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On Generalised Quasi-ideals in Ordered Ternary Semigroups

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Abstract. In this paper, we introduce generalised quasi-ideals in ordered ternary semigroups. Also, we define ordered m-right ideals, ordered (p, q)-lateral ideals and ordered n-left ideals in ordered ternary semigroups and studied the relation between them. Some intersection properties of ordered $(m, (p, q), n)$-quasi ideals are examined. We also characterize these notions in terms of minimal ordered $(m, (p, q), n)$-quasi-ideals in ordered ternary semigroups. Moreover, m-right simple, (p, q)-lateral simple, n-left simple, and $(m, (p, q), n)$-quasi simple ordered ternary semigroups are defined and some properties of them are studied.

1. Introduction

The idea of investigation of n-ary algebras *i.e.* the sets with one n-ary operation was given by Kasner's [10]. Dornte [6] introduced the notion of n-ary groups. A ternary semigroup is a particular case of an n-ary semigroup for n=3[14]. Ternary semigroups are universal algebra with one associative operation. Different applications of ternary structures in physics are described by Kerner [12]. Sioson [15] studied the ideal theory in ternary semigroup. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals. Dixit and Dewan [5] studied the properties of quasi-ideals and bi-ideals in ternary semigroups.

Steinfeld [16] introduced the notion of quasi-ideal for semigroups. It is a generalization of the notion of one sided ideal. The concept of the (m, n)-quasi-ideal in semigroups was given by Lajos [13]. It is studied by many researchers in different algebraic structures [1, 3]. Dubey and Anuradha [7] introduced generalised

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quasi-ideals and generalised bi-ideals in ternary semigroups and characterized these notions in terms of minimal quasi-ideals and minimal bi-ideals in ternary semigroups.

Kehayopulu build-up the theory of partially ordered semigroups. In [11] he introduced the concept and notion of ordered quasi-ideals in ordered semigroups. Abbasi and Basar [2] characterized intra-regular po-Γ-semigroups through ordered quasi-Γ-ideals, ordered right Γ-ideals and ordered left Γ-ideals. Iampan [8] introduced ordered ternary semigroup and characterized the minimality and maximality of ordered lateral ideals in ordered ternary semigroups. Daddi and Pawar [4] introduced the concepts of ordered quasi-ideals, ordered bi-ideals in ordered ternary semigroups and studied their properties. Jailoka and Iampan [9] studied some results on the minimality and maximality of ordered quasi-ideals in ordered ternary semigroups.

2. Preliminaries

Definition 2.1.([14]) A non-empty set S with a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1, x_2, x_3]$, is called a *ternary semigroup* if it satisfies the following identity, for any $x_1, x_2, x_3, x_4, x_5 \in S$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [[x_1x_2[x_3x_4x_5]].$$

For non-empty subsets A, B and C of a ternary semigroup S,

$$[ABC] := \{[abc] : a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we write $[\{a\}BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. For the sake of simplicity, we write $[x_1x_2x_3]$ as $x_1x_2x_3$ and $[ABC]$ as ABC.

Definition 2.2. A non-empty subset T of a ternary semigroup S is called a *ternary subsemigroup* of S if $TTT \subseteq T$.

For any positive integers m and n with $m \leq n$ and any elements $x_1, x_2, x_3 ....x_{2n}$ and x_{2n+1} of a ternary semigroup [15], we can write

$$[x_1x_2x_3.........x_{2n+1}] = [x_1, x_2, x_3..[[x_mx_{m+1}x_{m+2}]x_{m+3}x_{m+4}]...x_{2n+1}].$$

Example 2.3.([5]) Let $S = \{-i, 0, i\}$. Then S is a ternary semigroup under the multiplication over complex number while S is not a semigroup under complex number multiplication.

Definition 2.4.([8]) A ternary semigroup S is called a *partially ordered ternary semigroup* if there exits a partially ordered relation $\leq$ such that for any $a, b, x, y \in S$, $a \leq b \Rightarrow axy \leq bxy, xay \leq xby$, and $xya \leq xyb$.

Example 2.5. Let

$$S = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ c & 0 & d & e \\ f & g & 0 & h \\ i & 0 & 0 & j \end{pmatrix} : a, b, c, d, e, f, g, h, i, j \in \mathbb{N} \cup \{0\} \right\},$$

where $\mathbb{N} \cup \{0\}$ is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}}$ is "less than or equal to". Now we define partial order relation $\leq_S$ on S by, for any A, $B \in S$

$$A \leq_S B \text{ if and only if } a_{ij} \leq_{\mathbb{N}} b_{ij}, \text{ for all } i \text{ and } j.$$

Then it is easy to verify that S is an ordered ternary semigroup under usual multiplication of matrices over $\mathbb{N} \cup \{0\}$ with partial order relation $\leq_S$.

For a subset H of S, we denote $(H] := \{s \in S \mid s \leq h \text{ for some } h \in H\}$. If $H = \{a\}$, we also write $(\{a\}]$ as $(a]$.

Definition 2.6.([8]) A ternary subsemigroup T of S is called an *ordered ternary subsemigroup* of S if $(T] \subseteq T$.

Theorem 2.7.([4]) *Let S be an ordered ternary semigroup, then the following hold:*

(1) $A \subseteq (A]$, *for all* $A \subseteq S$.

(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(3) $((A]] = (A]$, *for all* $A \subseteq S$.

(4) $(A](B](C] \subseteq (ABC]$, *for all* $A, B, C \subseteq S$.

Definition 2.8.([8]) An element z of S is called a *zero element* if

(1) $zxy = xzy = xyz = z$ for all $x, y \in S$, and

(2) $z \leq x$ for all $x \in S$.

If $z \in S$ is a zero element, it is denoted by 0.

Definition 2.9.([4]) An element a of S is called *regular* if there exists an element x in S such that $a \leq axa$. S is called *regular ordered ternary semigroup* if every element of S is regular.

Theorem 2.10.([4]) *Let T be an ordered ternary subsemigroup of S. Then T is regular if and only if $a \in (aTa]$, for all $a \in T$.*

Definition 2.11.([4]) A non-empty subset I of S is called an *ordered right* (resp, *ordered left*, *ordered lateral*) *ideal* if

(1) $ISS \subseteq I$ (resp., $SSI \subseteq I$, $SIS \subseteq I$), and

(2) $(I] \subseteq I$.

Example 2.12. In Example 2.5, let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{N} \cup \{0\} \right\} \text{ s.t. } R \subseteq S.$$

Then R is an ordered right ideal of S.

A non-empty subset I of S is called an *ordered ideal* of S if I is an ordered left, an ordered right and an ordered lateral ideal of S.

Example 2.13. In Example 2.5, let

$$I = \left\{ \begin{pmatrix} a & 0 & 0 & e \\ b & 0 & 0 & f \\ c & 0 & 0 & g \\ d & 0 & 0 & h \end{pmatrix} : a, b, c, d, e, f, g, h \in \mathbb{N} \cup \{0\} \right\} \text{ s.t. } I \subseteq S.$$

Then I is an ordered ideal of S.

Definition 2.14.([4]) A non-empty subset Q of S is called an *ordered quasi-ideal* of S if

(1) $(SSQ] \cap (SQS] \cap (QSS] \subseteq Q$,

(2) $(SSQ] \cap (SSQSS] \cap (QSS] \subseteq Q$, and

(3) $(Q] \subseteq Q$.

Example 2.15. In Example 2.5, let

$$Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{N} \cup \{0\} \right\} \text{ s.t. } Q \subseteq S.$$

Then Q is an ordered quasi ideal of S which is not an ordered ideal of S.

We can easily prove that $\{0\}$ is the smallest ordered quasi-ideal of S with a zero element and it is called a zero ordered quasi-ideal of S. Moreover, $0 \in Q$ for all ordered quasi-ideal Q of S.

Definition 2.16.([4]) A non-empty subset B of S is called an *ordered bi-ideal* of S if,

(1) $BSBSB \subseteq B$,

(2) For $a \in B$, $b \in S$ such that $b \leq a$ implies $b \in B$. *i.e.* $(B] = B$.

Example 2.17. Consider $S = L_4(\mathbb{N}\cup\{0\})$, be the set of all strictly lower triangular 4×4 matrices over $\mathbb{N}\cup\{0\}$. As we know that $\mathbb{N}\cup\{0\}$ is an ordered ternary semigroup under the ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}}$ is "less than or equal to". Then S is an ordered ternary semigroup under the usual multiplication of matrices over $\mathbb{N}\cup\{0\}$ with partial order relation $\leq_S$, as defined in the Example 2.5. Let

$$B_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix} : a, b \in \mathbb{N}\cup\{0\} \right\}.$$

Clearly B_4 is a ternary subsemigroup of S. We have that $B_4SB_4SB_4 \subseteq B_4$ and $(B_4] \subseteq B_4$. But $(B_4SS] \cap (SB_4S \cup SSB_4SS] \cap (SSB_4] =$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & c & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{N}\cup\{0\} \right\} \nsubseteq B_4.$$

Therefore B_4 is an ordered bi-ideal of S which is not an ordered quasi-ideal of S.

3. Generalised Quasi-ideals

In this section, we define ordered $(m,(p,q),n)$-quasi-ideal of an ordered ternary semigroup and establish some of their elementary properties.

Definition 3.1. A ternary subsemigroup Q of S is called a *generalised quasi-ideal* or an *ordered* $(m,(p,q),n)$*-quasi-ideal* of S if

(1) $(Q(SS)^m] \cap ((S^pQS^q \cup S^pSQSS^q)] \cap ((SS)^nQ] \subseteq Q$, where m, n, p, q are positive integers greater than zero and $p+q =$ even,

(2) $(Q] \subseteq Q$.

Example 3.2. All the ordered quasi ideals of the Examples 2.12, 2.13 and 2.15 are ordered $(m,(p,q),n)$ quasi ideals of S.

Remark 3.3. Every ordered quasi-ideal of S is an ordered $(1,(1,1),1)$-quasi-ideal of S. But an ordered $(m,(p,q),n)$-quasi-ideal need not be an ordered quasi-ideal of S.

Example 3.4. Let

$$S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & j & 0 \end{pmatrix} : a, b, c, d, e, f, g, h, i, j \in \mathbb{N}\cup\{0\} \right\}.$$

As we know that $\mathbb{N}\cup\{0\}$ is an ordered ternary semigroup under ordinary multiplication of numbers with partial ordered relation $\leq_{\mathbb{N}}$ is "less than or equal to". Then S is an ordered ternary semigroup under the usual multiplication of matrices over $\mathbb{N}\cup\{0\}$ with partial order relation $\leq_S$, as defined in the Example 2.5. Let

$$Q_{gen}=\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ a & 0 & 0 & 0 & 0\\ 0 & b & 0 & 0 & 0\\ 0 & 0 & c & 0 & 0 \end{pmatrix} : a,b,c\in\mathbb{N}\cup\{0\}\right\}.$$

Then it is easy to see that Q_{gen} is a ternary subsemigroup of S and Q_{gen} is an ordered $(2,(2,2),2)$ quasi ideal of S. Now $(Q_{gen}SS]\cap(SQ_{gen}S\cup SSQ_{gen}SS]\cap(SSQ_{gen}]=$

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ a & 0 & 0 & 0 & 0 \end{pmatrix} : a\in\mathbb{N}\cup\{0\}\right\}\nsubseteq Q_{gen}.$$

Therefore Q_{gen} is not an ordered $(1,(1,1),1)$ quasi-ideal ideal of S. Although Q_{gen} is an ordered $(1,(1,1),1)$ bi-ideal of S.

Lemma 3.5. *Let $\{T_i \mid i\in I\}$ be the arbitrary collection of ordered ternary subsemigroups of S such that $\bigcap_{i\in I} T_i\neq\emptyset$. Then $\bigcap_{i\in I} T_i$ is an ordered ternary subsemigroup of S.*

Proof. Let T_i be an ordered ternary subsemigroup of S for all $i\in I$ such that $\bigcap_{i\in I} T_i\neq\emptyset$ and let $t_1,t_2,t_3\in\bigcap_{i\in I} T_i$ for all $i\in I$. As T_i is an ordered ternary subsemigroup of S for all $i\in I$, we have $t_1t_2t_3\in T_i$ for all $i\in I$. Therefore $t_1t_2t_3\in\bigcap_{i\in I} T_i$.
Now suppose that $x\in(\bigcap_{i\in I} T_i]$. Then $x\leq a$, for some $a\in\bigcap_{i\in I} T_i$. Now $a\in T_i$, for all $i\in I$, it implies $x\in(T_i]=T_i$, for all $i\in I$. Thus we have $x\in\bigcap_{i\in I} T_i$, which shows that $(\bigcap_{i\in I} T_i]\subseteq\bigcap_{i\in I} T_i$. Hence $\bigcap_{i\in I} T_i$ is an ordered ternary subsemigroup of S. □

Theorem 3.6. *Let S be an ordered ternary semigroup and Q_i be an ordered $(m,(p,q),n)$-quasi ideal of S such that $\bigcap_{i\in I} Q_i\neq\emptyset$. Then $\bigcap_{i\in I} Q_i$ is an ordered $(m,(p,q),n)$-quasi ideal of S.*

Proof. Let $\{Q_i \mid i\in I\}$ be a family of ordered $(m,(p,q),n)$-quasi ideal of S. Clearly $Q=\bigcap_{i\in I} Q_i$ is an ordered ternary subsemigroup of S by the Lemma 3.5. We claim that Q is an ordered $(m,(p,q),n)$-quasi ideal of S. Now

$$\begin{aligned}&(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ]\\ &= (\bigcap_{i\in I} Q_i(SS)^m] \cap (S^p \bigcap_{i\in I} Q_iS^q \cup S^pS \bigcap_{i\in I} Q_iSS^q] \cap ((SS)^n \bigcap_{i\in I} Q_i]\\ &\subseteq (Q_i(SS)^m] \cap (S^pQ_iS^q \cup S^pSQ_iSS^q] \cap ((SS)^nQ_i],\ for\ all\ i \in I.\\ &\subseteq Q_i,\ for\ all\ i \in I.\end{aligned}$$

Therefore $(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq \bigcap_{i\in I} Q_i$. Consequently Q is an ordered $(m,(p,q),n)$-quasi ideal of S. □

Definition 3.7. Let S be an ordered ternary semigroup. Then a ternary subsemigroup

(1) R of S is called an ordered m-right ideal of S if $R(SS)^m \subseteq R$ and $(R] = R$,

(2) M of S is called an ordered (p,q)-lateral ideal of S if $(S^pMS^q \cup S^pSMSS^q) \subseteq M$ and $(M] = M$,

(3) L of S is called an ordered n-left ideal of S if $(SS)^nL \subseteq L$ and $(L] = L$.

where m, n, p, q are positive integers and $p+q$ is an even positive integer.

Theorem 3.8. *Every ordered m-right, ordered (p,q)-lateral and ordered n-left ideal of S is an ordered $(m,(p,q),n)$-quasi ideal of S. But converse need not be true.*

Proof. Proof is straight forward. Conversely, take an ordered ternary semigroup S given in the Example 2.5. Let

$$H = \left\{ \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathbb{N} \cup \{0\} \right\}.$$

Then H is an ordered $(3,(2,2),3)$-quasi ideal of S. But it is not an ordered 3-right ideal, an ordered $(2,2)$-lateral ideal and an ordered 3-left ideal of S. □

Theorem 3.9. *Let S be an ordered ternary semigroup. Then the following statements hold:*

(1) *Let R_i be an ordered m-right ideal of S such that $\bigcap_{i\in I} R_i \neq \emptyset$. Then $\bigcap_{i\in I} R_i$ is an ordered m-right ideal of S.*

(2) *Let M_i be an ordered (p,q)-lateral ideal of S such that $\bigcap_{i\in I} M_i \neq \emptyset$. Then $\bigcap_{i\in I} M_i$ is an ordered (p,q)-lateral ideal of S.*

(3) *Let L_i be an ordered n-left ideal of S such that $\bigcap_{i\in I} Li \neq \emptyset$. Then $\bigcap_{i\in I} L_i$ is an ordered n-left ideal of S.*

Proof. Analogous to the proof of the Theorem 3.6. □

Theorem 3.10. *Let R be an ordered m-right ideal, M be an ordered (p,q)-lateral ideal and L be an ordered n-left ideal of S. Then $R \cap M \cap L$ is an ordered $(m,(p,q),n)$-quasi-ideal of S.*

Proof. Suppose that $Q = R \cap M \cap L$. By the Theorem 3.8, every ordered m-right, ordered (p,q)-lateral and ordered n-left ideal of S are ordered $(m,(p,q),n)$-quasi-ideals of S. Therefore R, M and L are ordered $(m,(p,q),n)$-quasi-ideals of S. If $R \cap M \cap L$ is non-empty. Then by the Theorem 3.6, we have $Q = R \cap M \cap L$ is an ordered $(m,(p,q),n)$-quasi-ideal of S. □

Theorem 3.11. *Let A be any non-empty subset of S. Then*

(1) $(A(SS)^m]$ *is an ordered m-right ideal of S,*

(2) $(S^pAS^q \cup S^pSASS^q]$ *is an ordered (p,q)-lateral ideal of S,*

(3) $((SS)^nA]$ *is an ordered n-left ideal of S,*

(4) $(A(SS)^m] \cap (S^pAS^q \cup S^pSASS^q] \cap ((SS)^nA]$ *is an ordered $(m,(p,q),n)$-quasi ideal of S.*

Proof. (1) It is easy to show that $(A(SS)^m]$ is a ternary subsemigroup and $((A(SS)^m]] = (A(SS)^m]$. Now

$$\begin{aligned}(A(SS)^m](SS)^m &\subseteq (A(SS)^m]((SS)^m] \\ &\subseteq (A(SS)^m(SS)^m] \\ &= (A(SSSS)^m] \\ &\subseteq (A(SS)^m].\end{aligned}$$

Therefore $(A(SS)^m]$ is an ordered m-right ideal of S.

(2), (3) and (4) can be proved analogously to (1). □

Theorem 3.12. *Let A be an ordered ternary subsemigroup of S. Then*

(1) $(A \cup A(SS)^m]$ *is an ordered m-right ideal of S containing A,*

(2) $(A \cup S^pAS^q \cup S^pSASS^q]$ *is an ordered (p,q)-lateral ideal of S containing A,*

(3) $(A \cup (SS)^nA]$ *is an ordered n-left ideal of S containing A,*

(4) *$((A(SS)^m] \cap (S^pAS^q \cup S^pSASS^q] \cap ((SS)^nA]) \cup (A]$ is an ordered $(m,(p,q),n)$-quasi ideal of S containing A.*

Proof. Proof is analogous to the Theorem 3.11. □

Theorem 3.13. *Let Q be an ordered $(m,(p,q),n)$-quasi ideal of S. Then*

(1) $R = (Q \cup Q(SS)^m]$ *is an ordered m-right ideal of S,*

(2) $M = (Q \cup S^pQS^q \cup S^pSQSS^q]$ *is an ordered (p,q)-lateral ideal of S,*

(3) $L = (Q \cup (SS)^n Q]$ *is an ordered n-left ideal of S.*

Proof. Proof is analogous to the Theorem 3.11. □

An ordered $(m, (p, q), n)$-quasi ideal Q has the $(m, (p, q), n)$ intersection property if Q is the intersection of an ordered m-right ideal, an ordered (p, q)-lateral and an ordered n-left ideal of S.

Remark 3.14 Every ordered m-right ideal, ordered (p, q)-lateral ideal and ordered n-left ideal have the intersection property.

Theorem 3.15. *Let S be an ordered ternary semigroup and Q be an ordered $(m, (p, q), n)$-quasi ideal of S. Then the following statements are equivalent:*

(1) *Q has the $(m, (p, q), n)$ intersection property;*

(2) $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] = Q$;

(3) $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap ((SS)^n Q] \subseteq Q$;

(4) $(Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] \subseteq Q$;

(5) $(Q \cup Q(SS)^m] \cap (S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] \subseteq Q$.

Proof. (1) $\Rightarrow$ (2) : Let Q has the $(m, (p, q), n)$ intersection property. It is obvious that $Q \subseteq (Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q]$....(i). Now to prove (2) we will show that $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] \subseteq Q$. As it is known that Q has $(m, (p, q), n)$ intersection property, it implies there exist an ordered m-right ideal R, an ordered (p, q)-lateral ideal M and an ordered n-left ideal L of S s.t. $R \cap M \cap L = Q$. Then $Q \subseteq R$, $Q \subseteq M$ and $Q \subseteq L$. Also we have that $((SS)^n Q] \subseteq ((SS)^n L] \subseteq L$ and in the similar way $(S^p QS^q \cup S^p SQSS^q] \subseteq M$ and $(Q(SS)^m] \subseteq R$ which implies $Q \cup ((SS)^n Q] = (Q \cup (SS)^n Q] \subseteq L$, $Q \cup (S^p QS^q \cup S^p SQSS^q] = (Q \cup S^p QS^q \cup S^p SQSS^q] \subseteq M$ and $Q \cup (Q(SS)^m] = (Q \cup Q(SS)^m] \subseteq R$. Hence we have $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] \subseteq L \cap M \cap R = Q$....(ii). From (i) and (ii), we have $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] = Q$.

(2) $\Rightarrow$ (1) : Consider $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] = Q$. By the Theorem 3.13, $(Q \cup Q(SS)^m]$ is an ordered m-right ideal of S, $(Q \cup S^p QS^q \cup S^p SQSS^q]$ is an ordered (p, q)-lateral ideal of S and $(Q \cup (SS)^n Q]$ is an ordered n-left ideal of S. Let $R = (Q \cup Q(SS)^m]$, $M = (Q \cup S^p QS^q \cup S^p SQSS^q]$ and $L = (Q \cup (SS)^n Q]$. Now $(Q(SS)^m] \cap (S^p QS^q \cup S^p SQSS^q] \cap ((SS)^n Q] \subseteq Q$, as Q is an ordered $(m, (p, q), n)$-quasi ideal of S. We have

$$\begin{aligned} L \cap M \cap R &= (Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q] \\ &= Q \cup (Q(SS)^m] \cap (S^p QS^q \cup S^p SQSS^q] \cap ((SS)^n Q] \\ &\subseteq Q \cup Q \\ &= Q. \end{aligned}$$

(2) $\Rightarrow$ (3) : Consider $Q = (Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q \cup S^p SQSS^q] \cap (Q \cup (SS)^n Q]$. As we know $((SS)^n Q] \subseteq (Q \cup (SS)^n Q]$, we have $(Q \cup Q(SS)^m] \cap (Q \cup S^p QS^q$

$\cup\, S^pSQSS^q] \cap ((SS)^nQ] \subseteq (Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap (Q \cup (SS)^nQ]$. Hence $(Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$.
(3) $\Rightarrow$ (2): Let $(Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$. Then $Q \subseteq (Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap (Q \cup (SS)^nQ]$. Now we have to show that $(Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$. For this suppose that $x \in (Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap (Q \cup (SS)^nQ]$. Then we have to show that $x \in Q$. Now $(Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$. We have $x \in Q$. Therefore $(Q \cup Q(SS)^m] \cap (Q \cup S^pQS^q \cup S^pSQSS^q] \cap (Q \cup (SS)^nQ]$=$Q$.

The proofs for (2) $\Rightarrow$ (4), (2)$\Rightarrow$ (5) and (4) $\Rightarrow$(2), (5) $\Rightarrow$ (2) are analogous to the proofs of (2)$\Rightarrow$(3) and (3) $\Rightarrow$ (2), respectively. □

Theorem 3.16. *Every regular ordered ternary semigroup S has the intersection property of ordered $(m,(p,q),n)$-quasi-ideals for any positive integer m,p,q,n and $p+q$ is even.*

Proof. Let S be a regular ordered ternary semigroup and Q be an ordered $(m,(p,q),n)$-quasi-ideal of S. Then by the Theorem 3.13, $R = (Q \cup Q(SS)^m]$, M=$(Q \cup S^pQS^q \cup S^pSQSS^q]$ and L=$(Q \cup (SS)^nQ]$ are an ordered m-right, an ordered (p,q)-lateral and an ordered n-left ideal of S respectively. Clearly $Q \subseteq R, Q \subseteq M$ and $Q \subseteq L$ implies $Q \subseteq R \cap M \cap L$. As S is regular, we have $Q \subseteq (Q(SS)^m]$, $Q \subseteq (S^pQS^p \cup S^pSQSS^q]$ and $Q \subseteq ((SS)^nQ]$. Therefore $R = (Q(SS)^m]$, $M = (S^pQS^q \cup S^pSQSS^q]$ and $L = ((SS)^nQ]$. Hence we have $R \cap M \cap L = (Q(SS)^m] \cap ((S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$. It implies $Q = R \cap M \cap L$. Therefore Q has the $(m,(p,q),n)$ intersection property. □

4. Generalised Minimal Quasi-ideals and $(m,(p,q),n)$-Quasi Simple Ordered Ternary Semigroups

In this section, we introduce the concept of a minimal ordered $(m,(p,q),n)$-quasi-ideal, a minimal ordered m-right ideal, a minimal ordered (p,q)-lateral ideal and a minimal ordered n-left ideal in ordered ternary semigroups and study the relationship between them. Also m-right simple, (p,q)-lateral simple, n-left simple and $(m,(p,q),n)$-quasi-simple ordered ternary semigroups are defined and some properties of them are investigated.

Definition 4.1. An ordered m-right ideal R of S is called *minimal ordered m-right ideal* if it does not properly contain any ordered m-right ideal of S.

Definition 4.2. An ordered (p,q)-lateral ideal M of S is called *minimal ordered (p,q)-lateral ideal* if it does not properly contain any ordered (p,q)-lateral ideal of S.

Definition 4.3. An ordered n-left ideal L of S is called *minimal ordered n-left ideal* if it does not properly contain any ordered n-left ideal of S.

Definition 4.4. An ordered $(m,(p,q),n)$-quasi ideal Q of S is called *minimal ordered $(m,(p,q),n)$-quasi ideal* if it does not properly contain any ordered

$(m,(p,q),n)$-quasi ideal of S.

Theorem 4.5. *Let S be an ordered ternary semigroup and Q be an ordered $(m,(p,q),n)$-quasi-ideal of S. Then Q is minimal if and only if Q is the intersection of some minimal ordered m-right ideal R, minimal ordered (p,q)-lateral ideal M and minimal ordered n-left ideal L of S.*

Proof. Assume that Q is minimal ordered $(m,(p,q),n)$-quasi ideal of S. Then

$$(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q.$$

By the Theorem 3.11, $(Q(SS)^m]$,$(S^pQS^q \cup S^pSQSS^q]$, $((SS)^nQ]$ are an ordered m-right, an ordered (p,q)-lateral and an ordered n-left ideal of S and by Theorem 3.10, intersection of an ordered m-right, an ordered (p,q)-lateral and an ordered n-left ideal is an ordered $(m,(p,q),n)$-quasi ideal of S. As Q is minimal, we have

$$(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] = Q.$$

To show that $((SS)^nQ]$ is an minimal ordered n-left ideal of S. Let L be an ordered n-left ideal of S contained in $((SS)^nQ]$. Then $((SS)^nL] \subseteq (L] = L \subseteq ((SS)^nQ]$. Thus, $(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nL] \subseteq (Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nQ] \subseteq Q$
Now $(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nL]$ is an ordered $(m,(p,q),n)$-quasi ideal of S and Q is a minimal ordered $(m,(p,q),n)$-quasi ideal of S. We have $(Q(SS)^m] \cap (S^pQS^q \cup S^pSQSS^q] \cap ((SS)^nL] = Q$. Then $Q \subseteq ((SS)^nL]$ and we have $((SS)^nQ] \subseteq ((SS)^n((SS)^nL]] \subseteq ((SS)^n(SS)^nL] \subseteq ((SS)^nL] \subseteq L$. It implies $L = ((SS)^nQ]$. Therefore $((SS)^nQ]$ is a minimal ordered n-left ideal of S. Similarly other cases can be proved.

Conversely, suppose $Q = L \cap M \cap R$, where L, M and R are minimal ordered n-left, minimal ordered (p,q)-lateral and minimal ordered m-right ideals of S, respectively. Then $Q \subseteq L, Q \subseteq M$ and $Q \subseteq R$. By the Theorem 3.10, Q will be an ordered $(m,(p,q),n)$-quasi ideal of S. Now we have to show that Q is minimal. For this let $Q^{'}$ be an ordered $(m,(p,q),n)$-quasi ideal of S contained in Q. By the Theorem 3.11, $(Q^{'}(SS)^m]$, $(S^pQ^{'}S^q \cup S^pSQ^{'}SS^q]$, $((SS)^nQ^{'}]$ are an ordered m-right, an ordered (p,q)-lateral and an ordered n-left ideal of S, respectively. Now,

$$((SS)^nQ^{'}] \subseteq ((SS)^nQ] \subseteq ((SS)^nL] \subseteq L.$$

But L is minimal, it implies $((SS)^nQ^{'}] = L$. Similarly $(Q^{'}(SS)^m] = R$ and $(S^pQ^{'}S^q \cup S^pSQ^{'}SS^q] = M$. As $Q^{'}$ is an ordered $(m,(p,q),n)$-quasi ideal of S. We have

$$Q = L \cap M \cap R = ((SS)^nQ^{'}] \cap (S^pQ^{'}S^q \cup S^pSQ^{'}SS^q] \cap (Q^{'}(SS)^m] \subseteq Q^{'}.$$

It implies $Q = Q^{'}$.

Therefore Q is a minimal ordered $(m,(p,q),n)$-quasi ideal of S. □

Theorem 4.6. *Let S be an ordered ternary semigroup. Then the following holds:*

(1) *An ordered m-right ideal R is minimal if and only if* $(a(SS)^m] = R$ *for all* $a \in R$*;*

(2) *An ordered* (p,q)*-lateral ideal M is minimal if and only if* $(S^p aS^q \cup S^p SaSS^q] = M$ *for all* $a \in M$*;*

(3) *An ordered n-left ideal L is minimal if and only if* $((SS)^n a] = L$ *for all* $a \in L$*;*

(4) *An ordered* $(m,(p,q),n)$*-quasi-ideal Q is minimal if and only if* $(a(SS)^m] \cap (S^p aS^q \cup S^p SaSS^q] \cap ((SS)^n a] = Q$ *for all* $a \in Q$*.*

Proof. (2) Suppose that an ordered (p,q)-lateral ideal M is minimal. Let $a \in M$. Then $(S^p SaSS^q \cup S^p aS^q] \subseteq (S^p SMSS^q \cup S^p MS^q] \subseteq M$. By the Theorem 3.11(2), we have $(S^p SaSS^q \cup S^p aS^q]$ is an ordered (p,q)-lateral ideal of S. As M is minimal ordered (p,q)-lateral ideal of S. We have $(S^p SaSS^q \cup S^p aS^q] = M$.

Conversely, suppose that $(S^p SaSS^q \cup S^p aS^q] = M$ for all $a \in$ M. Let $M^{'}$ be any ordered (p,q)-lateral ideal of S contained in M. Let $m \in M^{'}$. Then $m \in M$. By assumption, we have $(S^p SmSS^q \cup S^p mS^q] = M$ for all $m \in M$. $M = (S^p SmSS^q \cup S^p mS^q] \subseteq (S^p SM^{'}SS^q \cup S^p M^{'}S^q] \subseteq M^{'}$. It implies $M \subseteq M^{'}$. Thus, $M= M^{'}$. Hence, M is minimal ordered (p,q)-lateral ideal of S.

Analogously we can prove (1), (3) and (4). □

Definition 4.7 Let S be an ordered ternary semigroup. Then S is called an *m-right simple* if S is a unique ordered m-right ideal of S.

Definition 4.8. Let S be an ordered ternary semigroup. Then S is called an (p,q)*-lateral simple* if S is a unique ordered (p,q)-lateral ideal of S.

Definition 4.9. Let S be an ordered ternary semigroup. Then S is called an *n-left simple* if S is a unique ordered n-left ideal of S.

Definition 4.10. Let S be an ordered ternary semigroup. Then S is called an $(m,(p,q),n)$*-quasi simple* if S is a unique ordered $(m,(p,q),n)$-quasi ideal of S.

Theorem 4.11. *Let S be an ordered ternary semigroup. The following statements hold true:*

(1) *S is an m-right simple if and only if* $(a(SS)^m] = S$ *for all* $a \in S$*;*

(2) *S is an* (p,q)*-lateral simple if and only if* $(S^p aS^q \cup S^p SaSS^q] = S$ *for all* $a \in S$*;*

(3) *S is an n-left simple if and only if* $((SS)^n a] = S$ *for all* $a \in S$*;*

(4) *S is an* $(m,(p,q),n)$*-quasi simple if and only if* $(a(SS)^m] \cap (S^p aS^q \cup S^p SaSS^q] \cap ((SS)^n a] = S$ *for all* $a \in S$*.*

Proof. (1) Assume that S is an m-right simple, we have that S is a minimal ordered m-right ideal of S. By the Theorem 4.6(1), $(a(SS)^m] = S$ for all $a \in S$.

Conversely, suppose that $(a(SS)^m] = S$ for all $a \in S$. By the Theorem 4.6(1), S is a minimal ordered m-right ideal of S, and therefore S is an m-right simple.

(2), (3) and (4) can be proved analogously to (1). □

Theorem 4.12. *Let S be an ordered ternary semigroup. The following statements hold true:*

(1) *If an ordered m-right ideal R of S is an m-right simple, then R is a minimal ordered m-right ideal of S;*

(2) *If an ordered (p, q)-lateral ideal M of S is an (p, q)-lateral simple, then M is a minimal ordered (p, q)-lateral ideal of S;*

(3) *If an ordered n-left ideal L of S is an n-left simple, then L is a minimal ordered n-left ideal of S;*

(4) *If an ordered $(m, (p, q), n)$-quasi ideal Q of S is an $(m, (p, q), n)$-quasi simple, then Q is a minimal ordered $(m, (p, q), n)$-quasi ideal of S.*

Proof. (1) Let R be an m-right simple. By the Theorem 4.11(1), we have $(a(RR)^m] = R$ for all $a \in R$. For every $a \in R$, we have $R = (a(RR)^m] \subseteq (a(SS)^m] \subseteq (R(SS)^m] \subseteq R$. Then $(a(SS)^m] = R$ for all $a \in R$. By the Theorem 4.6(1), we have R is minimal.

(2), (3) and (4) can be proved analogously to (1). □

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