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On Graded 2-Absorbing and Graded Weakly 2-Absorbing Primary Ideals

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Abstract. Let G be an arbitrary group with identity e and let R be a G-graded ring. In this paper, we define the concept of graded 2-absorbing and graded weakly 2-absorbing primary ideals of commutative G-graded rings with non-zero identity. A number of results and basic properties of graded 2-absorbing primary and graded weakly 2-absorbing primary ideals are given.

1. Introduction

Prime ideals (resp,. primary ideals) play an important role in commutative ring theory. Also, weakly prime ideals have been introduced and studied by D. D. Anderson and E. Smith in [1] and weakly primary ideals have been introduced and studied by S. E. Atani and F. Farzalipour in [2]. Later, A. Badawi in [6] generalized the concept of prime ideals in a different way. He defined a non-zero proper ideal I of commutative ring R to be a 2-*absorbing ideal*, if whenever $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. This concept has a generalization that is called *weakly* 2-*absorbing ideal*, which has studied by A. Badawi and A. Y. Darani in [9]. A proper ideal I of R is said to be a *weakly* 2-*absorbing ideal*, if whenever $a, b, c \in R$ with $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Recently, A. Badawi, U. Tekir and E. Yetkin in [7] and [8] have generalized the

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concepts of primary and weakly primary ideals. A proper ideal I of R is said to be a 2-*absorbing* (*resp., weakly* 2-*absorbing*) *primary ideal*, if whenever $a, b, c \in R$ such that $abc \in I$ (resp, $0 \neq abc \in I$), then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. A proper ideal I of R is said to be an *irreducible ideal*, if for ideals J, K of R with $I = J \cap K$, then $I = J$ or $I = K$. In [12], H. Mostafanasab and A. Y. Darani defined a new characterization of irreducible ideals of a commutative ring. So let R be a commutative ring and I be a proper ideal. Then I is called a 2-*irreducible ideal*, if for ideals $J, K, L \in R$ with $I = J \cap K \cap L$, then either $I = J \cap K$ or $I = K \cap L$ or $I = J \cap L$. For further study on these concepts see (c.f [3], [4], [13],[16] and [17]). In this paper we define *graded* 2-*absorbing ideals* and *graded* 2-*absorbing* (*resp,. weakly* 2-*absorbing*) *primary ideals* of a G-graded commutative ring.

Throughout this work all rings are commutative G-graded ring with non-zero identity such that G is a non-finitely generated abelian group unless explicitly mentioned otherwise. Before we state some results, let us introduce some notations. Let G be an arbitrary monoid with identity e. Then R is a G-*graded ring*, if there exists additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. It is denoted by $G(R)$. The elements of R_g are called *homogeneous of degree* g, where R_g are additive subgroups of R indexed by element $g \in G$. Here $R_g R_h$ denotes the additive subgroup of $G(R)$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in R_h$. We consider $SuppG(R) = \{g \in G | R_g \neq 0\}$. If $a \in R$, then a can be written uniquely as $\sum_{g \in G} a_g$, where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g \in G} R_g$. Let I be an ideal of $G(R)$. Then I is a *graded ideal* of $G(R)$ if $I = \bigoplus_{g \in G}(I \cap R_g)$. Clearly $\bigoplus_{g \in G}(I \cap R_g) \subseteq I$. Hence I is a graded ideal of $G(R)$ if $I \subseteq \bigoplus_{g \in G}(I \cap R_g)$. For $g \in G$, let $I_g = I \cap R_g$. Then I is a graded ideal of $G(R)$ if $I = \bigoplus_{g \in G} I_g$. In this case, I_g is called the g-*component* of I for $g \in G$. Let R be a G-graded ring and I be a graded ideal of $G(R)$. *The quotient ring* R/I is a G-graded ring. Indeed, $R/I = \bigoplus_{g \in G}(R/I)_g$ where $R/I = \bigoplus_{g \in G}(R/I)_g = \{a + I | a \in R_g\}$. Suppose that $R = \bigoplus_{g \in G} R_g$ and $S = \bigoplus_{g \in G} S_g$ are two graded rings. Then a mapping $f : R \longrightarrow S$ with $f(1_R) = 1_S$ is called a *graded homomorphism* (*G-homomorphism*), if $f(R_g) \subseteq S_g$. A subset $S \subseteq h(R)$ is said to be *multiplicatively closed subset*, if for any $a, b \in S$ implies that $ab \in S$. Assume that R is a G-graded ring and $S \subseteq h(R)$ is a multiplicatively closed subset R. Then the ring of fraction $S^{-1}R$ is a graded ring and said to be *the graded ring of fraction*. Indeed, $S^{-1}R = \bigoplus_{g \in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s | r \in R, s \in S \ \ and \ \ g = (degs)^{-1}(degr)\}$. We write $h(S^{-1}R) = \bigcup_{g \in G}(S^{-1}R)_g$. Let $\varphi : R \longrightarrow S^{-1}R$ be a G-homomorphism with $\varphi(r) = r/1$. Then for any graded ideal I of $G(R)$, the graded ideal of $S^{-1}R$ generated by $\varphi(I)$, that is denoted by $S^{-1}I$. It is easy to see that $S^{-1}R$ is a graded ring and for every ideal I of R, $S^{-1}I$ is an ideal of $S^{-1}R$ and $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \emptyset$. Indeed, $S^{-1}I = \{\lambda \in S^{-1}R | \lambda = r/s \ \ for \ \ r \in I \ \ and \ \ s \in S\}$. If $\mathfrak{J}$ is a graded ideal of $S^{-1}R$, then $\mathfrak{J} \cap R$ will denote the graded ideal $\varphi(\mathfrak{J})$ of R. Furthermore, if R_1 is a G_1-graded ring and R_2 is a G_2-graded ring, then $R = R_1 \times R_2$ is a $G = G_1 \times G_2$-graded ring. Indeed, $(R_1 \times R_2)_{(g,h)} = \{(a_g, b_h) | a_g \in R_{1g} \ \ and \ \ b \in R_{2h}\}$

for $(g,h) \in G_1 \times G_2$ and $h(R) = h(R_1) \times h(R_2)$. Here a graded ideal of $G = G_1 \times G_2$-graded ring $R = R_1 \times R_2$ is $I = I_1 \times I_2$ such that I_1 is a graded ideal of $G_1(R_1)$ and I_2 is a graded ideal of $G_2(R_2)$.

A graded ideal I of $G(R)$ is said to be a *graded prime* (*G-prime*) (resp., *graded weakly prime*) *ideal*, if $I \neq R$ and whenever $a, b \in h(R)$ with $ab \in I$ ($0 \neq ab \in I$), then either $a \in I$ or $b \in I$. *The graded radical* of I, which is denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of $G(R)$, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$. It is easy to see that if I is a graded ideal of $G(R)$, then $Gr(I)$ is a graded ideal of $G(R)$, [17, Proposition 2.3]. A graded ideal I of $G(R)$ is said to be *graded primary* (*G-primary*) (resp,. *graded weakly primary*) *ideal*, if $I \neq R$ and whenever $a, b \in h(R)$ such that $ab \in I$ ($0 \neq ab \in I$), then either $a \in I$ or $b \in Gr(I)$. If $Gr(I)$ is a G-prime ideal, then I is called a *graded P-primary* (*G-P-primary*) *ideal*. Let I be a proper ideal of $G(R)$. Then I is called a *G-irreducible graded ideal*, if whenever $I_1, I_2 \in R(G)$ such that $I = I_1 \cap I_2$, then either $I = I_1$ or $I = I_2$, (c.f [3], [4] and [16]).

2. Properties of Graded 2-Absorbing Primary Ideals

In this section we will define the concept of graded 2-absorbing primary ideals and study some basic properties.

Definition 2.1. Let I be a graded ideal of $G(R)$.

(1) I is called a *graded 2-absorbing ideal of $G(R)$* (*denoted by* G(2)-absorbing ideal), if I is a proper graded ideal of $G(R)$ and whenever $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $bc \in I$ or $ac \in I$.

(2) I is called a *graded 2-absorbing primary ideal of $G(R)$* (*denoted by* G(2)-absorbing primary ideal), if I is a proper graded ideal of $G(R)$ and whenever $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$.

In the following we show that some straightforward results.

Lemma 2.2. *Let R be a G-graded ring.*

(1) *Every G-prime ideal is a $G(2)$-absorbing ideal of $G(R)$;*

(2) *Every G-primary ideal is a $G(2)$-absorbing primary ideal of $G(R)$;*

(3) *Every $G(2)$-absorbing ideal is a $G(2)$-absorbing primary ideal of $G(R)$.*

Example 2.3. Let $G = \mathbb{Z}_2$, $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Then $R = \mathbb{Z}[i] = \{a+bi | a, b \in \mathbb{Z}\}$ is a G-graded ring. Assume that $I =< 6 > \oplus < 0 >$ is a graded ideal of $G(R)$. Hence I is a $G(2)$-absorbing primary ideal, but it is not a G-primary ideal. This example shows that a $G(2)$-absorbing primary ideal of a G-graded ring R is not necessarily a G-primary ideal of $G(R)$.

Example 2.4. Let $G = \mathbb{Z}_2$, $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Then $R = \mathbb{Z}[i] = \{a+bi|a, b \in \mathbb{Z}\}$ is a G-graded ring. Let $J =< 12 > \oplus < 0 >$. Then J is a graded ideal of $G(R)$ and a $G(2)$-absorbing primary ideal of $G(R)$. Although J is not a $G(2)$-absorbing ideal of $G(R)$. Since $(2, 0)(2, 0)(3i, 0) \in J$ but $(2, 0)(2, 0) \notin J$ and $(2, 0)(3i, 0) \notin Gr(J)$. This example shows that a $G(2)$-absorbing primary ideal of a G-graded ring R need not be a $G(2)$-absorbing ideal of $G(R)$.

Theorem 2.5. *Let R be a G-graded ring. Then the intersection of each pair of distinct G-prime (resp., G-primary) ideals of $G(R)$ is a $G(2)$-absorbing (resp., $G(2)$-absorbing primary) ideal.*

Proof. Assume that I_1 and I_2 are two distinct G-prime ideals of $G(R)$. Suppose that $a, b, c \in h(R)$ such that $abc \in I_1 \cap I_2$ with $ab \notin I_1 \cap I_2$ and $ac \notin I_1 \cap I_2$. Now we can consider two cases. **Case one,** assume that $ab \notin I_1$ and $ac \notin I_1$. Since $abc \in I_1$ and I_1 is a G-prime ideal, we can conclude that $c \in I_1$ and so $bc \in I_1$. A similar argument for the case $b \in I_1$ implies that $bc \in I_1$. **Case two,** let $ab \notin I_2$ and $ac \notin I_2$. Since $abc \in I_2$ and I_2 is a G-prime ideal, we conclude that $bc \in I_2$. Hence $bc \in I_1 \cap I_2$ and then $I_1 \cap I_2$ is a $G(2)$-absorbing ideal, as required. □

In the following result, we show that under certain conditions, a $G(2)$-absorbing primary ideal is a $G(2)$-absorbing ideal.

Theorem 2.6. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If I is a G-primary ideal and R/I has no non-zero homogeneous nilpotent element, then I is a $G(2)$-absorbing ideal.*

Proof. Assume that $abc \in I$ for some $a, b, c \in h(R)$ with $ab \notin I$ and $bc \notin I$. Since I is a G-primary ideal and $abc \in I$, we conclude that $c \in Gr(I)$ and $a \in Gr(I)$. Then $(ac)^n \in I$ for some positive integer n. Since R/I has no non-zero homogeneous nilpotent element, $ac \in I$ and then I is $G(2)$-absorbing. □

Theorem 2.7. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If I is a $G(2)$-absorbing primary ideal and R/I has no non-zero homogeneous nilpotent element, then I is a $G(2)$-absorbing ideal.*

Proof. Assume that $abc \in I$ for some $a, b, c \in h(R)$ with $ab \notin I$. Since I is a $G(2)$-absorbing primary ideal and $abc \in I$, we conclude that $bc \in Gr(I)$ or $ac \in Gr(I)$. Then $(bc)^n \in I$ or $(ac)^n \in I$ for some positive integer n. Since R/I has no non-zero homogeneous nilpotent element, $bc \in I$ or $ac \in I$ and then I is $G(2)$-absorbing. □

Lemma 2.8. *Let I and J be graded ideals of $G(R)$.*

(1) $I \subseteq Gr(I)$;

(2) $Gr(Gr(I)) = Gr(I)$;

(3) $Gr(I) = R$ *if and only if* $I = R$;

(4) $Gr(IJ) = Gr(I \cap J) = Gr(I) \cap Gr(J)$;

(5) *If P is a G-prime ideal of $G(R)$, then $Gr(P^n) = P$ for all $n > 0$.*

Proof. [17, Proposition 2.4]. □

Proposition 2.9. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. Then $Gr(I)$ is a $G(2)$-absorbing ideal if and only if $Gr(I)$ is a $G(2)$-absorbing primary ideal.*

Proof. By Lemma 2.8, $Gr(Gr(I)) = Gr(I)$. Hence the proof is straightforward. □

Theorem 2.10. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If I is a $G(2)$-absorbing primary ideal, then $Gr(I)$ is a $G(2)$-absorbing ideal of $G(R)$.*

Proof. Assume that $a, b, c \in h(R)$ such that $abc \in Gr(I)$ with $bc \notin Gr(I)$ and $ac \notin Gr(I)$. Then there exists $n \in \mathbb{N}$ such that $(abc)^n = a^n b^n c^n \in I$. Since I is a $G(2)$-absorbing primary, $(bc)^n \notin I$ and $(ac)^n \notin I$, we can conclude that $a^n b^n = (ab)^n \in I$ and then $ab \in Gr(I)$. Hence $Gr(I)$ is a $G(2)$-absorbing ideal of $G(R)$. □

Example 2.11. Assume that R is a $\mathbb{Z}$-graded ring. Let K be a field and $R = K[X, Y]$ with $degX = 1 = degY$. Suppose that $I = (X^2, XY)$ is a graded ideal. Then I is a G-prime ideal and $Gr(I) = (X)$. However, I is not a G-P-primary ideal, [16, p.220], that is a $G(2)$-absorbing primary ideal. Since for $X.Y.X = XYX \in I$, either $X.Y = XY \in I$ or $Y.X = XY \in Gr(I)$ or $X.X = X^2 \in Gr(I)$. This example shows that a $G(2)$-absorbing primary ideal of a G-graded ring R is not necessarily a G-P-primary ideal.

Theorem 2.12. *Let R be a G-graded ring, I_1, I_2 be graded ideals and P_1, P_2 be G-prime ideals of $G(R)$. Suppose that I_1 is a G-P_1-primary ideal of $G(R)$ and I_2 is a G-P_2-primary ideal of $G(R)$. Then the following statements hold:*

(1) *$I_1 I_2$ is a $G(2)$-absorbing primary ideal;*

(2) *$I_1 \cap I_2$ is a $G(2)$-absorbing primary ideal.*

Proof. (1) Assume that $a, b, c \in h(R)$ with $abc \in I_1 I_2$ but $ac, bc \notin Gr(I_1 I_2)$. Since $Gr(I_1 I_2) = P_1 \cap P_2$, we get that $a, b, c \notin Gr(I_1 I_2) = P_1 \cap P_2$ and $Gr(I_1 I_2) = P_1 \cap P_2$ is a 2-absorbing ideal of R, by Theorem 2.5. Then $ab \in Gr(I_1 I_2)$. Now it is enough that is shown $ab \in I_1 I_2$. Since $ab \in Gr(I_1 I_2) = P_1 \cap P_2 \subseteq P_1$, we can conclude $a \in P_1$. On the other hand, $a \notin Gr(I_1 I_2)$ and $ab \in Gr(I_1 I_2) = P_1 \cap P_2 \subseteq P_2$ and $a \in P_1$, then we conclude that $a \notin P_2$ and $b \in P_2$. Furthermore, $b \in P_2$ and $b \notin Gr(I_1 I_2)$, so we have $b \notin P_1$. Hence $a \in P_1$ and $b \in P_2$. Now if $a \in I_1$ and $b \in I_2$, then $ab \in I_1 I_2$. So we can assume that $a \notin I_1$. Then as I_1 is a G-P_1-primary ideal of R, we have $bc \in P_1 = Gr(I_1)$. Since $b \in P_2$, we conclude that $bc \in Gr(I_1 I_2)$, which is a contradiction. Hence $a \in I_1$. By similar way, we get that $b \in I_2$. Therefore $ab \in I_1 I_2$ and so $I_1 I_2$ is a $G(2)$-absorbing primary ideal.
(2) Assume that $a, b, c \in h(R)$ with $abc \in I_1 \cap I_2$ but $bc \notin Gr(I_1 \cap I_2)$ and $ac \notin Gr(I_1 \cap I_2)$. Since I_1 is a G-P_1-primary ideal and I_2 is a G-P_2-primary ideal, we have $Gr(P) := Gr(I_1 \cap I_2) = P_1 \cap P_2$. Then $a, b, c \notin Gr(P) = P_1 \cap P_2$. Now the proof is completely similar to that of part (1). □

Corollary 2.13. *If P_1^n is a G-P_1-primary and P_2^m is a G-P_2-primary ideals for every $n, m \geq 1$, then $P_1^n P_2^m$ and $P_1^n \cap P_2^m$ are $G(2)$-absorbing primary ideals of $G(R)$.*

Theorem 2.14. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If I is a $G(2)$-absorbing primary ideal of $G(R)$, then $I \cap R_e$ is a $G(2)$-absorbing primary ideal of R_e.*

Proof. Assume that $a, b, c \in R_e$ such that $abc \in I \cap R_e$. Since I is a $G(2)$-absorbing primary ideal, we get that $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Then $ab \in I \cap R_e$ or $bc \in Gr(I) \cap R_e$ or $ac \in Gr(I) \cap R_e$. Since R_e is a subring of $G(R)$ and $1 \in R_e$. Hence $I \cap R_e$ is a $G(2)$-absorbing primary ideal of R_e. □

Recall that M is a graded maximal (G-maximal) ideal of G-graded ring R, if $M \neq R$ and there is no graded ideal I of $G(R)$ such that $M \subset I \subset R$. Let R be a G-graded ring and $I \neq R$ be a graded ideal of $G(R)$. If $Gr(I) = M$ is a G-maximal ideal of $G(R)$, then I is a G-M-primary ideal of $G(R)$, [16, Proposition 1.11]. In the following result we can use it to characterize $G(2)$-absorbing primary ideals.

Theorem 2.15. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If $Gr(I)$ is a G-prime ideal, then I is a $G(2)$-absorbing primary ideal. Assuming further that $Gr(I)$ is a G-primary ideal, then I is a $G(2)$-absorbing primary ideal.*

Proof. Assume that $a, b, c \in h(R)$ such that $abc \in I$ with $ab \notin I$. Since $abc^2 = (ac)(bc) \in Gr(I)$ and $Gr(I)$ is a G-prime ideal, we conclude that $ac \in Gr(I)$ or $bc \in Gr(I)$. Hence I is a $G(2)$-absorbing primary ideal of $G(R)$. Suppose now $Gr(I)$ is a G-primary ideal. Clearly $Gr(I)$ is a G-prime ideal. Thus I is so. □

Corollary 2.16. *If P is a G-prime ideal of $G(R)$, then P^n is a $G(2)$-absorbing primary ideal of $G(R)$ for each positive integer n.*

In view of Theorem 2.10, the following is a definition and more results.

Definition 2.17. Let I be a $G(2)$-absorbing primary ideal of $G(R)$. If $Gr(I) = P$ is a $G(2)$-absorbing ideal as it is shown in Theorem 2.10, we say I is a *G-P-2-absorbing primary ideal of $G(R)$.*

Theorem 2.18. *Let $I_1, \cdots, I_r$ be G-P-2-absorbing primary ideals of $G(R)$ where P is a $G(2)$-absorbing ideal of $G(R)$. Then $I = \bigcap_{i=1}^n I_i$ is a G-P-2-absorbing primary ideal of $G(R)$.*

Proof. Assume that $abc \in I$ for some $a, b, c \in h(R)$ and $ab \notin I$. Then $ab \notin I_i$ for some $1 \leq i \leq n$. Since $abc \in I_i$ and I_i is a G-P-2-absorbing primary ideal for every $1 \leq i \leq n$, we can conclude that $bc \in Gr(I_i) = P$ or $ac \in Gr(I_i) = P$. Hence I is a G-P-2-absorbing primary ideal of $G(R)$. □

Let R be a G-graded ring. The ring is called a Gr-Noetherian ring if it satisfies the ascending chain condition on graded ideals of $G(R)$. Let I be a proper graded ideal of $G(R)$. A graded primary G-decomposition of I is an intersection finitely many graded primary ideals of $G(R)$.

Proposition 2.19. *Let R be a G-graded ring. If R is a Gr-Noetherian ring, then the following statements holds:*

(1) *Every proper graded ideal of $G(R)$ is a $G(2)$-absorbing primary ideal of $G(R)$;*

(2) *Every proper graded ideal of $G(R)$ has a graded 2-absorbing primary G-decomposition.*

Proof. (1) Assume that I is a G-P-primary ideal of $G(R)$. As $Gr(I) = P$, we conclude that I is a $G(2)$-absorbing primary ideal of R, by Theorem 2.15.
(2) Assume that I is a graded ideal of $G(R)$. Since R is a Gr-Noetherian ring, I has a graded primary decomposition, by [16, Corollary 2.16]. Then I has a graded 2-absorbing primary G-decomposition. □

Definition 2.20. Let R be a G-graded ring and I be a proper graded ideal of $G(R)$. We say I is a graded 2-irreducible ideal ($G(2)$-irreducible ideal), if whenever for ideals J, K and L of $G(R)$ with $I = J \cap K \cap L$, then $I = J \cap K$ or $I = K \cap L$ or $I = J \cap L$.

Theorem 2.21. *Let R be a Gr-Noetherian ring. If I is a $G(2)$-irreducible ideal of $G(R)$, then either I is a G-irreducible or I is the intersection of two G-irreducible ideals.*

Proof. Assume that I is a $G(2)$-irreducible ideal of $G(R)$. Then I can be an intersection of finitely many G-irreducible ideals of $G(R)$, say $I = I_1 \cap I_2 \cap \cdots \cap I_n$, by [16, Proposition 2.14]. Assume that $n > 2$. Since I is $G(2)$-irreducible, we may assume that $I = I_i \cap I_j$ for some $1 \leq i, j \leq n$. Say $i = 1$ and $j = 2$. Thus $I_1 \cap I_2 \subseteq I_3$, which is a contradiction. Hence $n = 1$ or $n = 2$, as needed. □

Theorem 2.22. *Let R be a Gr-Noetherian ring. If I is a $G(2)$-irreducible ideal of $G(R)$, then I is a $G(2)$-absorbing primary ideal.*

Proof. Since I is a $G(2)$-irreducible ideal, I is G-irreducible, by Theorem 2.21. Then I is a G-primary ideal of $G(R)$, by [16, Proposition 2.15]. Since every G-primary ideal is a $G(2)$-absorbing primary and the intersection of two G-primary ideals is a $G(2)$-absorbing primary ideal, by Lemma 2.2 and Theorem 2.5, we get that I is $G(2)$-absorbing primary. □

Theorem 2.23. *Let R be a G-graded ring and I, J be graded ideals of $G(R)$ with $J \subseteq I$. If I is a $G(2)$-absorbing primary ideal of $G(R)$, then I/J is a $G(2)$-absorbing primary ideal of R/J.*

Proof. Assume that $(a+J)(b+J)(c+J) = abc + J \in I/J$ for some $a, b, c \in h(R)$. Then $abc \in I$. Since I is a $G(2)$-absorbing primary ideal, we conclude that $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Hence $ab + J \in I/J$ or $(bc)^n + J = (bc+J)^n \in I/J$ or $(ac)^n + J = (ac+J)^n \in I/J$ for some positive integer n. It implies that $bc + J \in Gr(I/J)$ or $ac + J \in Gr(I/J)$ and so I/J is a $G(2)$-absorbing primary ideal of R/J. □

Let R be a G-graded ring. *A graded zero-divisor on* R is an element $r \in h(R)$ such that $rs = 0$ for some $0 \neq s \in h(R)$. The set of all graded zero-divisor on R is denoted by $Gz(R)$.

The following result and its proof is an analogue of [7, Theorem 2.22].

Theorem 2.24. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of homogeneous element R. Then the following statements hold:*

(1) *If I is a $G(2)$-absorbing primary ideal of $G(R)$ with $S \cap I = \emptyset$, then $S^{-1}I$ is a $G(2)$-absorbing primary ideal of $S^{-1}R$;*

(2) *If $S^{-1}I$ is a $G(2)$-absorbing primary ideal of $S^{-1}R$ with $S \cap Gz(R/I) = \emptyset$, then I is a $G(2)$-absorbing primary ideal of R.*

Proof. (1) Assume that $a, b, c \in R$ and $r, s, t \in S$ such that $(a/r)(b/s)(c/t) \in S^{-1}I$. Then there exists $u \in S$ such that $(ua)bc \in I$. Since I is a $G(2)$-absorbing primary ideal of $G(R)$, we conclude that $(ua)b \in I$ or $bc \in Gr(I)$ or $(ua)c \in Gr(I)$. If $(ua)b \in I$, then $(a/r)(b/s) = uab/urs \in S^{-1}I$. If $bc \in Gr(I)$, then $(b/s)(c/t) \in S^{-1}(Gr(I)) = Gr(S^{-1}I)$. If $(ua)c \in Gr(I)$, then $(a/r)(c/t) = uac/urt \in S^{-1}(Gr(I)) = Gr(S^{-1}I)$. Hence $S^{-1}I$ is a $G(2)$-absorbing primary ideal of $S^{-1}R$.
(2) Assume that $a, b, c \in h(R)$ such that $abc \in I$. Then $(a/1)(b/1)(c/1) \in S^{-1}I$. Since $S^{-1}I$ is a $G(2)$-absorbing primary ideal of $S^{-1}R$, we conclude that $(a/1)(b/1) \in S^{-1}I$ or $(b/1)(c/1) \in Gr(S^{-1}I)$ or $(a/1)(c/1) \in Gr(S^{-1}I)$. If $(a/1)(b/1) \in S^{-1}I$, then there exists $t \in S$ such that $tab \in I$. Since $S \cap Gz(R/I) = \emptyset$, we can conclude that $ab \in I$. If $(b/1)(c/1) \in Gr(S^{-1}I) = S^{-1}(Gr(I))$, then there exists $s \in S \subseteq h(R)$ such that $(sbc)^n = s^n b^n c^n \in I$ for some positive integer n. Thus $b^n c^n \in I$ and then $bc \in Gr(I)$. Since $s \in S$ and $s^n \notin Gz(R/I)$. By the similar argument for the case $(a/1)(c/1) \in Gr(S^{-1}I)$, we get that $ac \in Gr(I)$. Hence I is a $G(2)$-absorbing primary ideal of $G(R)$. □

As it is well-known, if $R = R_1 \oplus R_2$ is a commutative ring with identity where R_i is a commutative ring (for $i = 1, 2$), then I is a 2-absorbing primary ideal of R if and only if either $I = P_1 \oplus R_2$ for some 2-absorbing primary ideal P_1 of R_1 or $I = R_1 \oplus P_2$ for some 2-absorbing primary ideal P_2 of R_2 or $I = P_1 \oplus P_2$ for some primary ideal P_1 of R_1 and some primary ideal P_2 of R_2, [7, Theorem 2.23]. We have the following result.

Theorem 2.25. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring (for $i = 1, 2$). Then the following statements hold:*

(1) *I_1 is a $G_1(2)$-absorbing primary ideal of $G_1(R_1)$ if and only if $I_1 \times R_2$ is a $G(2)$-absorbing primary ideal of $G(R)$;*

(2) *I_2 is a $G_2(2)$-absorbing primary ideal of $G_2(R_2)$ if and only if $R_1 \times I_2$ is a $G(2)$-absorbing primary ideal of $G(R)$.*

Proof. (1) Let I_1 be a $G_1(2)$-absorbing primary ideal of $G_1(R_1)$. Assume that $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in h(R_1)\times h(R_2)$ such that $(a_1,b_1)(a_2,b_2)(a_3,b_3) \in I_1 \times R_2$. Then $a_1a_2a_3 \in I_1$. Since I_1 is $G_1(2)$-absorbing primary, we conclude that $a_1a_2 \in I_1$ or $a_2a_3 \in Gr(I_1)$ or $a_1a_3 \in Gr(I_1)$. Hence $(a_1,b_1)(a_2,b_2) \in I_1 \times R_2$ or $(a_2,b_2)(a_3,b_3) \in Gr(I_1 \times R_2) = Gr(I_1)\times R_2$ or $(a_1,b_1)(a_3,b_3) \in Gr(I_1\times R_2) = Gr(I_1)\times R_2$. Conversely, suppose that $I_1 \times R_2$ is a $G(2)$-absorbing primary ideal of $G(R)$. Let I_1 is not a $G(2)$-absorbing primary ideal of $G_1(R_1)$. Then there exists $a,b,c \in h(R_1)$ with $abc \in I_1$ but neither $ab \in I_1$ nor $bc \in Gr(I_1)$ nor $ac \in Gr(I_1)$. Since $(a,1)(b,1)(c,1) \in I_1 \times R_2$ and $I_1 \times R_2$ is a $G(2)$-absorbing primary ideal of $G(R)$, we conclude that $(a,1)(b,1) \in I_1 \times R_2$ or $(b,1)(c,1) \in Gr(I_1 \times R_2) = Gr(I_1)\times R_2$ or $(a,1)(c,1) \in Gr(I_1\times R_2) = Gr(I_1)\times R_2$. Hence $ab \in I_1$ or $bc \in Gr(I_1)$ or $ac \in Gr(I_1)$, which is a contradiction. Therefore I_1 is $G(2)$-absorbing primary.
(2) The proof is similar part(1), so we omit that. □

Theorem 2.26. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_1, R_2 are G_1-graded ring and G_2-graded ring respectively with $rank(G_i) = 2$ (for $i = 1,2$). Suppose that I_1 and I_2 are distinct graded ideals of $G_1(R_1)$ and $G_2(R_2)$ respectively. Then the following statements are equivalent:*

(1) *I is a $G(2)$-absorbing primary ideal of $G(R)$;*

(2) *$I = I_1 \times R_2$ for some $G_1(2)$-absorbing primary ideal I_1 of $G_1(R_1)$ or $I = R_1 \times I_2$ for some $G_2(2)$-absorbing primary ideal I_2 of $G_2(R_2)$ or $I = I_1 \times I_2$ for some G_1-primary ideal I_1 of $G_1(R_1)$ and some G_2-primary ideal I_2 of $G_2(R_2)$.*

Proof. (1) $\Rightarrow$ (2) Let $g_i \in G_1$ and $h_i \in G_2$ for $i = 1,2$. Assume that $I_1 = P_{g_1} \oplus P_{g_2}$ and $I_2 = Q_{h_1} \oplus Q_{h_2}$. If $I_2 \neq R_2$, then I is $G(2)$-absorbing primary, by Theorem 2.25(2). If $I_1 \neq R_1$, then I is $G(2)$-absorbing primary, by Theorem 2.25(1). Suppose that neither $I_1 = R_1$ nor $I_2 = R_2$. Assume that I_1 is not a G_1-primary ideal of $G_1(R_1)$. So there exists $a,b \in h(R_1)$ such that $ab \in I_1$ where $a = (a_{g_1}, a_{g_2})$ and $b = (b_{g_1}, b_{g_2})$ but $a \notin I_1$ and $b^n \notin I_1$ for a smallest positive integer $n > 1$. Assume that $x,y,z \in h(R)$. Then there is $x = \big((a_{g_1}, a_{g_2}),(1,1)\big)$, $y = \big((1,1),(0,0)\big)$ and $z = \big((1,1),(b_{g_1}, b_{g_2})\big)$. Hence $xyz = \big((a_{g_1}, a_{g_2}),(1,1)\big)\big((1,1),(0,0)\big)\big((1,1),(b_{g_1}, b_{g_2})\big) = \big((a_{g_1}b_{g_1}, a_{g_2}b_{g_2}),(0,0)\big) \in I$. But neither $\big((a_{g_1}, a_{g_2}),(1,1)\big) \in I$ nor $\big((1,1),(b^n_{g_1}, b^n_{g_2})\big) \in I$ nor $\big((a^n_{g_1}, a^n_{g_2}),(b^n_{g_1}, b^n_{g_2})\big) \in I$ for a smallest positive integer $n > 1$, which is a contradiction. Thus I_1 is a G_1-primary ideal of $G_1(R_1)$. Now by similar argument we can conclude that I_2 is a G_2-primary ideal of $G_2(R_2)$.
(2) $\Rightarrow$ (1) If $I_2 = R_2$ and I_1 is a $G_1(2)$-absorbing primary ideal of $G_1(R_1)$, then $I = I_1 \times I_2$ is a $G(2)$-absorbing primary ideal of $G(R)$. By similar way if $I_1 = R_1$ and I_2 is a $G_2(2)$-absorbing primary ideal of $G_2(R_2)$, then $I = I_1 \times I_2$ is a $G(2)$-absorbing primary ideal of $G(R)$, by Theorem 2.25. Now we may assume that $I = I_1 \times I_2$ for some G_1-primary ideal I_1 of $G_1(R_1)$ and some G_2-primary ideal I_2 of $G_2(R_2)$. So $P = I_1 \times R_2$ and $Q = R_1 \times I_2$ are $G(2)$-absorbing primary ideals.

Hence

$$\begin{aligned}P \cap Q &= \big(I_1 \times R_2\big) \cap \big(R_1 \times I_2\big)\\ &= \big((P_{g_1} \oplus P_{g_2}) \times (R_2h_1 \oplus R_2h_2)\big) \cap \big((R_1g_1 \oplus R_1g_2) \times (Q_{h_1} \oplus Q_{h_2})\big)\\ &= \big((P_{g_1} \oplus P_{g_2}) \cap (R_1g_1 \oplus R_1g_2)\big) \times \big((P_{g_1} \oplus P_{g_2}) \cap (Q_{h_1} \oplus Q_{h_2})\big)\\ &\times \big((R_1g_1 \oplus R_1g_2) \cap (R_2h_1 \oplus R_2h_2)\big) \times \big((R_2h_1 \oplus R_2h_2) \cap (Q_{h_1} \oplus Q_{h_2})\big)\\ &= (P_{g_1} \oplus P_{g_2}) \times (I_1 \cap I_2) \times (R_1 \cap R_2) \times (Q_{h_1} \oplus Q_{h_2})\\ &= I_1 \times I_2 \times (R_1 \cap R_2).\end{aligned}$$

If $R_1 = R_2$, then $P \cap Q = I_1 \times (I_2 \times R_2)$ and so I is a $G(2)$-absorbing primary ideal of $G(R)$, by Theorem 2.25 and Theorem 2.5. If $R_1 \neq R_2$ ($R_1 \cap R_2 = \emptyset$), then $P \cap Q = I_1 \times I_2$ is a $G(2)$-absorbing primary ideal, by Theorem 2.5. □

The following result and its proof is an analogue of [7, Theorem 2.24].

Theorem 2.27. *Let $R = R_1 \times \cdots \times R_n$ be a $G = G_1 \times \cdots \times G_n$-graded ring, where R_i is a G_i-graded ring, rank$(G_i) = 2$ and $\Lambda = \{i\}_{i=1}^n$ for every $n \geq 2$. Suppose that I_j is distinct graded ideal of $G(R_j)$ for every $j \in \Lambda$. Then the following statements are equivalent:*

(1) *I is a $G(2)$-absorbing primary ideal of $G(R)$;*

(2) *$I = \prod_{j\in\Lambda} I_j$ such that for some $G_j(2)$-absorbing primary ideal I_j of $G_j(R_j)$ and $I_k = R_k$ for every $j \in \Lambda\setminus\{k\}$ or $I = \prod_{j\in\Lambda} I_j$ such that for some $k, t \in \Lambda$, I_k is a G_k-primary ideal of $G_k(R_k)$, I_t is a G_t-primary ideal of $G_t(R_t)$ and $I_j = R_j$ for every $j \in \Lambda \setminus \{k, t\}$. Assuming further that rank$(G_i) = m$ for every $i \in \Lambda$ and $2 \leq m \leq \infty$, then I is so.*

Proof. Let $n = 2$. Then the result is true, by Theorem 2.26. Assume that $n \geq 3$ and the result is true for $S = R_1 \times \cdots \times R_{n-1}$. Now we must show that the result is true when $R = S \times R_n$. Hence I is a $G(2)$-absorbing primary ideal of $G(R)$ if and only if $I = J \times R_n$ for some $G(2)$-absorbing primary ideal J of $G(S)$ or $I = S \times K_n$ for some $G_n(2)$-absorbing primary ideal K_n of $G_n(R_n)$ or $I = J \times K_n$ for some G-primary ideal J of $G(S)$ and some G_n-primary ideal K_n of $G_n(R_n)$, by Theorem 2.26. Then Q is a G-primary ideal of $G(S)$ if and only if $Q = \prod_{t\in\Lambda} I_t$ such that $\Lambda = \{i\}_{i=1}^{n-1}$ and for some $k \in \Lambda$, I_k is a G_k-primary ideal of $G_k(R_k)$ and $I_t = R_t$ for every $t \in \Lambda \setminus \{k\}$ and so we are done. □

The following result is an analogue of [7, Lemma 2.18].

Proposition 2.28. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. Suppose that I is a $G(2)$-absorbing ideal of $G(R)$ and $abJ \subseteq I$ for some graded ideal J of $G(R)$ and elements a, b of $h(R)$. If $ab \notin I$, then $aJ \subseteq Gr(I)$ or $bJ \subseteq Gr(I)$.*

Proof. Assume that $aJ \nsubseteq Gr(I)$ and $bJ \nsubseteq Gr(I)$. There exists $j_1, j_2 \in J$ such that $aj_1 \notin Gr(I)$ and $bj_2 \notin Gr(I)$. Then since I is a $G(2)$-absorbing primary ideal, $abj_1 \in I$, $ab \notin I$ and $aj_1 \notin Gr(I)$, we get that $bj_1 \in Gr(I)$. By the similar sense, since $abj_2 \in I$, $ab \notin I$, $bj_2 \notin Gr(I)$ and I is a $G(2)$-absorbing primary ideal, we also get that $aj_2 \in Gr(I)$. Now since $ab(j_1 + j_2) \in I$ and $ab \notin I$, we conclude that $a(j_1 + j_2) \in Gr(I)$ or $b(j_1 + j_2) \in Gr(I)$. If $a(j_1 + j_2) = aj_1 + aj_2 \in Gr(I)$, then $aj_1 \in Gr(I)$, which is a contradiction. If $b(j_1 + j_2) = bj_1 + bj_2 \in Gr(I)$, then $bj_2 \in Gr(I)$, which is a contradiction. Hence $aJ \subseteq Gr(I)$ or $bJ \subseteq Gr(I)$, as needed. □

The following result is an analogue of [7, Theorem 2.19].

Theorem 2.29. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. Then the following statements are equivalent:*

(1) *I is a $G(2)$-absorbing primary ideal of $G(R)$;*

(2) *If $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of $G(R)$, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq Gr(I)$ or $I_1I_3 \subseteq Gr(I)$.*

Proof. (1) ⇒ (2) Assume that I is a $G(2)$-absorbing primary ideal of $G(R)$ and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2 and I_3 of $G(R)$. Let $I_1I_2 \nsubseteq I$. It is enough to show that $I_2I_3 \subseteq Gr(I)$ or $I_1I_3 \subseteq Gr(I)$. Suppose that neither $I_2I_3 \subseteq Gr(I)$ nor $I_1I_3 \subseteq Gr(I)$. Then there exists $r_1 \in I_1$ and $r_2 \in I_2$ such that $r_1r_2I_3 \subseteq I$ but neither $r_1I_3 \subseteq Gr(I)$ nor $r_2I_3 \subseteq Gr(I)$. Hence $r_1r_2 \in I$, by Proposition 2.28. Since $I_1I_2 \nsubseteq I$, there exists $a \in I_1$ and $b \in I_2$ such that $ab \notin I$. Since $abI_3 \subseteq I$, $ab \notin I$ and I is $G(2)$-absorbing primary, by Proposition 2.28, we have $aI_3 \subseteq Gr(I)$ or $bI_3 \subseteq Gr(I)$. Now we may assume that three cases:
Case I. Let $aI_3 \subseteq Gr(I)$ but $bI_3 \nsubseteq Gr(I)$. Since $r_1bI_3 \subseteq I$ but neither $bI_3 \subseteq Gr(I)$ nor $r_1I_3 \subseteq Gr(I)$, we have $r_1b \in I$, by Proposition 2.28. We have $aI_3 \subseteq Gr(I)$ but $r_1I_3 \nsubseteq Gr(I)$, then $(a + r_1)I_3 \nsubseteq Gr(I)$. Since $(a + r_1)bI_3 \subseteq I$, $bI_3 \nsubseteq Gr(I)$ and $(a+r_1)I_3 \nsubseteq Gr(I)$, we conclude that $(a+r_1)b = ab+r_1b \in I$, by Proposition 2.28. Then $ab \in I$, which is a contradiction.
Case II. Let $aI_3 \nsubseteq Gr(I)$ but $bI_3 \subseteq Gr(I)$. Hence the complete proof is the same way by Case I.
Case III. Let $aI_3 \subseteq Gr(I)$ and $bI_3 \subseteq Gr(I)$. At the first we consider $aI_3 \subseteq Gr(I)$. Since $aI_3 \subseteq Gr(I)$ and $r_1I_3 \nsubseteq Gr(I)$, we have $(a + r_1)I_3 \nsubseteq Gr(I)$. Since $(a+r_1)r_2I_3 \subseteq I$ but neither $(a+r_1)I_3 \subseteq Gr(I)$ nor $r_2I_3 \subseteq Gr(I)$, we conclude that $(a + r_1)r_2 = ar_2 + r_1r_2 \in I$, by Proposition 2.28. Then $ar_2 \in I$. Now we consider $bI_3 \subseteq Gr(I)$. Since $bI_3 \subseteq Gr(I)$ and $r_2I_3 \nsubseteq Gr(I)$, we have $(b + r_2)I_3 \nsubseteq Gr(I)$. Since $r_1(b + r_2)I_3 \subseteq I$ but neither $r_1I_3 \subseteq Gr(I)$ nor $(b + r_2)I_3 \subseteq Gr(I)$, we have $r_1(b + r_2) = r_1b + r_1r_2 \in I$, by Proposition 2.28. Then $r_1b \in I$. Now since $(a + r_1)(b + r_2)I_3 \subseteq I$ but neither $(a + r_1)I_3 \subseteq Gr(I)$ nor $(b + r_2)I_3 \subseteq Gr(I)$, we can conclude that $(a+r_1)(b+r_2) = ab+ar_2+r_1b+r_1r_2 \in I$ and so $ab \in I$, which is a contradiction. Therefore $I_2I_3 \subseteq Gr(I)$ or $I_1I_3 \subseteq Gr(I)$, as needed.
(2) ⇒ (1) The proof is straightforward. □

3. Graded Weakly 2-Absorbing Primary Ideals

In this section we define the concept of graded weakly 2-absorbing primary ideals and weakly 2-absorbing primary subgroups and study some basic properties. Let I be a graded ideal of $G(R)$ and $x \in G$. The set $\{a \in R_x | a^n \in I$ for some positive integer n$\}$, is a subgroup of R_x and it is called the *x-radical* of I, denoted by $xr(I)$. Obviously, $I_x \subseteq xr(I)$ and if $r \in R_x$ such that $r \in Gr(I)$, then $r \in xr(I)$. The graded *nilradical* of $G(R)$ is denoted by G-$nil(R)$ that is the set of all $x \in R$ such that x_g is a nilpotent element of R for every $g \in G$. It is easy to show that a homogenous element of R is belong to G-$nil(R)$ if and only if it is nilpotent, (c.f. [17]). Let R be a G-graded ring. A proper graded ideal I of $G(R)$ is called *prime subgroup* of R_g if whenever for $g \in G$ and $a, b \in R_g$ such that $ab \in I_g$, then either $a \in I_g$ or $b \in I_g$. A proper graded ideal I of $G(R)$ is said to be a *primary subgroup* (*weakly primary subgroup*) of R_g if whenever for $g \in G$ and $a, b \in R_g$ with $ab \in I_g$ ($0 \neq ab \in I_g$), then either $a \in I_g$ or $b \in gr(I)$, (c.f [3], [4]). Now we use it to define the following definition.

Definition 3.1. *Let R be a G-graded ring, I be a graded ideal and $g \in G$.*

(1) *I_g is called a* 2-absorbing primary subgroup ($g(2)$-absorbing primary subgroup)*, if $I_g \neq R_g$ and whenever $a, b, c \in R_g$ with $abc \in I_g$, then either $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$.*

(2) *I_g is called a* weakly 2-absorbing primary subgroup*, if $I_g \neq R_g$ and whenever $a, b, c \in R_g$ with $0 \neq abc \in I_g$, then either $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$.*

(3) *I is called a* graded weakly 2-absorbing primary ideal*, if $I \neq R$ and whenever $a, b, c \in h(R)$ with $0 \neq abc \in I$, then either $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$.*

In the following results we show that some straightforward properties of graded weakly 2-absorbing primary ideals and 2-absorbing primary (resp., weakly 2-absorbing primary) subgroups.

Lemma 3.2. *Let R be a G-graded ring and I be a graded ideal of $G(R)$. If I is a graded weakly 2-absorbing primary ideal, then for $g \in G$, I_g is a weakly 2-absorbing primary subgroup.*

Proof. Assume that $g \in G$ and $a, b, c \in R_g$ such that $0 \neq abc \in I_g$. Since I is a graded weakly 2-absorbing primary ideal of $G(R)$ and $I_g \subseteq I$, we conclude that $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Since a, b and c are homogenous elements, there exists positive integer n such that $(bc)^n = b^n c^n \in I$. Then it implies that $bc \in gr(I)$. By the similar argument for the case $ac \in Gr(I)$, we get that $ac \in gr(I)$. Hence either $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$. Therefore I_g is a weakly 2-absorbing primary subgroup of R_g. □

Lemma 3.3. *Let R be a G-graded ring, I be a proper graded ideal and $g \in G$. Then the following statements hold:*

(1) *Every graded weakly primary ideal of $G(R)$ (resp., weakly primary subgroup of R_g) is a graded weakly 2-absorbing primary ideal (resp., weakly 2-absorbing primary subgroup);*

(2) *Every $G(2)$-absorbing ideal of $G(R)$ is a graded weakly 2-absorbing primary ideal;*

(3) *Every $G(2)$-absorbing primary ideal of $G(R)$ is a graded weakly 2-absorbing primary ideal.*

Lemma 3.4. *Let R be a G-graded ring, I be a proper graded ideal and $g \in G$. Then the following statements hold:*

(1) *If I_g is a weakly 2-absorbing subgroup of R_g, then I_g is a weakly 2-absorbing primary subgroup;*

(2) *$gr(I)$ is a weakly 2-absorbing subgroup of R_g if and only if $gr(I)$ is a weakly 2-absorbing primary subgroup.*

Theorem 3.5. *Let I be a proper graded ideal of $G(R)$ and $g \in G$. If $gr(I)$ is a weakly primary subgroup of R_g, then I_g is weakly 2-absorbing primary.*

Proof. Assume that $0 \neq abc \in I_g$ for some $a, b, c \in R_g$ and $ab \notin I_g$. If $ab \notin gr(I)$, since $gr(I)$ is a weakly primary subgroup of R_g and $0 \neq abc \in gr(I)$, we conclude that $c \in I_g \subseteq gr(I)$. Then $ac \in gr(I)$ for some homogeneous element a. If $ab \in gr(I)$, then $0 \neq ab \in gr(I)$. Since $gr(I)$ is a weakly primary subgroup of R_g and $0 \neq abc \in I_g$. Hence either $a \in gr(I)$ or $b \in gr(gr(I)) = gr(I)$. Thus $ac \in gr(I)$ or $bc \in gr(I)$ for some homogeneous element c. Therefore I_g is weakly 2-absorbing primary. □

The following results are analogue of [8, Theorems 2.18 and 2.20].

Theorem 3.6. *Let R be a G-graded ring and I, J be proper graded ideals of $G(R)$ with $J \subseteq I$. Then the following statements hold:*

(1) *If I is a graded weakly 2-absorbing primary ideal of $G(R)$, then I/J is a graded weakly 2-absorbing primary ideal of R/J;*

(2) *If J is a $G(2)$-absorbing primary ideal of $G(R)$ and I/J is a graded weakly 2-absorbing primary ideal of R/J, then I is a $G(2)$-absorbing primary ideal of $G(R)$;*

(3) *If J and I/J are graded weakly 2-absorbing primary ideals, then I is a $G(2)$-absorbing primary ideal of $G(R)$.*

Proof. (1) Assume that $(a+J), (b+J), (c+J) \in R/J$ with $J \neq (a+J)(b+J)(c+J) = abc + J \in I/J$. Then the complete proof is similar with Theorem 2.23.
(2) Assume that $abc \in I$ for some $a, b, c \in h(R)$ with $J \subseteq I$. Now we may suppose that two cases.

Case one. If $abc \in J$, then either $ab \in J \subseteq I$ or $bc \in Gr(J) \subseteq Gr(I)$ or $ac \in Gr(J) \subseteq Gr(I)$. Since J is a $G(2)$-absorbing primary ideal of $G(R)$.
Case two. If $abc \notin J$, then $J \neq abc+J = (a+J)(b+J)(c+J) \in I/J$. Since I/J is graded weakly 2-absorbing primary, we conclude that $(a+J)(b+J) = ab+J \in I/J$ or $(b+J)(c+J) = bc+J \in Gr(I/J)$ or $(a+J)(c+J) = ac+J \in Gr(I/J)$. It implies that $ab + J \in I/J$ or $(ac + J)^n = (ac)^n + J \in I/J$ or $(bc + J)^n = (bc)^n + J \in I/J$ for some positive n. Hence $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Therefore I is a $G(2)$-absorbing primary ideal of $G(R)$.
(3) Assume that $0 \neq abc \in I$ for some $a, b, c \in h(R)$. Then $(a + J)(b + J)(c + J) \in I/J$. If $abc \in J$, then either $ab \in J \subseteq I$ or $bc \in Gr(J) \subseteq Gr(I)$ or $ac \in Gr(J) \subseteq Gr(I)$. Since J is graded weakly 2-absorbing primary. So we may assume that $abc \notin J$. Then either $(a + J)(b + J) = ab + J \in I/J$ or $(b + J)(c + J) = bc + J \in Gr(I/J)$ or $(a + J)(c + J) = ac + J \in Gr(I/J)$. It implies that $ab + J \in I/J$ or $(ac + J)^n = (ac)^n + J \in I/J$ or $(bc + J)^n = (bc)^n + J \in I/J$ for some positive n. Hence either $ab \in I$ or $bc \in Gr(I)$ or $ac \in Gr(I)$. Then I is a $G(2)$-absorbing primary ideal of $G(R)$, as required. □

Theorem 3.7. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of homogeneous element R. Then the following statements hold:*

(1) *If I is a graded weakly 2-absorbing primary ideal of $G(R)$ with $S \cap I = \emptyset$, then $S^{-1}I$ is a graded weakly 2-absorbing primary ideal of $S^{-1}R$;*

(2) *If $S^{-1}I$ is a graded weakly 2-absorbing primary ideal of $S^{-1}R$ with $S \cap Gz(R/I) = \emptyset$ and $S \cap Gz(R) = \emptyset$, then I is a graded weakly 2-absorbing primary ideal of R.*

Proof. The proof is similar with Theorem 2.24 and so we omit that. □

Remark 3.8. A graded 2-absorbing primary ideal of $G(R)$ is a graded weakly 2-absorbing primary ideal, by Lemma 3.3(3). However, since 0 is always a graded weakly 2-absorbing primary ideal by definition, a graded weakly 2-absorbing primary ideal need not be a graded 2-absorbing primary ideal. For instance, let $S = Mat_2(R)$ be a $\mathbb{Z}_2$-graded ring where $R = \mathbb{Z}_{12}$. Then $S_0 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ and $S_1 = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$. It is easy to see that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is graded weakly 2-absorbing primary but it is not $g(2)$-absorbing primary.
The following result is an analogue of [8, Theorem 2.10].

Theorem 3.9. *Let $I = \bigoplus_{g \in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$. Then for $g \in G$ either I_g is a $g(2)$-absorbing primary subgroup of R_g or $I_g^3 = 0$.*

Proof. Assume that $I_g^3 \neq 0$. We show that I_g is a $g(2)$-absorbing primary subgroup of R_g. Suppose that $a, b, c \in R_g$ such that $abc \in I_g$ for $g \in G$. If $abc \neq 0$, then $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$. Since I_g is a weakly 2-absorbing primary

subgroup of R_g, by Lemma 3.2. So we suppose that $abc = 0$. Now we may assume that three cases:
Case I. Let $abI_g \neq 0$. Say $abr_0 \neq 0$ for some $r_0 \in I_g$. Then $0 \neq abr_0 = ab(c+r_0) \in I$. Since I is a graded weakly 2-absorbing primary ideal of R, we conclude that $ab \in I$ or $b(c+r_0) \in Gr(I)$ or $a(c+r_0) \in Gr(I)$. Then $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$, by Lemma 2.8(1). So we can suppose that $abI_g = 0$. Likewise we can assume for two next forms of this case $acI_g = 0 = bcI_g$.
Now since $I_g^3 \neq 0$, there exists $a_0, b_0, c_0 \in I_g \subseteq I \subseteq Gr(I)$ with $a_0b_0c_0 \neq 0$.
Case II. Let $ab_0c_0 \neq 0$. Then $a(b+b_0)(c+c_0) \in I$. It implies that $a(b+b_0) \in I$ or $(b+b_0)(c+c_0) \in Gr(I)$ or $a(c+c_0) \in Gr(I)$. Hence $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$, by Lemma 2.8(1). So we can assume that $ab_0c_0 = 0$. Likewise we can consider for two next forms of this case $a_0bc_0 = 0 = a_0b_0c$.
Case III. Let $0 \neq a_0b_0c_0 = (a+a_0)(b+b_0)(c+c_0) \in I$. Thus $(a+a_0)(b+b_0) \in I$ or $(b+b_0)(c+c_0) \in Gr(I)$ or $(a+a_0)(c+c_0) \in Gr(I)$. Hence $ab \in I$ or $bc \in gr(I)$ or $ac \in gr(I)$, by Lemma 2.8(1). Therefore I_g is a $g(2)$-absorbing primary subgroup of R_g, as needed. □

Corollary 3.10. *Let I be a graded weakly 2-absorbing primary ideal of $G(R)$ such that I_g is not a $g(2)$-absorbing primary subgroup of R_g for $g \in G$. Then $Gr(I) = Gr(0)$.*

Proof. Clearly $Gr(0) \subseteq Gr(I)$. Assume that $a \in I$ for some $a \in h(R)$. By Theorem 3.9, $a_g^3 = 0$ for every $g \in G$ and so $a \in Gr(0)$. Then $I \subseteq Gr(0)$ and so $Gr(I) \subseteq Gr(0)$, by Lemma 2.8(2). Hence $Gr(I) = Gr(0)$. □

Corollary 3.11. *Let I be a graded weakly 2-absorbing primary ideal of $G(R)$ such that I_g is not a $g(2)$-absorbing primary subgroup of R_g for $g \in G$. Then I_g is nilpotent.*

Theorem 3.12. *Let $\{I_i\}_{i\in\Lambda}$ be a family of graded weakly 2-absorbing primary ideals of G-graded ring R such that for every $i \in \Lambda$, $(I_i)_g$ is not a $g(2)$-absorbing primary subgroup of R_g for each $g \in G$. Then $I = \bigcap_{i\in\Lambda} I_i$ is a graded weakly 2-absorbing primary ideal of $G(R)$.*

Proof. We claim that $Gr(I) = \bigcap_{i\in\Lambda} Gr(I_i)$. Let $a \in \bigcap_{i\in\Lambda} Gr(I_i)$. Since by Corollary 3.10, $\bigcap_{i\in\Lambda} Gr(I_i) = Gr(0)$, we get that for each $g \in G$, $a_g^{n_g} = 0$ for some smallest n_g. Then $a_g^{n_g} \in I_i$ for every $i \in \Lambda$. Thus $a \in Gr(I)$ and so $\bigcap_{i\in\Lambda} Gr(I_i) \subseteq Gr(I)$. Assume that $a, b, c \in h(R)$ such that $0 \neq abc \in I$ and $ab \notin I$. Then there exists an element $i \in \Lambda$ such that $ab \notin I_i$. Thus $bc \in Gr(I_i) = Gr(0) = Gr(I)$ or $ac \in Gr(I_i) = Gr(0) = Gr(I)$. Hence I is a graded weakly 2-absorbing primary ideal of $G(R)$. □

Theorem 3.13. *Let $Gr(0)$ be a G-prime (G-primary) ideal of $G(R)$. Suppose that I is a graded ideal. If I is a graded weakly 2-absorbing primary ideal of $G(R)$, then for every $g \in G$, I_g is a $g(2)$-absorbing primary subgroup of R_g.*

Proof. By Lemma 3.2, I_g is a weakly 2-absorbing primary subgroup of R_g for every $g \in G$. Let $a, b, c \in R_g$ such that $abc \in I_g$. If $0 \neq abc \in I_g$, then $ab \in I_g$ or

$bc \in gr(I)$ or $ac \in gr(I)$. Hence we may assume that $abc = 0$ and $ab \notin I_g$. Since $Gr(0)$ is a G-prime ideal and $abc = 0$, we conclude that $a \in Gr(0)$ or $b \in Gr(0)$ or $c \in Gr(0)$. Since $Gr(0) \subseteq Gr(I)$, we can conclude that $bc \in Gr(0) \subseteq Gr(I)$ or $ac \in Gr(0) \subseteq Gr(I)$. Then $bc \in gr(I)$ or $ac \in gr(I)$. Hence I_g is a $g(2)$-absorbing primary subgroup of R_g. □

The view of the Theorem 3.9 leads us to have the following definition.

Definition 3.14. Let $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$. We say (a, b, c) is a *triple-zero* of I_g if $abc = 0$, $ab \notin I_g$, $bc \notin gr(I)$ and $ac \notin gr(I)$.

Theorem 3.15. *Let I be a graded weakly 2-absorbing primary ideal of $G(R)$. Suppose that (a, b, c) is a triple-zero of I_g for some $a, b, c \in R_g$ and for every $g \in G$. Then the following statements hold:*

(1) $abI_g = acI_g = bcI_g = 0$;

(2) $aI_g^2 = bI_g^2 = cI_g^2 = 0$.

Proof. (1) By Theorem 3.9, the result is true.
(2) Assume that $ar_1r_2 \neq 0$ for some $r_1, r_2 \in I_g$. Then $0 \neq a(b+r_1)(c+r_2) = ar_1r_2 \in I$. Since I is a graded weakly 2-absorbing primary ideal of $G(R)$, we conclude that $a(b + r_1) \in I$ or $(b + r_1)(c + r_2) \in Gr(I)$ or $a(c + r_2) \in Gr(I)$. Thus $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$, which is a contradiction. Hence $aI_g^2 = 0$. By similar way, we get that $bI_g^2 = 0 = cI_g^2$, as needed. □

Definition 3.16. Let $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$ and $g \in SuppG(R)$. Suppose that $I_{g_1}I_{g_2}I_{g_3} \subseteq I_g$ for some ideal I_{g_i} of R_{g_i} and for every $g_i \in G$ $(1 \leq i \leq 3)$. We say I is a *free triple-zero with respect to* $I_{g_1}I_{g_2}I_{g_3}$, if (a, b, c) is not a triple-zero of I_g for every $a \in I_{g_1}$, $b \in I_{g_2}$ and $c \in I_{g_3}$.

Proposition 3.17. *Let $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$ and $g \in SuppG(R)$. Suppose that for $h \in G$, $abI_h \subseteq I_g$ for some ideal I_h of R_h and $a, b \in R_g$ such that (a, b, c) is not a triple-zero of I_g for every $c \in I_h$. If $ab \notin I_g$, then $aI_h \subseteq gr(I)$ or $bI_h \subseteq gr(I)$.*

Proof. Assume that $aI_h \nsubseteq gr(I)$ and $bI_h \nsubseteq gr(I)$. Then there exists $i_1, i_2 \in I_h$ such that $ai_1 \notin gr(I)$ and $bi_2 \notin gr(I)$. Since (a, b, i_1) is not a triple-zero of I_g, $ab \notin I_g$ and $ai_1 \notin gr(I)$, we get that $bi_1 \in gr(I)$. Also since (a, b, i_1) is not a triple-zero of I_g, $ab \notin I_g$ and $bi_2 \notin gr(I)$, we get that $ai_2 \in gr(I)$. Now since $(a, b, i_1 + i_2)$ is not a triple-zero of I_g and $ab \notin I_g$, we conclude that $a(i_1 + i_2) \in gr(I)$ or $b(i_1 + i_2) \in gr(I)$. If $a(i_1 + i_2) = ai_1 + ai_2 \in gr(I)$, then $ai_1 \in gr(I)$, which is a contradiction. If $b(i_1 + i_2) = bi_1 + bi_2 \in gr(I)$, then $bi_2 \in gr(I)$, which is a contradiction. Hence $aI_h \subseteq gr(I)$ or $bI_h \subseteq gr(I)$, as needed. □

Remark 3.18. Let $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$. It is well known that I is a free triple-zero with respect to $I_{g_1}I_{g_2}I_{g_3}$ for some ideal I_{g_i} of R_{g_i} $(1 \leq i \leq 3)$ if and only if $ab \in I_g$ or $bc \in gr(I)$ or $ac \in gr(I)$ for every $a \in I_{g_1}$, $b \in I_{g_2}$ and $c \in I_{g_3}$.

The following result is an analogue of [8, Theorem 2.30].

Theorem 3.19. *Let $I = \bigoplus_{g\in G} I_g$ be a graded weakly 2-absorbing primary ideal of $G(R)$ and $g \in SuppG(R)$. Suppose that I is a free triple-zero with respect to $I_{g_1}I_{g_2}I_{g_3}$ such that $0 \neq I_{g_1}I_{g_2}I_{g_3} \subseteq I_g$ for some ideal I_{g_i} of R_{g_i} $(1 \leq i \leq 3)$. Then $I_{g_1}I_{g_2} \subseteq I_g$ or $I_{g_2}I_{g_3} \subseteq gr(I)$ or $I_{g_1}I_{g_3} \subseteq gr(I)$.*

Proof. Assume that I is a $G(2)$-absorbing primary ideal of $G(R)$ and I is a free triple-zero with respect to $I_{g_1}I_{g_2}I_{g_3}$ such that $0 \neq I_{g_1}I_{g_2}I_{g_3} \subseteq I_g$ for some ideal I_{g_i} of R_{g_i} $(1 \leq i \leq 3)$. Then I_g is a weakly 2-absorbing primary subgroup of R_g, by Lemma 3.2. Let $I_{g_1}I_{g_2} \nsubseteq I_g$. It is enough to show that $I_{g_2}I_{g_3} \subseteq gr(I)$ or $I_{g_1}I_{g_3} \subseteq gr(I)$. Suppose that neither $I_{g_2}I_{g_3} \subseteq gr(I)$ nor $I_{g_1}I_{g_3} \subseteq gr(I)$. Then there exists $r_1 \in I_{g_1}$ and $r_2 \in I_{g_2}$ such that $r_1r_2I_{g_3} \subseteq I_g$ but neither $r_1I_{g_3} \subseteq gr(I)$ nor $r_2I_{g_3} \subseteq gr(I)$. Hence $r_1r_2 \in I_g$, by Proposition 3.17. Since $I_{g_1}I_{g_2} \nsubseteq I_g$, there exists $a \in I_{g_1}$ and $b \in I_{g_2}$ such that $ab \notin I_g$. Since $abI_{g_3} \subseteq I_g$, $ab \notin I_g$ and by Proposition 3.17, we have $aI_{g_3} \subseteq gr(I)$ or $bI_{g_3} \subseteq gr(I)$. Now we may assume that three cases:
Case I. Let $aI_{g_3} \subseteq gr(I)$ but $bI_{g_3} \nsubseteq gr(I)$. Since $r_1bI_{g_3} \subseteq I_g$ but neither $bI_{g_3} \subseteq gr(I)$ nor $r_1I_{g_3} \subseteq gr(I)$, we have $r_1b \in I_g$, by Proposition 3.17. We have $aI_{g_3} \subseteq gr(I)$ but $r_1I_{g_3} \nsubseteq gr(I)$, then $(a + r_1)I_{g_3} \nsubseteq gr(I)$. Since $(a + r_1)bI_{g_3} \subseteq I$, $bI_{g_3} \nsubseteq gr(I)$ and $(a + r_1)I_{g_3} \nsubseteq gr(I)$, we conclude that $(a + r_1)b = ab + r_1b \in I$, by Proposition 3.17. Then $ab \in I$, which is a contradiction.
Case II. Let $aI_{g_3} \nsubseteq gr(I)$ but $bI_{g_3} \subseteq gr(I)$. Since $ar_2I_{g_3} \subseteq I_g$ but neither $aI_{g_3} \subseteq gr(I)$ nor $r_2I_{g_3} \subseteq gr(I)$, we have $ar_2 \in I_g$, by Proposition 3.17. We have $bI_{g_3} \subseteq gr(I)$ but $r_2I_{g_3} \nsubseteq gr(I)$, then $(b + r_2)I_{g_3} \nsubseteq gr(I)$. Since $a(b + r_2)I_{g_3} \subseteq I$, $aI_{g_3} \nsubseteq gr(I)$ and $(b + r_2)I_{g_3} \nsubseteq gr(I)$, we conclude that $a(b + r_2) = ab + ar_2 \in I$, by Proposition 3.17. Then $ab \in I$, which is a contradiction.
Case III. Let $aI_{g_3} \subseteq gr(I)$ and $bI_{g_3} \subseteq gr(I)$. Since $aI_{g_3} \subseteq gr(I)$ and $r_1I_{g_3} \nsubseteq gr(I)$, we have $(a+r_1)I_{g_3} \nsubseteq gr(I)$. Since $(a+r_1)r_2I_{g_3} \subseteq I$ but neither $(a+r_1)I_{g_3} \subseteq gr(I)$ nor $r_2I_{g_3} \subseteq gr(I)$, we conclude that $(a+r_1)r_2 = ar_2+r_1r_2 \in I$, by Proposition 3.17. Then $ar_2 \in I$. Now since $bI_{g_3} \subseteq gr(I)$ and $r_2I_{g_3} \nsubseteq gr(I)$, we have $(b + r_2)I_{g_3} \nsubseteq gr(I)$. Since $r_1(b + r_2)I_{g_3} \subseteq I$ but neither $r_1I_{g_3} \subseteq gr(I)$ nor $(b + r_2)I_{g_3} \subseteq gr(I)$, we have $r_1(b + r_2) = r_1b + r_1r_2 \in I$, by Proposition 3.17. Then $r_1b \in I$. Now since $(a + r_1)(b + r_2)I_{g_3} \subseteq I$ but neither $(a + r_1)I_{g_3} \subseteq gr(I)$ nor $(b + r_2)I_{g_3} \subseteq gr(I)$, we can conclude that $(a + r_1)(b + r_2) = ab + ar_2 + r_1b + r_1r_2 \in I$ and so $ab \in I$, which is a contradiction. Therefore $I_{g_2}I_{g_3} \subseteq gr(I)$ or $I_{g_1}I_{g_3} \subseteq gr(I)$, as needed. □

Theorem 3.20. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring for $i = 1, 2$. Then the following statements hold:*

(1) *If $I \times R_2$ is a graded weakly 2-absorbing primary ideal of $G(R)$, then $I_g \times R_{2_h}$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$;*

(2) *If I is a $G_1(2)$-absorbing primary ideal of $G_1(R_1)$, then $I \times R_2$ is a graded weakly 2-absorbing primary ideal of $G(R)$;*

(3) *If J is a $G_2(2)$-absorbing primary ideal of $G_2(R_2)$, then $R_1 \times J$ is a graded weakly 2-absorbing primary ideal of $G(R)$;*

(4) *$I_g \times R_h$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$ if and only if I_g is a $g(2)$-absorbing primary subgroup of R_g;*

(5) *$R_1 \times J_h$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$ if and only if J_h is a $g(2)$-absorbing primary subgroup of R_h.*

Proof. (1) Since $I \times R_2$ is a graded weakly 2-absorbing primary, $I_g \times R_h \nsubseteq Gr(0)$, by Corollary 3.10. Then $I_g \times R_{2_h}$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$.
(2) and (3) By Lemma 3.3 and Theorem 2.25.
(4) and (5) By Theorem 2.25. □

Theorem 3.21. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring and $I = I_1 \times I_2$ is a graded ideal of $G(R)$ where I_i is a graded ideal of $G_i(R_i)$ and $rankG_i = n$ for every $i = 1, 2$. If I is a graded weakly 2-absorbing primary ideal of $G(R)$, then $I_{(g,h)}$ is a $g(2)$-absorbing primary subgroup or $I_{(g,h)} = 0$.*

Proof. Suppose that $I = I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal of $G(R)$ and $I_{(g,h)} \neq 0$. We may assume that $I_{1_g} = R_{1_g}$ and I_{2_h} is a $g(2)$-absorbing primary subgroup of R_{2_h} or $I_{2_h} = R_{2_h}$ and I_{1_g} is a $g(2)$-absorbing primary subgroup of R_{1_g} or I_{1_g} and I_{2_h} are primary subgroups of R_{1_g} and R_{2_h} respectively. If $I_{1_g} = R_{1_g}$, then I_{2_h} is a $g(2)$-absorbing primary subgroup of R_{2_h}. If $I_{2_h} = R_{2_h}$, then I_{1_g} is a $g(2)$-absorbing primary subgroup of R_{1_g}, by Theorem 3.20(4),(5). Now we suppose that neither $I_{1_g} = R_{1_g}$ nor $I_{2_h} = R_{2_h}$. It can be shown that I_{1_g} and I_{2_h} are primary subgroups of R_{1_g} and R_{2_h} respectively. Without loss of generality assume that $a, b \in R_{1_g}$ such that $ab \in I_{1_g}$ where $a = \sum_{i=1}^n a_{g_i}$, $b = \sum_{i=1}^n b_{g_i}$ and $0 \neq y = \sum_{i=1}^n y_{h_i} \in I_{2_h}$ with $y_{h_i} \neq 0$ for every $1 \leq i \leq n$. Then $0 \neq (a,1)(b,1)(1,y) = (ab, y) = \left(\sum_{i=1}^n a_{g_i} b_{g_i}, \sum_{i=1}^n y_{h_i}\right) = \sum_{i=1}^n (a_{g_i} b_{g_i}, y_{h_i}) \in I_{1_g} \times I_{2_h}$. Since $I_1 \times I_2$ is a graded weakly 2-absorbing primary ideal and $I_2 \neq R_2$, we conclude that $(ab, 1) = (\sum_{i=1}^n a_{g_i} b_{g_i}, 1) \notin Gr(I_1 \times I_2)$. Thus we have $(a, y) = (a,1)(1,y) = (\sum_{i=1}^n a_{g_i}, \sum_{i=1}^n y_{h_i}) = \sum_{i=1}^n (a_{g_i}, y_{h_i}) \in I_{1_g} \times I_{2_h}$ or $(b, y) = (b,1)(1,y) = (\sum_{i=1}^n b_{g_i}, \sum_{i=1}^n y_{h_i}) = \sum_{i=1}^n (b_{g_i}, y_{h_i}) \in Gr(I_1 \times I_2)$(so $(b, y) \in gr(I_1 \times I_2)$) and so $a \in I_{1_g}$ or $b \in gr(I)$. Hence I_{1_g} is a primary subgroup of R_{1_g}. By similar argument we can also show that I_{2_h} is a primary subgroup of R_{2_h}. □

Theorem 3.22. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring and $I = I_1 \times I_2$ is a graded ideal of $G(R)$ where I_i is a graded ideal of $G_i(R_i)$ and $rankG_i = n$ for every $i = 1, 2$. If $I_{1_g} = R_{1_g}$ and I_{2_h} is a $g(2)$-absorbing primary subgroup of R_{2_h} or $I_{2_h} = R_{2_h}$ and I_{1_g} is a $g(2)$-absorbing primary subgroup of R_{1_g} or I_{1_g} and I_{2_h} are primary subgroups of R_{1_g} and R_{2_h} respectively, then $I_{(g,h)}$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$.*

Proof. The proof is straightforward by Theorem 2.26 and Theorem 3.20. □

The following result is an analogue of [8, Theorem 2.23].

Theorem 3.23. *Let $R = R_1 \times R_2$ be a $G = G_1 \times G_2$-graded ring where R_i is a G_i-graded ring and $I = I_1 \times I_2$ is a graded ideal of $G(R)$ where I_i is a graded ideal of $G_i(R_i)$ and $rankG_i = n$ for every $i = 1, 2$. If I is a graded weakly 2-absorbing primary ideal of $G(R)$ such that $I_{(g,h)}$ is not a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$. Then one of the following statements must hold:*

(1) *$I_{(g,h)} = I_{1_g} \times I_{2_h}$, where $I_{1_g} \neq R_{1_g}$ is a weakly primary subgroup that is not a primary subgroup of R_{1_g} and $I_{2_h} = 0$ is a primary subgroup of R_{2_h}.*

(2) *$I_{(g,h)} = I_{1_g} \times I_{2_h}$, where $I_{2_h} \neq R_{2_h}$ is a weakly primary subgroup that is not a primary subgroup of R_{2_h} and $I_{1_g} = 0$ is a primary subgroup of R_{1_g}.*

Proof. Assume that I is a graded weakly 2-absorbing primary ideal of $G(R)$ such that $I_{(g,h)}$ is not a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$. Suppose that neither $I_{1_g} = 0$ nor $I_{2_h} = 0$. Then $I_{(g,h)}$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$, by Theorem 3.21, which is a contradiction. Therefore $I_{1_g} \neq 0$ or $I_{2_h} \neq 0$. Without loss of generality assume that $I_{2_h} = 0$. Let $cd \in I_{2_h}$ and $0 \neq x \in I_{1_g}$ where $c = \sum_{i=1}^{n} c_{h_i}$, $d = \sum_{i=1}^{n} d_{h_i}$ of R_{2_h} and $x = \sum_{i=1}^{n} x_{g_i}$ for every $1 \leq i \leq n$. Since $0 \neq (x,1)(1,a)(1,,b) = (x,ab) = \left(\sum_{i=1}^{n} x_{g_i}, \sum_{i=1}^{n} c_{h_i} d_{h_i}\right) = \sum_{i=1}^{n}(x_{g_i}, c_{h_i} d_{h_i}) \in I = I_1 \times I_2$ and I is a graded weakly 2-absorbing primary, we get that $(1,ab) \notin Gr(I)$. Then $(x,1)(1,a) = (x,a) \in I$ $\big($so $(x,a) \in I_{(g,h)}\big)$ or $(x,1)(1,b) = (x,b) \in Gr(I)$ $\big($so $(x,b) \in gr(I)\big)$. Hence $a \in I_{2_h}$ or $b \in gr(I_2)$ and so $I_{2_h} = 0$ is a primary subgroup of R_{2_h}. Now we show that I_{1_g} is a weakly primary subgroup of R_{1_g}. Clearly $I_{1_g} \neq R_{1_g}$. Otherwise, if $I_{1_g} = R_{1_g}$, then $R_{1_g} \times 0$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$, by Theorem 3.22. Suppose that $0 \neq ab \in I_{1_g}$ for some $a = \sum_{i=1}^{n} a_{g_i}$ and $b = \sum_{i=1}^{n} b_{g_i}$ of R_{1_g}. Since $0 \neq (a,1)(1,0)(b,1) \in I_1 \times 0$ and $(ab,1) \notin Gr(I_1 \times 0)$, we conclude that $(b,1) = (b,1)(1,0) \in Gr(I_1 \times 0)$ $\big($so $(b,1) \in gr(I_1 \times 0)\big)$ or $(a,1) = (a,1)(1,0) \in I_1 \times 0$ $\big($so $(a,0) \in I_{1_g} \times 0\big)$. Hence $a \in I_{1_g}$ or $b \in gr(I_1)$. Then I_{1_g} is a weakly primary subgroup of R_{1_g}. So we show that I_{1_g} is not a graded primary subgroup. Suppose that I_{1_g} is a primary subgroup. As we are shown that $I_{2_h} = 0$ is a primary subgroup of R_{2_h}, we can get that $I_{(g,h)} = I_{1_g} \times I_{2_h}$ is a $g(2)$-absorbing primary subgroup of $R_{(g,h)}$, by Theorem 3.21, which is a contradiction. Therefore I_{1_g} is a weakly primary subgroup that is not a primary subgroup of R_{1_g}. □

4. Some Properties on Graded Principal Ideal Domains

In this section we study some structures of graded rings which are easily investigated and much like the structure of ungraded rings, e.g. Gr-Dedekind and Gr-Principal ideal rings for arbitary $\mathbb{Z}$-gradations. Throughout this section $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a commutative ring without zero divisor, K will be a field of fractions. A graded ring R is said to be a *division ring* if every non-zero homogeneous element of R is invertible and a commutative graded division ring is called a *graded field*. A graded field has the form $K[t, t^{-1}]$, where the part of degree 0, say K, which is a field and t a variable of degree $e > 0$. We write $h^*(R)$ for the set of non-zero element of $h(R)$. An R-submodule K, say I, which is said to be a *fractional ideal* of R if there exists $0 \neq x \in R$ such that $xI \subset R$. A fractional ideal I of R is said to be *invertible* if there exists a fractional ideal $J \subset K$ such that $IJ = R$, we say $J = I^{-1}$. A graded domain R is said to be a *graded principal* (*Gr-principal*) *ideal ring* if every graded ideal is principal. A graded domain is a *Gr-Dedekind ring* if every graded ideal of R is a projective module, (c.f [13], [15]).

Theorem 4.1. *Let R be a Gr-Noetherian graded integral domain and I be a non-zero graded ideal of R. Suppose that R is a Gr-Dedekind domain. Then I is a $G(2)$-absorbing primary ideal if and only if either $I = P^n$ for some G-prime ideal P of R and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some G-prime ideals P_1, P_2 of R and some positive integer $n, m \geq 1$.*

Proof. Assume that R is a Gr-Noetherian ring that is a Gr-Dedekind domain. Let I be a graded ideal of R. Then $I = P_1^{n_1} P_2^{n_2} \cdots P_t^{n_t}$ for some distinct G-prime ideals $P_1, \cdots, P_t$ of R and some positive integer $n_1, \cdots, n_t \geq 1$, by [13, p 179, Theorem II.2.1]. If $t = 1$, we are done. We show that $t = 2$. Suppose that $t > 2$. Then $P_1^{n_1} P_2^{n_2}(P_3^{n_3} \cdots P_t^{n_t}) \in I$ but neither $P_1^{n_1} P_2^{n_2} \in I$ nor $P_2^{n_2}(P_3^{n_3} \cdots P_t^{n_t}) \in Gr(I)$ nor $P_1^{n_1}(P_3^{n_3} \cdots P_t^{n_t}) \in Gr(I)$, which is a contradiction, and so $t \leq 2$. Conversely, if $I = P^{n_1}$ for some G-prime ideal P of R or $I = P_1^{n_1} P_2^{n_2}$ for some G-prime ideals P_1, P_2 of R and some positive integer $n_1, n_2 \geq 1$, then I is a $G(2)$-absorbing primary ideal of R, by Theorem 2.15 and Corollary 2.13. □

A Gr-principal ideal ring is a Gr-Dedekind ring, by [15, Corollary 1.2]. In the following we use it to characterize next result on graded principal ideal domains.

Corollary 4.2. *Let R be a Gr-principal domain and I be a non-zero graded ideal of R. Then I is a $G(2)$-absorbing primary ideal of R if and only if either $I = p^n R$ for some prime member p of R or $I = p_1^n p_2^m R$ for some distinct prime elements p_1 and p_2 of R where $n, m \geq 1$. Assuming further that $R = K[t]$ such that K is a field and t is a variable, then I is so.*

Example 4.3. Let $G = \mathbb{Z}_2$ and $R_0 = \mathbb{Z} = R_1$. Then $R = \mathbb{Z} \oplus \mathbb{Z}$ is a G-graded ring. Assume that $I_1 = 10\mathbb{Z} \oplus 6\mathbb{Z}$ and $I_2 = 6\mathbb{Z} \oplus 15\mathbb{Z}$ are $G(2)$-absorbing primary ideals of $G(R)$. However, $J = I_1 \cap I_2 = 30\mathbb{Z} \oplus 6\mathbb{Z}$ is not $G(2)$-absorbing primary ideal. Since $(2,2)(3,2)(5,3) \in J$ but neither $(2,2)(3,2) = (6,4) \in J$ nor $(3,2)(5,3) = (15,6) \in Gr(J) = Gr(30\mathbb{Z} \oplus 6\mathbb{Z}) = 30\mathbb{Z} \oplus 6\mathbb{Z}$ nor $(2,2)(5,3) = (10,6) \in Gr(J) = Gr(30\mathbb{Z} \oplus 6\mathbb{Z}) = 30\mathbb{Z} \oplus 6\mathbb{Z}$. This example shows that the Theorem 2.5 is not satisfying in general.

The following result is an analogue of [7, Theorem 2.11].

Theorem 4.4. *Let R be a Gr-Noetherian graded integral domain such that R is not a field and I be a non-zero graded ideal of R. Suppose that R is a Gr-Dedekind domain. Then the following statements are equivalent:*

(1) *R_0 is Dedekind domain;*

(2) *I is a 2-absorbing primary ideal of R_0 if and only if either $I = M^n$ for some maximal ideal M of R_0 and some positive integer $n \geq 1$ or $I = M_1^n M_2^m$ for some maximal ideals M_1, M_2 of R_0 and some positive integer $n, m \geq 1$.*

(3) *I is a 2-absorbing primary ideal of R_0 if and only if either $I = P^n$ for some prime ideal P of R_0 and some positive integer $n \geq 1$ or $I = P_1^n P_2^m$ for some prime ideals P_1, P_2 of R_0 and some positive integer $n, m \geq 1$.*

Proof. Since R is a Gr-dedekind domain, R_0 is a Dedekind ring, by [13, Lemma II.2.3]. Then the complete proof is satisfied by [7, Theorem 2.11]. □

Remark 4.5. If R is a graded field, the Theorem 4.4 is not satisfying in general. For example, let $R = \mathbb{Q}[t, t^{-1}]$ with $degt = 1$ and $degt^{-1} = -1$. Hence R is a graded field. Since $1 + t$ is not unit, R is not a field. Then a graded field is not necessary a field.

Theorem 4.6. *Let I be a proper graded ideal of R. If I is a G-P-primary ideal of R with $P^2 \subseteq I$, then I is a $G(2)$-absorbing ideal of R.*

Proof. Assume that $abc \in I$ for some $a, b, c \in h^*(R)$. If $a \in I$ or $bc \in I$, there is nothing to prove. We may assume that neither $a \in I$ nor $bc \in I$. Since I is a G-P-primary ideal of R such that $P^2 \subseteq I$, we conclude that $a \in P$ and $bc \in P$. Thus $a, b \in P$ or $a, c \in P$. Hence $ab \in I$ or $ac \in I$. Then I is a $G(2)$-absorbing ideal. □

Let R be a graded field. A graded subring V of R is said to be a *Gr-valuation ring* R, if for every homogenous $x \in R$ either $x \in V$ or $x^{-1} \in V$.

Theorem 4.7. *Let V be a Gr-valuation ring of the graded field, I be a non-zero graded ideal of V and P is a G-prime ideal with $P^2 \subseteq I$. If $I = Q$ for some G-P-primary ideal Q or $I = Q_1Q_2$ for some G-P-primary ideals Q_1 and Q_2 of V, then I is a $G(2)$-absorbing primary ideal.*

Proof. Assume that Q_1, Q_2 are G-P-primary ideals of V. If $I = Q_1$, then I is a $G(2)$-absorbing primary ideal, by Theorem 4.6 and Lemma 2.2. Let V be a Gr-valuation ring. Then V is a Gr-Dedekind domain, by [13, Proposition II.2.2]. Now since Q_1, Q_2 are G-P-primary ideals, for every G-maximal ideal M of V such that $P \subseteq M$, Q_1R_M and Q_2R_M are G-PR_M-primary ideals of V. Thus $Q_1R_MQ_2R_M = Q_1Q_2R_M$ is a G-PR_M-primary. Then Q_1Q_2 is a G-P-primary ideal of V. Hence I is a $G(2)$-absorbing primary ideal, by Theorem 2.12. □

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