{3.11}$$

Appyling Lemma 1.1 for the coefficients p_2 and q_2, we obtain

$$|a_2| \le \sqrt{\frac{2\,(1-\beta)}{\lambda(2\lambda-1)2^{2k}}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (3.8) from (3.6), we obtain

$$2\,(3\lambda-1)\,3^k a_3 - 2\,(3\lambda-1)\,2^{2k} a_2^2 = (1-\beta)\,(p_2-q_2)\,.$$

Then, in view of (1.3) and (3.11), we have

$$|a_3| \le \frac{2(1-\beta)}{(3\lambda-1)3^k} + \frac{4(1-\beta)^2}{(2\lambda-1)^2 2^{2k}}.$$

This completes the proof of Theorem 3.2. □

Putting $k = 0$ in Theorem 3.2, we have

Remark 3.3.([10]) Let f given by (1.1) be in the class $S_\Sigma^\lambda(\beta),\ 0 \le \beta < 1$. Then

$$|a_2| \le \sqrt{\frac{2\,(1-\beta)}{\lambda(2\lambda-1)}}$$

and

$$|a_3| \le \frac{2(1-\beta)}{(3\lambda-1)} + \frac{4(1-\beta)^2}{(2\lambda-1)^2}.$$

Taking $k = 0$ and $\lambda = 1$ in Theorems 2.2 and 3.2 one can get the following corollaries.

Corollary 3.4. *Let f given by (1.1) be in the class $S_\Sigma(\alpha),\ 0 < \alpha \le 1$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}}$$

and

$$|a_3| \le \alpha + 4\alpha^2.$$

Corollary 3.5. *Let f given by (1.1) be in the class $S_\Sigma(\beta),\ 0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{2\,(1-\beta)}$$

and

$$|a_3| \le (1-\beta) + 4(1-\beta)^2.$$

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