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On Semisimple Representations of the Framed g-loop Quiver

Jaeyoo Choy
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail : chovy@knu.ac.kr

Abstract. Let Q be the frame g-loop quiver, i.e. a generalized ADHM quiver obtained by replacing the two loops into g loops. The vector space $\mathbf{M}$ of representations of Q admits an involution $*$ if orthogonal and symplectic structures on the representation spaces are endowed. We prove equivalence between semisimplicity of representations of the $*$-invariant subspace $\mathbf{N}$ of $\mathbf{M}$ and the orbit-closedness with respect to the natural adjoint action on $\mathbf{N}$. We also explain this equivalence in terms of King's stability [8] and orthogonal decomposition of representations.

1. Introduction

Quiver representations mathematically formulate Yang-Mills instantons in four-dimensional $\mathcal{N} = 1, 2, 4$ gauge theory in physics context. A basic type of the theory is the ADHM data originally given by Atiyah, Drinfeld, Hitchin and Manin [1] to describe the self-dual equations of $\mathrm{SU}(N)$-instantons with instanton number k over the standard space-time $\mathbb{R}^4$. Donaldson [6] generalizes the ADHM description to the $\mathrm{USp}(N/2)$- and $\mathrm{SO}(N, \mathbb{R})$-instantons.

The ADHM data are quadruples of linear maps (B_1, B_2, i, j) in $\mathrm{End}(V)^{\oplus 2} \oplus \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(V, W)$, $V = \mathbb{C}^k$, $W = \mathbb{C}^N$ satisfying $\mu(B_1, B_2, i, j) := [B_1, B_2] + ij = 0$ in $\mathrm{End}(V)$. In other words they are the representations of the ADHM quiver satisfying the holomorphic moment map 0 equation $\mu = 0$ (see Figure 1). The coarse moduli space of such ADHM data is defined as the closed subscheme $\mu = 0$ of the (algebro-geometric) GIT quotient $\mathbf{M} /\!\!/ \mathrm{GL}(V)$. Therefore it parametrizes the closed $\mathrm{GL}(V)$-orbits of (B_1, B_2, i, j) with $[B_1, B_2] + ij = 0$.

In this paper we generalize the ADHM quiver in two ways: first g loops, secondly symmetry on the quiver representations. See Figure 2 for the framed 3-loop quiver. For symmetry we endow orthogonal or symplectic structures $(\ ,\ )_V, (\ ,\ )_W$ on V, W as follows: Let $(\ ,\ )_\varepsilon$ denote an orthogonal (resp. symplectic) structure on a vector

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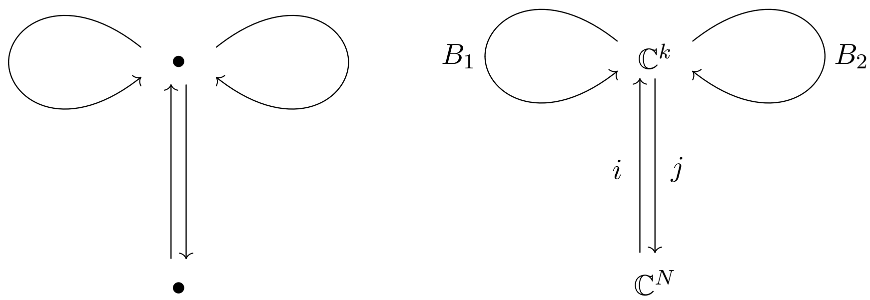

Figure 1: the ADHM quiver and a representation

space if $\varepsilon = +1$ (resp. $\varepsilon = -1$). Let $(\ ,\ )_V = (\ ,\ )_\varepsilon$, $(\ ,\ )_W = (\ ,\ )_{-\varepsilon}$. If V_1, V_2 are one of V, W, the right adjoint $f^* \in \mathrm{Hom}(V_2, V_1)$ of $f \in \mathrm{Hom}(V_1, V_2)$ is defined in a way that $(v_1, f^*(v_2))_{V_1} = (f(v_1), v_2)_{V_2}$. Now the symmetric representations of the framed g-loop quiver form a subspace $\mathbf{N}$ in $\mathbf{M}$, where

$$\mathbf{M} := \mathrm{End}(V)^{\oplus g} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W),$$
$$\mathbf{N} := \{(B_1, B_2, ..., B_g, i, j) |\, B_n = B_n^*, j = i^*, 1 \leq n \leq g\}.$$

Let us use an obvious isomorphism

$$\text{(1.1)} \quad c\colon \mathrm{End}(V)^{\oplus g} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \cong \mathrm{End}(V)^{\oplus g} \oplus V^{\oplus N} \oplus (V^\vee)^{\oplus N}$$

via identification $W = \mathbb{C}^N$. The latter vector space is the set of representations of the *deframed* quiver (see Figure 2). Let $G(V)$ be the subgroup of $\mathrm{GL}(V)$ preserving $(\ ,\ )_V$. Thus $G(V)$ is either $\mathrm{SO}(V)$ (if $\varepsilon = +1$) or $\mathrm{Sp}(V)$ (if $\varepsilon = -1$). The main theorem of the paper is the following:

Theorem 1.1. *Let* $x := (B_n, i, j)_{n=1,2,...,g} \in \mathbf{N}$. *Then the following are equivalent.*

(a) $c(x)$ *is semisimple (i.e. the direct sum of simple quiver representations).*

(b) *There exists a* $(\ ,\ )_\varepsilon$*-orthogonal decomposition*

$$V = V^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus}_a V_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus}_b (V_b \oplus V_b')$$

such that

(1) V_b *and* V_b' *are dual isotropic subspaces of* V *for each index* b*;*

(2) $c(B_n|_{V^s}, i, j|_{V^s})$, $(B_n|_{V_a}, 0, 0)$, $(B_n|_{V_b}, 0, 0)$ *and* $(B_n|_{V_b'}, 0, 0)$ *are simple quiver representations.*

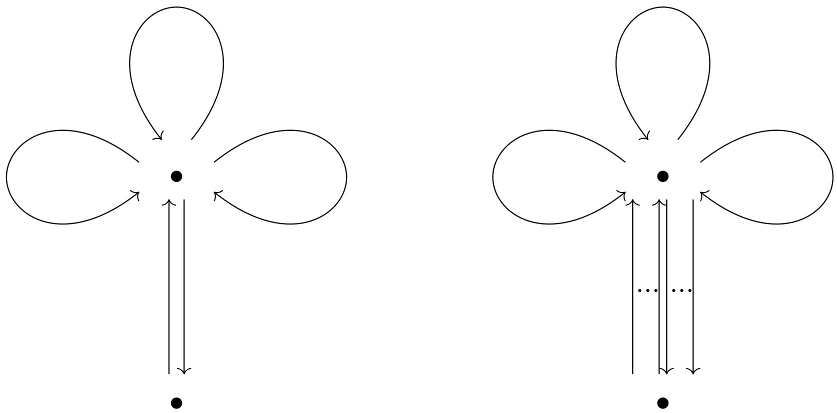

Figure 2: the framed g-loop quiver ($g = 3$) and its deframed quiver

(c) *The orbit* $G(V).x$ *is closed in* $\mathbf{N}$.

We give further motivation of the paper regarding the above theorem. The ADHM quiver arises also from two-dimensional conformal field theory over elliptic curves (cf. e.g. [11]). The two loops in the quiver represent the longitude and latitude of an elliptic curve, while the other two arrows do some matters in physics context. In general for even g, the framed g-loop quiver corresponds to conformal field theory over genus $g/2$ Riemann surfaces. On the other hand, when $g = 2$, the symmetry condition comes from Donaldson's realization of USp($N/2$)- and SO($N, \mathbb{R}$)-instantons. At this moment the author does not know the explicit meaning of symmetry in the case $g \geq 3$ in both gauge theory and conformal field theory. In contrast the antisymmetry (e.g. $B = -B^*$) corresponds to the conformal field theory of principal Sp- and SO-bundles over genus $g/2$ Riemann surfaces, a.k.a. the additive Deligne-Simpson problem [5].

In the case $g = 2$, the above theorem appears in [2, Theorem 3.1] in context of USp($N/2$)- and SO($N, \mathbb{R}$)-instantons, and also in a published article [3, Theorem 2.2] without proof.

Contents of the paper. The generality of quiver representations, in particular correspondence between semisimple representations and closed orbits, is reviewed in Section 2. Representations of the framed g-loop quiver are studied in Section 3. In this quiver case, semisimplicity amounts to King's stability (and costability) for the deframed quiver. In Section 4, we prove the main theorem.

2. Semisimple Quiver Representations

2.1. Generality of Quiver Representations

Let Q be any finite quiver. Let I and E be the sets of vertices and arrows of Q respectively. For $a \in E$, let $t(a)$ and $h(a)$ be the tail and head vertex respectively, i.e.

$$t(a) \bullet \xrightarrow{a} \bullet\, h(a)$$

Let us assign to a finite dimensional vector space V_v for each vertex $v \in I$. Let $\mathbf{V} := \bigoplus_{v \in I} V_v$. Frequently we denote $\mathbf{V}$ by (V_v) for short if there is no confusion for the vertex set I. Let

$$\mathbf{M_V} := \bigoplus_{a \in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)}).$$

Definition 2.1. An element of $\mathbf{M_V}$ is called a *representation of Q with the representation space* $\mathbf{V}$.

A representation $\mathbf{B}$ of Q is written as $\mathbf{B} = (B_a)_{a \in E} \in \mathbf{M_V}$. Let $\mathbf{M_{V'}}$ be another vector space of representations of Q, where $\mathbf{V}' := (V'_v)$. Let $\mathbf{B}' = (B'_a)_{a \in E} \in \mathbf{M_{V'}}$.

Definition 2.2. A *homomorphism* $\sigma\colon \mathbf{B} \to \mathbf{B}'$ is a collection $(\sigma_v)_{v \in I}$ such that $\sigma_v\colon V_v \to V'_v$ are linear maps satisfying the commutativity of the following diagram:

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{B_a} & V_{h(a)} \\ \downarrow{\scriptstyle \sigma_{t(a)}} & & \downarrow{\scriptstyle \sigma_{h(a)}} \\ V'_{t(a)} & \xrightarrow{B'_a} & V'_{h(a)} \end{array}$$

for each $a \in E$.

The category of the objects $\mathbf{B}$ is an abelian category. We use the notation $\mathbf{B}' \subset \mathbf{B}$ as a subrepresentation if $V'_v \subset V_v$ and $B_a|_{V'_{t(a)}} = B'_a$ for all $v \in I$ and $a \in E$. So a subrepresentation of $\mathbf{B}$ always comes from a $\mathbf{B}$-invariant subspace $\mathbf{V}'$ of $\mathbf{V}$.

2.2. Semisimple Quiver Representations and Closed Orbits

Definition 2.3. A nonzero representation $\mathbf{B}$ of Q is called *simple* if any subrepresentation of $\mathbf{B}$ is either $\mathbf{B}$ itself or 0. A direct sum of simple representations is called a *semisimple* representation.

Let $\mathrm{GL}(\mathbf{V}) := \prod_{v \in I} \mathrm{GL}(V_v)$. Then, $\mathrm{GL}(\mathbf{V})$ acts on $\mathbf{M_V}$ by

$$(g_v).(B_a) := (g_{h(a)} B_a g_{t(a)}^{-1}).$$

The following lemma is well-known. To the author's best knowledge, this is first mentioned in [9] without proof.

Lemma 2.4. *Let* $\mathbf{V} \neq 0$ *and* $\mathbf{B} \in \mathbf{M}_{\mathbf{V}}$*. Then* $\mathbf{B}$ *is semisimple if and only if the orbit* $\mathrm{GL}(\mathbf{V}).\mathbf{B}$ *is closed in* $\mathbf{M}_{\mathbf{V}}$*.*

We will not prove the lemma itself, but prove an essential ingredient of the proof:

Lemma 2.5. *Let* $\mathbf{V} \neq 0$ *and* $\mathbf{B} \in \mathbf{M}_{\mathbf{V}}$ *be semisimple. Let* $\mathbf{W}$ *and* $\mathbf{W}'$ *be* $\mathbf{B}$*-invariant subspaces of* $\mathbf{V}$*. Then there exists a* $\mathbf{B}$*-invariant* $\mathbf{W}''$ *such that* $\mathbf{W} = \mathbf{W}' \oplus \mathbf{W}''$*. Hence any nonzero subquotient of* $\mathbf{B}$ *is semisimple.*

Proof. We may assume $\mathbf{W} \neq 0$. Let $\mathbf{B}_{\mathbf{W}} := \mathbf{B}|_{\mathbf{W}}$ and $\mathbf{B}_{\mathbf{W}'} := \mathbf{B}|_{\mathbf{W}'}$. Let $\mathbf{B} = \bigoplus_{m=1}^{n} \mathbf{B}^m$ be a decomposition by simple representations. We denote by $\mathbf{V}^m$ the representation space of $\mathbf{B}^m$. Let $[n] := \{1, 2, ..., n\}$ for short.

We use the induction on n. If $n = 1$, then $\mathbf{B}$ is simple and the claim is obvious.

Let $p_S : \mathbf{V} \to \oplus_{m\in S}\mathbf{V}^m$ be the projection where S is any nonempty subset of $[n]$. We denote by $p_{S*} : \mathbf{B} \to \bigoplus_{m\in S} \mathbf{B}^m$ the induced homomorphism by the projection. Note that there exists $n_0 \in [n]$ such that $p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}}) \neq 0$, since otherwise $\mathbf{W} = 0$. We denote by $\hat{n}_0 := [n] \setminus n_0$. It is clear that $\mathrm{Ker}(p_{\hat{n}_0 *})$ is equal to $\mathbf{B}^{n_0}$ and thus simple. So if $p_{\hat{n}_0 *}|_{\mathbf{B}_{\mathbf{W}}}$ is non-injective for all $n_0 \in [n]$ then we have $\mathbf{B}_{\mathbf{W}} = \mathbf{B}$ and thus $\mathbf{W} = \mathbf{V}$.

We assume first that $\mathbf{W} \neq \mathbf{V}$. Then there exists $n_0 \in [n]$ such that $\mathbf{B}_{\mathbf{W}} \cong p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}})$ via $p_{\hat{n}_0 *}$. Now we have subrepresentations

$$p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}'}) \subset p_{\hat{n}_0 *}(\mathbf{B}_{\mathbf{W}}) \subset \mathrm{Im}(p_{\hat{n}_0 *}) = \bigoplus_{m\in\hat{n}_0} \mathbf{B}^m.$$

By the induction on n, there exists a $\left(\bigoplus_{m\in\hat{n}_0} \mathbf{B}^m\right)$-invariant subspace ${}''\mathbf{W}$ in $p_{\hat{n}_0}(\mathbf{W})$ complementary to $p_{\hat{n}_0}(\mathbf{W}')$. We set $\mathbf{W}''$ to be the pull-back of ${}''\mathbf{W}$ via $p_{\hat{n}_0}|_{\mathbf{W}}$. We are done in the case $\mathbf{W} \neq \mathbf{V}$.

We need to prove the case when $\mathbf{W} = \mathbf{V}$ (i.e., $\mathbf{B}_{\mathbf{W}} = \mathbf{B}$). We assume $\mathbf{B}_{\mathbf{W}'} \neq \mathbf{B}$ since otherwise we set $\mathbf{W}'' = 0$. By a similar argument as before, there exists $n_0 \in [n]$ such that $\mathbf{B}_{\mathbf{W}'} \cong p_{\hat{n}_0}(\mathbf{B}_{\mathbf{W}'})$ via $p_{\hat{n}_0 *}$. And there exists a $\left(\bigoplus_{m\in\hat{n}_0} \mathbf{B}^m\right)$-invariant subspace ${}''\mathbf{W}$ in $\bigoplus_{m\in\hat{n}_0} \mathbf{V}^m$ complementary to $p_{\hat{n}_0}(\mathbf{W}')$. By setting $\mathbf{W}'' := {}''\mathbf{W} \oplus \mathbf{V}^{n_0}$, we are done. □

Let $\lambda \in \mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(\mathbf{V}))$ where $\mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(\mathbf{V}))$ denotes the set of group homomorphisms. Write $\lambda(t) = (\lambda_v(t))_{v\in I}$ where $\lambda_v \in \mathrm{Hom}(\mathbb{C}^*, \mathrm{GL}(V_v))$. Let $\mathbf{V}_{\mathrm{wt}n} := \bigoplus_{v\in I}(V_v)_{\mathrm{wt}n}$ where $(V_v)_{\mathrm{wt}n} := \{a \in V_v |\, \lambda_v(t).a = t^n a,\, t \in \mathbb{C}^*\}$ (the weight n subspace). The decreasing filtration by weight defines a graded vector space

$$\mathrm{gr}^{\lambda}\mathbf{V} := \bigoplus_{v\in I}\left(\bigoplus_{n}(V_v)_{\mathrm{wt}\geq n}/(V_v)_{\mathrm{wt}\geq n+1}\right).$$

Lemma 2.6. *Let* $\mathbf{B} \in \mathbf{M}_{\mathbf{V}}$*. The following two are equivalent:*

(i) $\mathbf{B}_0 := \lim_{t\to 0}\lambda(t).\mathbf{B}$ *exists in* $\mathbf{M}$.

(ii) $\mathbf{V}_{\mathrm{wt}\geq n}$ *is* $\mathbf{B}$*-invariant for each* $n \in \mathbb{Z}$.

Hence if $\mathbf{B}_0$ *exists,* $\mathbf{B}_{\mathrm{wt}\geq n} := \mathbf{B}|_{\mathbf{V}_{\mathrm{wt}\geq n}}$ *is a subrepresentation of* $\mathbf{B}$ *and the obvious isomorphism* $\mathbf{V} \cong \mathrm{gr}^{\lambda}\mathbf{V}$ *induces*

$$\mathbf{B}_0 \cong \bigoplus_{n\in\mathbb{Z}} \mathbf{B}_{\mathrm{wt}\geq n-1}/\mathbf{B}_{\mathrm{wt}\geq n}.$$

Proof. If B_v maps $(V_v)_{\mathrm{wt}n}$ to $(V_v)_{\mathrm{wt}\geq n+1}$ but not to $(V_v)_{\mathrm{wt}n}$, $\lambda(t).B_v$ has a weight -1 summand in the weight decomposition of $\mathrm{End}(V_v)$, which implies $\lim_{t\to 0}\lambda(t).B_v$ does not exist. This proves the implication from (ii) to (i).

The converse is similar. □

Lemmas 2.5 and 2/6 immediately yield the following corollary.

Corollary 2.7. *If* $\mathbf{B}$ *is semisimple, so is* $\mathbf{B}_0$ (*if it exists*).

3. Representations of the Framed g-loop Quiver

3.1. King's Stability

We set Q to be the framed g-loop quiver in the rest of the paper. There are precisely two vertices in Q. The upper one (resp. lower one) in Figure 2 is called *loop* vertex (resp. *frame* vertex).

Definition 3.1. Let x be a representation of the deframed quiver of Q. If the representation space of x at the frame vertex is $\mathbb{C}$, we say x is *of frame* $\mathbb{C}$. If the representation space of x at the frame vertex is 0, we say x is *of frame* 0.

Let us recall the two GIT stability conditions for the $\mathrm{GL}(V)$-actions on

$$\mathbf{M} := \mathrm{End}(V)^{\oplus g} \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W)$$

due to A. King [8] (see also [10, Chap. 2]). We also recall the obvious isomorphism c in (1.1) which maps $x \in \mathbf{M}$ to a representation of the deframed quiver of Q.

Remark 3.2. The modification from x to $c(x)$ is called Crawley-Boevey's trick [4, p.57].

Definition 3.3. $x = (B_n, i, j) \in \mathbf{M}$ is *stable* (resp. *costable*) if the following condition holds:

1. (*stability*) there is no proper subrepresentation of frame $\mathbb{C}$ of $c(x)$,
2. (*costability*) there is no nonzero subrepresentation of frame 0 of $c(x)$.

Remark 3.4. The stability of x amounts to that there is no subspace $S \subsetneq V$ such that $B_n(S) \subset S$ and $\mathrm{Im} i \subset S$. The costability of x amounts to that there is no subspace $T \neq 0 \subset V$ such that $B_n(T) \subset T$ and $T \subset \mathrm{Ker} j$.

Lemma 3.5. *Suppose $W \neq 0$. Then $x \in \mathbf{M}$ is stable and costable if and only if $c(x)$ is simple.*

Proof. Since any subrepresentation of $c(x)$ is either of frame 0 or of frame $\mathbb{C}$, the stability-costability is equivalent to that $c(x)$ is simple. □

Let

$$*_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{M}, \quad x = (B_n, i, j) \mapsto x^* := (B_n^*, -j^*, i^*).$$

Note that $*_{\mathbf{M}}$ is an involution and that $\mathbf{N}$ is the $*_{\mathbf{M}}$-invariant subspace of $\mathbf{M}$. The following lemma is obvious.

Lemma 3.6. *If $x \in \mathbf{M}$ is stable (resp. costable) then x^* is costable (resp. stable). Hence stability and costability are equivalent on $\mathbf{N}$. In particular, $x \in \mathbf{N}$ is stable if and only if $c(x)$ is simple when $W \neq 0$.*

3.2. Decomposition of Semisimple Quiver Representations of the Framed g-loop Quiver

In this subsection we give a statement equivalent to (b) in terms of orthogonal decomposition of a representation $x \in \mathbf{N}$ into simple subrepresentations. As we will see, there are three types of simple subrepresentations.

Lemma 3.7. *Let $x \in \mathbf{N}$. Let $V' \subset V$ and $W' \subset W$ be subspaces such that (V', W') is x-invariant. Then $(V'^{\perp}, W'^{\perp})$ is x-invariant.*

Proof. Let $x = (B_n, i, i^*)$. We need to show $B_n((V')^{\perp}) \subset (V')^{\perp}$, $i((W')^{\perp}) \subset (V')^{\perp}$ and $i^*((V')^{\perp}) \subset (W')^{\perp}$. For $v \in (V')^{\perp}, v' \in V'$, we have $(B_n(v), v')_V = (v, B_n(v'))_V = 0$ so that $B_n((V')^{\perp}) \subset (V')^{\perp}$.

For $v \in V'$ and $w \in (W')^{\perp}$, we have $(i(w), v)_V = (w, i^*(v))_W = 0$ so that $i((W')^{\perp}) \subset (V')^{\perp}$.

For $v \in (V')^{\perp}$ and $w \in W'$, we have $(w, i^*(v))_W = (i(w), v)_V = 0$ so that $i^*((V')^{\perp}) \subset (W')^{\perp}$. □

For $x \in \mathbf{N}$ and a subrepresentation $y = x|_{(V', W')}$, we denote the subrepresentation in the above lemma by

$$y^{\perp} := x|_{(V'^{\perp}, W'^{\perp})}.$$

Definition 3.8. A subspace of V is *nondegenerate* (resp. *isotropic*) if the restriction of $(\ ,\ )_V$ is nondegenerate (resp. 0). Two isotropic subspaces are *dual*, if their direct sum is nondegenerate.

Let V', V'' be subspaces of V. Let $y \in \mathbf{M}_{(V',0)}$ and $z \in \mathbf{M}_{(V'',0)}$. We say y is *nondegenerate* (resp. *isotropic*) if so is V'. We say y, z are *dual isotropic* if so are V', V''.

Let y, z be any subrepresentations of $x \in \mathbf{N}$. We write $x = y \overset{\perp}{\oplus} z$, if y and z satisfies $x = y \oplus z$ and $y^{\perp} = z$.

Lemma 3.9. *Let y be a simple subrepresentation of $x \in \mathbf{N}$. If y is of frame 0, it is either nondegenerate or isotropic.*

Proof. By Lemma 3.7 and the simplicity of y, $T \cap T^{\perp} = 0$ or T. In the first case (resp. the second case), $(\ ,\ )_T$ is nondegenerate (resp. 0). □

Lemma 3.10. *Let y, z be any subrepresentations of frame* 0 *of $x \in \mathbf{N}$. If y and z are dual isotropic then $y = z^*$.*

Proof. Let us write $y = (B'_n, 0, 0)$, $z = (B''_n, 0, 0)$ and $\tilde{B}_n := B'_n \oplus B''_n$. Since $\tilde{B}_n = \tilde{B}^*_n$, we have $(B'_n v', v'')_V = (\tilde{B}_n v', v'')_V = (v', \tilde{B}_n v'')_V = (v', B''_n v'')_V$ where $v' \in V'$ and $v'' \in V''$. Thus $B'_n = B''^*_n$. □

Definition 3.11. Let $x = (B_n, i, j) \in \mathbf{N}$. We define

$$V^s := \sum_P P(B_1, B_2, ..., B_g)i(W)$$

where P runs over all the g-variable polynomials. Let

$$x^s := x|_{(V^s,W)} \in \mathbf{M}_{(V^s,W)}.$$

It is clear that $c(x^s)$ is stable because the stability does not change under c.

Lemma 3.12. *Let $x \in \mathbf{N}$. If $c(x)$ is semisimple, V^s is nondegenerate.*

Proof. Let $Z := V^s \cap (V^s)^{\perp}$. By Lemma 3.7, Z is an x^s-invariant subspace and thus we have a subrepresentation $z := x|_{(Z,0)}$ of x^s. By Lemma 2.5, $c(x^s)$ is semisimple and there exists $z' \subset c(x^s)$ such that $c(x^s) = z \oplus z'$. Since z is of frame 0, z' is of frame $\mathbb{C}$ and thus $z' = c(x^s)$. This implies $z = 0$ and thus $Z = 0$. This proves V^s is nondegenerate. □

For the actual proof of Theorem 1.1, it is convenient to replace (b) into the following equivalent statement:

(b') x is decomposed as

$$x = x^s \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_b} (z_b \oplus z'_b),$$

where $c(x^s)$ and all the other summands are simple and moreover z_b, z'_b are dual isotropic.

We also assume for convenience that the indices b are arranged as $1, 2, 3, ...$ consecutively in $\mathbb{Z}_{\geq 1}$ unless empty.

4. Proof of Theorem 1.1

4.1. Proof of the Equivalence of (a) and (b)

It is clear that (b) implies (a). We prove the opposite direction (a)⇒(b').

By Lemma 3.12, we have $x = x^s \overset{\perp}{\oplus} (x^s)^\perp$. Our goal is to find $y_a, z_b, z'_b \subset (x^s)^\perp$ satisfying the statement (b'). Let $z \subset (x^s)^\perp$ be a simple subrepresentation. Since z is of frame 0, it is either nondegenerate or isotropic by Lemma 3.9. We assume first that z is nondegenerate. Let $y_1 := z$. Since $c(z^\perp)$ is semisimple (Lemma 2.5), we are done by the induction on $\dim V$. We assume that z is isotropic.

Lemma 4.1. *If z is isotropic, there exists a subrepresentation z' of frame 0 which is dual isotropic to z, simple and $z' \subset (x^s)^\perp$.*

Proof. By Lemma 3.12, V^s and thus $(V^s)^\perp$ are nondegenerate. Since $(x^s)^\perp$ is semisimple it has a simple direct summand $w := (B_n|_T, 0, 0)$ such that T is not orthogonal to Z. Thus $Z \neq Z \cap T^\perp$ and $T \neq T \cap Z^\perp$. By Lemma 3.7 and simplicity of z, w, we have $Z \cap T^\perp = T \cap Z^\perp = 0$. This means $(\ ,\ )_V$ on $Z \times T$ is nondegenerate.

If T is isotropic, then we are done by $z' := x|_{(T,0)}$.

If T is not isotropic, then it is nondegenerate by Lemma 3.9. By simplicity of z, w, we have $Z \cap T = 0$. Hence $Z \oplus T$ is nondegenerate.

We will find a B_n-invariant isotropic subspace Z' in $Z \oplus T$ which is complementary to Z. Let T' be the orthogonal complement of T in $Z \oplus T$. Since $(\ ,\ )_V$ is nondegenerate on $Z \times T$, we have $Z \cap T' = 0$.

By Lemma 3.7, we have a subrepresentation $w' := x|_{(T',0)}$. Let $p_1 \colon T \oplus T' \to T$ and $p_2 \colon T \oplus T' \to T'$ be the projections. Since $Z \cap T = Z \cap T' = 0$, $p_1|_Z$ and $p_2|_Z$ induce isomorphisms $z \cong w$ and $z \cong w'$ respectively. So we have $w \cong w'$. Let $h \in \mathrm{Isom}(T, T')$ such that $h.w' = w$. Then we have $Z = \{(v, h(v)) |\, v \in T\}$. This means $hB_n(v) = B_n h(v)$, where $v \in T$.

Let

$$Z' := \{(v, -h(v)) | v \in T\}.$$

Since $-hB_n = -B_n h$, we have $B_n(Z') \subset Z'$. Let $z' := x|_{(Z',0)}$. Since $((v, -h(v)), (v', -h(v'))_V = ((v, h(v)), (v', h(v'))_V = 0$ for $v, v' \in V$, Z' (and thus z') is isotropic. Since $Z \cap Z' = 0$, $Z \oplus Z'$ is nondegenerate.

It remains to show that z' is simple. Since $z' \cong w$ by the isomorphism induced by $p_1|_{Z'}$, we are done. □

Let $z_1 := z$ and $z'_1 := z'$ where z' is given as in Lemma 4.1. By the induction on $\dim V$, we have the decomposition

$$(z_1 \oplus z'_1)^\perp = x^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{b \neq 1}} (z_b \oplus z'_b).$$

As a result we have

$$x = x^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_a} y_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{b \geq 1}} (z_b \oplus z'_b).$$

This proves (b'). □

4.2. Proof of (a,b) to (c)

The strategy is to show that whenever $x_0 := \lim_{t\to 0}\lambda(t).x$ exists for any given group homomorphism $\lambda \in \mathrm{Hom}(\mathbb{C}^*, G)$, it is contained in the orbit $G(V).x$. Then Iwahori's theorem [7] asserts that $G(V).x$ is closed.

Let λ, x_0 be given as above. Let h be the highest weight. Then $h \geq 0$ and $-h$ is the lowest height, which are left as an exercise. If $h = 0$ then λ is trivial. So we may assume $h > 0$. Then $x|_{(V_{\mathrm{wt}h},0)}$ is a subrepresentation of x.

Let Z be a subspace $V_{\mathrm{wt}h}$ such that $x|_{(Z,0)}$ is simple. Let $z := x|_{(Z,0)}$. It is a subrepresentation of both x, x_0. We showed in the above proof of (a)⇒(b') that there exists a decomposition of x as in (b') with $z_1 = z$. Similarly there exists a decomposition of x_0 with $w_1 = z$ as in (b'):

$$x_0 = x_0^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\alpha}} u_\alpha \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\beta\geq 1}} (w_\beta \oplus w'_\beta).$$

Here we used that x_0 is semisimple (Corollary 2.7). We denote by $V_0^s, U_\alpha, W_\beta, W'_\beta$ (the factor of) the representation spaces of $x_0^s, u_\alpha, w_\beta, w'_\beta$ respectively.

The canonical isomorphism $V^s \oplus \bigoplus_a Y_a \oplus \bigoplus_{b\neq 1}(Z_b \oplus Z'_b) \overset{\cong}{\to} Z^\perp/Z$ is a symplectic isomorphism and induces an isomorphism

$$x^s \oplus \bigoplus_a y_a \oplus \bigoplus_{b\neq 1}(z_b \oplus z'_b) \cong z^\perp/z. \tag{4.1}$$

We need a similar isomorphism for a limit of $z^\perp/z$ with respect to λ. Let us define such a limit first. Since $\lambda(t)(Z) \subset Z$ and $\lambda(t)(Z^\perp) \subset Z^\perp$ for any $t \in \mathbb{C}^*$, we have the induced group homomorphism $\overline{\lambda}(t) \in \mathrm{Hom}(\mathbb{C}^*, G(Z^\perp/Z))$. The existence of the limit x_0 asserts $(z^\perp/z)_0 := \lim_{t\to 0}\overline{\lambda}(t).z^\perp/z$ also exists in $\mathbf{M}_{(Z^\perp/Z,W)}$. Now the canonical isomorphism $V_0^s \oplus \bigoplus_\alpha U_\alpha \oplus \bigoplus_{\beta\neq 1}(W_\beta \oplus W'_\beta) \overset{\cong}{\to} Z^\perp/Z$ is a symplectic isomorphism and induces an isomorphism

$$x_0^s \oplus \bigoplus_\alpha u_\alpha \oplus \bigoplus_{\beta\neq 1}(w_\beta \oplus w'_\beta) \cong (z^\perp/z)_0. \tag{4.2}$$

By the dimension induction we have $(z^\perp/z)_0 \in G(Z^\perp/Z).z^\perp/z$. Therefore by (4.1) and (4.2), there exists a $(\ ,\ )_V$-preserving isomorphism

$$f\colon V^s \oplus \bigoplus_a Y_a \oplus \bigoplus_{b\neq 1}(Z_b \oplus Z'_b) \overset{\cong}{\to} V_0^s \oplus \bigoplus_\alpha U_\alpha \oplus \bigoplus_{\beta\neq 1}(W_\beta \oplus W'_\beta)$$

such that

$$x_0^s \oplus \bigoplus_\alpha u_\alpha \oplus \bigoplus_{\beta\neq 1}(w_\beta \oplus w'_\beta) = f.\left(x^s \oplus \bigoplus_a y_a \oplus \bigoplus_{b\neq 1}(z_b \oplus z'_b)\right).$$

Now it suffices to prove that there exists a $(\ ,\ )_V$-preserving $f' \in \mathrm{Isom}(Z_1 \oplus Z_1', W_1 \oplus W_1')$ such that $f'.(z_1 \oplus z_1') = w_1 \oplus w_1'$. For, we will have $f \oplus f' \in G(V)$ and $x_0 = f.x$ so that Iwahori's theorem asserts $G(V).x$ is closed in $\mathbf{M}$. The proof comes from that z and z' are dual. More precisely it is reduced to the following lemma:

Lemma 4.2. *Let (L, L') and (M, M') be pairs of dual isotropic subspaces of V. Let $B \in \mathrm{End}(L)$ and $f \in \mathrm{Isom}(L, M)$. Then we have the following:*

(1) *There exists a unique $\tilde{B} \in \mathfrak{p}(L \oplus L')$ such that $\tilde{B}(L) \subset L$, $\tilde{B}(L') \subset L'$ and $\tilde{B}|_L = B$.*

(2) *There exists a unique form-preserving $\widetilde{f} \in \mathrm{Isom}(L \oplus L', M \oplus M')$ such that $\widetilde{f}(L) = M$, $\widetilde{f}(L') = M'$ and $\widetilde{f}|_L = g$.*

Proof. We omit the proof. □

This completes the proof of (a,b')⇒(c). □

4.3. Proof of (c) to (a)

We will prove (a) semisimplicity assuming (c) the orbit-closedness.

Let x' be a maximal direct summand of x such that the representation space of x' is either $(V', 0)$ or (V', W) for some $V' \subset V$ and $c(x')$ is 0 or semisimple. Note that such x' is unique, because if x'' is another one, $x' + x''$ is a also direct summand of x with $c(x' + x'')$ being semisimple. Here we used the canonical isomorphism $x' + x'' \cong (x' \oplus x'')/(x' \cap x'')$ and the splitting of $x' + x'' \subset x$ by the composite $x \to x' \oplus x'' \to (x' \oplus x'')/(x' \cap x'')$.

We need to prove that x' is equal to x. We suppose the contrary $x' \neq x$. We divide the proof into the cases: (i) V' is nondegenerate, (ii) V' is not nondegenerate.

Case (i). We have a decomposition $x = x' \oplus (x')^\perp$. Let z be a subrepresentation of x with the representation space $(Z, 0)$ or (Z, W) for some $Z \subset V$ such that $c(z)$ is simple. If z has the representation space $(Z, 0)$, it is either nondegenerate or isotropic by Lemma 3.9.

If z is nondegenerate we have decomposition $x = x' \oplus z \oplus (z^\perp \cap (x')^\perp)$. This is absurd to maximality of x'.

Suppose z is isotropic. Let $\lambda \in \mathrm{Hom}(\mathbb{C}^*, G(V))$ such that

$$V_{\mathrm{wt}1} = Z, \quad V_{\mathrm{wt}\geq 0} = z^\perp, \quad V_{\mathrm{wt}\geq -1} = V.$$

The limit $x_0 = \lim_{t\to 0} \lambda(t).x$ is isomorphic to $z \oplus z^\perp/z \oplus x/z^\perp$. Note that x' is a direct summand of $z^\perp/z$. Since x is isomorphic to x_0 by the orbit-closedness, it has a direct summand isomorphic to $z \oplus x'$. This is absurd to maximality of x'.

If z has the representation space (Z, W), Z is nondegenerate since $z \cap z^\perp = 0$. Thus we have decomposition $x = x' \oplus z \oplus (z^\perp \cap (x')^\perp)$. This is absurd to maximality of x'.

Case (ii). In the case, $V' \cap (V')^\perp$ is a nonzero isotropic subspace of V. Let $z := x' \cap (x')^\perp$. Then z is a semisimple isotropic representation of frame 0. Let

$\lambda \in \mathrm{Hom}(\mathbb{C}^*, G(V))$ such that

$$V_{\mathrm{wt}1} = Z, \quad V_{\mathrm{wt}\geq 0} = z^{\perp}, \quad V_{\mathrm{wt}\geq -1} = V.$$

The limit $x_0 = \lim_{t\to 0} \lambda(t).x$ is isomorphic to $z \oplus z^{\perp}/z \oplus x/z^{\perp}$. Note that x'/z is a direct summand of $z^{\perp}/z$ and that $x/z^{\perp}$ and z are dual to each other (so that $x/z^{\perp}$ is semisimple). Since x is isomorphic to x_0 by the orbit-closedness, it has a direct summand isomorphic to $z \oplus x'/z \oplus x/z$. This is absurd to maximality of x'.

This finishes the proof of $x' = x$. □

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