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Some Inequalities for Derivatives of Polynomials

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Abstract. In this paper, we generalize some earlier well known results by considering polynomials of lacunary type having some zeros at origin and rest of the zeros on or outside the boundary of a prescribed disk.

1. Introduction

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and $P'(z)$ its derivative, then it is known that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above result, which is an immediate consequence of Bernstein's inequality on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial $P(z) = \lambda z^n$, where λ is a complex number.

For the class of polynomials having all their zeros in $|z| \geq 1$, inequality (1.1) can be sharpened. In fact, Erdös conjectured and later Lax [7] proved that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

On the other hand, if the polynomial $P(z)$ of degree n has all its zeros in $|z| \leq 1$,

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then it was proved by Turán [12], that

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.3}$$

Inequalities (1.2) and (1.3) are best possible and become equality for polynomials which have all zeros on $|z|=1$.

Aziz and Dawood [1] improved inequality (1.2) and under the same hypothesis proved:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\Big\}. \tag{1.4}$$

Equality in (1.4) holds for $P(z)=\alpha+\beta z^n$, $|\alpha|\geq|\beta|$.

For the class of polynomials $P(z)$ of degree n having all their zeros in $|z|\geq k$, $k\geq 1$, Malik [8] proved:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{1.5}$$

Inequality (1.5) was further improved by Govil [6] who under the same hypothesis proved:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k}\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\Big\}. \tag{1.6}$$

Chan and Malik [2] obtained a generalization of (1.5) by considering the lacunary type of polynomials and obtained the following:

Theorem A. *Let $P(z) := a_0+\sum_{j=\mu}^{n} a_j z^j$, $1\leq\mu\leq n$ be a polynomial of degree n having all its zeros in $|z|\geq k$, $k\geq 1$, then*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^\mu}\max_{|z|=1}|P(z)|. \tag{1.7}$$

The result is best possible and extremal polynomial is $P(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

The next result was proved by Pukhta [10], who infact proved:

Theorem B. *Let $P(z) := a_0+\sum_{j=\mu}^{n} a_j z^j$, $1\leq\mu\leq n$ be a polynomial of degree n having all its zeros in $|z|\geq k$, $k\geq 1$, then*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k^\mu}\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\Big\}. \tag{1.8}$$

The result is best possible and extremal polynomial is $P(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Theorem C. *Let $P(z)$ be a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |P(z)|. \tag{1.9}$$

The above result was independently given by Govil [5]. For the polynomials of type $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu < n$, Theorem C was further generalized by Dewan and Hans [3] and proved the following:

Theorem D. *Let $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu < n$, be a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |P(z)|. \tag{1.10}$$

2. Lemmas

For the proofs of above theorems, we need the following lemmas. The first result is due to Qazi [11, Lemma 1].

Lemma 2.1. *If $P(z) := a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$ is a polynomial of degree n having all its zeros in $|z| \geq k$, $k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq n \frac{n|a_0| + \mu|a_\mu| k^{\mu+1}}{n|a_0|(1 + k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |P(z)|.$$

The next lemma which we need is due to Dewan and Hans [4].

Lemma 2.2. *Let $P(z) := a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu < n$, be a polynomial of degree n, having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^{n-\mu+1}} \left(\frac{n|a_n| k^{2\mu} + \mu|a_{n-\mu}| k^{\mu-1}}{\mu|a_{n-\mu}|(1 + k^{\mu-1}) + n|a_n| k^{\mu-1}(1 + k^{\mu+1})} \right) \max_{|z|=1} |P(z)|.$$

Theorem 2.3. *Let $P(z) := z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j \right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ be a polynomial of degree n having s-fold zeros at the origin and remaining $n-s$ zeros in $|z| > k$, $k \geq 1$, then*

$$\begin{aligned} &\max_{|z|=1} |P'(z)| \leq \\ &\frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu| k^{\mu+1} + s(n-s)|a_0|(1 + k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1} + k^{2\mu})}{(n-s)|a_0|(1 + k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |P(z)| \\ &\qquad - \frac{1}{k^s} \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu| k^{\mu+1}}{(n-s)|a_0|(1 + k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1} + k^{2\mu})} \min_{|z|=k} |P(z)|. \end{aligned}$$

If we take $\mu = 1$ in Theorem 2.3, we have the following result which is an improvement of a result of Mir [9, Theorem 1.6]

Proof. Let

$$P(z) = z^s H(z) \tag{2.1}$$

where

$$H(z) = a_0 + \sum_{j=\mu}^{n-s} a_j z^j$$

is a polynomial of degree $n-s$ having all its zeros in $|z| > k$, $k \geq 1$. From (2.1), we have

$$\begin{aligned} zP'(z) &= sz^s H(z) + z^{s+1} H'(z) \\ &= sP(z) + z^{s+1} H'(z). \end{aligned}$$

This gives for $|z| = 1$,

$$|P'(z)| \leq s|P(z)| + |H'(z)|.$$

The above inequality holds for all points on $|z| = 1$ and hence

$$|P'(z)| \leq s|P(z)| + \max_{|z|=1} |H'(z)|. \tag{2.2}$$

Let $m = \min_{|z|=k} |H(z)|$, then $m \leq |H(z)|$ for $|z| = k$. As all $n-s$ zeros of $H(z)$ lie in $|z| > k$, $k \geq 1$, therefore for every complex number λ such that $|\lambda| < 1$, it follows by Rouche's Theorem that all the zeros of the polynomial $H(z) - \lambda m$ of degree $n-s$ lie in $|z| > k$, $k \geq 1$. By using Lemma 2.1 to the polynomial $H(z) - \lambda m$ of degree $n-s$, we get

$$\max_{|z|=1} |H'(z)| \leq (n-s)\Big(\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\Big) \max_{|z|=1} |H(z) - \lambda m|. \tag{2.3}$$

Choosing the argument of λ such that

$$|H(z) - \lambda m| = |H(z)| - |\lambda| m \quad for \quad |z| = 1, \tag{2.4}$$

and letting $|\lambda| \to 1$, we get from (2.3) and (2.4)

$$\max_{|z|=1} |H'(z)| \leq (n-s)\Big(\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\Big) \max_{|z|=1} (|H(z)| - m). \tag{2.5}$$

Combining the inequalities (2.2) and (2.5), we obtain

$$\begin{aligned} |P'(z)| \leq{}& (n-s)\Big(\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\Big) \max_{|z|=1} |H(z)| \\ & - (n-s)\Big(\frac{(n-s)|a_0| + \mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\Big) m + s|P(z)|. \end{aligned} \tag{2.6}$$

From (2.1), we have on $|z| = 1$, $|P(z)| = |H(z)|$ and that one can easily obtain

$$m = \min_{|z|=k} |H(z)| = \frac{1}{k^s} \min_{|z|=k} |P(z)|.$$

This gives from (2.6)

$$\max_{|z|=1} |P'(z)| \leq$$

$$\frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1} + s(n-s)|a_0|(1+k^{\mu+1}) + s\mu|a_\mu|(k^{\mu+1}+k^{2\mu})}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \max_{|z|=1} |P(z)|$$
$$- \frac{1}{k^s} \frac{(n-s)^2|a_0| + (n-s)\mu|a_\mu|k^{\mu+1}}{(n-s)|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})} \min_{|z|=k} |P(z)|.$$

This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let* $P(z) := z^s(a_{n-s}z^{n-s} + \sum_{j=\mu}^{n-s} a_{n-s-j}z^{n-s-j})$, $1 \leq \mu < n-s$, $0 \leq s \leq n-1$, *be a polynomial of degree* n, *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros on* $|z| = k$, $k \leq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \Big\{ \frac{(n-s)}{k^{n-s-\mu+1}} \Big[\frac{(n-s)|a_{n-s}|k^{2\mu} + \mu|a_{n-s-\mu}|k^{\mu-1}}{\mu|a_{n-s-\mu}|(1+k^{\mu-1}) + (n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})} \Big] + s \Big\} \max_{|z|=1} |P(z)|.$$

If we put $\mu = 1$ *in Theorem 2.4, we get the following :*

Proof. Let

$$P(z) = z^s H(z) \tag{2.7}$$

where

$$H(z) = a_{n-s}z^{n-s} + \sum_{j=\mu}^{n-s} a_{n-s-j}z^{n-s-j}$$

is a polynomial of degree $n-s$ having all its zeros on $|z| = k$, $k \leq 1$. By using Lemma 2.2 to the polynomial $H(z)$ of degree $n-s$, we get

$$\max_{|z|=1} |H'(z)| \leq \frac{(n-s)}{k^{n-s-\mu+1}} \Big[\frac{(n-s)|a_{n-s}|k^{2\mu} + \mu|a_{n-s-\mu}|k^{\mu-1}}{\mu|a_{n-s-\mu}|(1+k^{\mu-1}) + (n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})} \Big] \max_{|z|=1} |H(z)|. \tag{2.8}$$

From (2.7), it easily follows that for $|z| = 1$,

$$|P'(z)| \leq s|P(z)| + \max_{|z|=1} |H'(z)|. \tag{2.9}$$

Combining the inequalities (2.8) and (2.9) and using the fact that for $|z| = 1$, $|P(z)| = |H(z)|$, we get

$$\max_{|z|=1} |P'(z)| \leq \Big\{ \frac{(n-s)}{k^{n-s-\mu+1}} \Big[\frac{(n-s)|a_{n-s}|k^{2\mu} + \mu|a_{n-s-\mu}|k^{\mu-1}}{\mu|a_{n-s-\mu}|(1+k^{\mu-1}) + (n-s)|a_{n-s}|k^{\mu-1}(1+k^{\mu+1})} \Big] + s \Big\} \max_{|z|=1} |P(z)|.$$

This completes the proof of the Theorem 2.4. □

Corollary 2.5. *Let* $P(z) := z^s\Big(\sum_{j=0}^{n-s} a_j z^j\Big)$, $0 \leq s \leq n-1$ *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| > k$, $k \geq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{(n-s)^2|a_0| + (n-s)|a_1|k^2 + s(n-s)|a_0|(1+k^2) + s|a_1|(k^2+k^4)}{(n-s)|a_0|(1+k^2) + |a_1|(k^2+k^4)} \max_{|z|=1} |P(z)| - \frac{1}{k^s} \frac{(n-s)^2|a_0| + (n-s)|a_1|k^2}{(n-s)|a_0|(1+k^2) + |a_1|(k^2+k^4)} \min_{|z|=k} |P(z)|.$$

On taking $k = 1$ in Theorem 2.3, we get the following:

Corollary 2.6. *Let* $P(z) := z^s\Big(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\Big)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$ *be a polynomial of degree* n *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros in* $|z| > 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{(n+s)}{2} \max_{|z|=1} |P(z)| - \frac{(n-s)}{2} \min_{|z|=1} |P(z)|.$$

Remark 2.7. Inequality (1.4) that is the result of Aziz and Dawood is a special case of Theorem 2.3 when $\mu = k = 1$ and $s = 0$.

If we put $\mu = 1$ in Theorem 2.4, we get the following :

Corollary 2.8. *Let* $P(z) := z^s(a_{n-s}z^{n-s} + \sum_{j=1}^{n-s} a_{n-s-j} z^{n-s-j})$, $0 \leq s \leq n-1$, *be a polynomial of degree* n, *having* s*-fold zeros at the origin and remaining* $n-s$ *zeros on* $|z| = k$, $k \leq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \left\{ \frac{(n-s)}{k^{n-s}} \Big[\frac{(n-s)|a_{n-s}|k^2 + |a_{n-s-1}|}{2|a_{n-s-1}| + (n-s)|a_{n-s}|(1+k^2)} \Big] + s \right\} \max_{|z|=1} |P(z)|.$$

On taking s=0 and $\mu = 1$ in Theorem 2, we get the following result:

Corollary 2.9. *Let* $P(z) := \sum_{j=0}^{n} a_{n-j} z^{n-j}$, *be a polynomial of degree* n, *having all its zeros on* $|z| = k$, $k \leq 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{k^n} \frac{n|a_n|k^2 + |a_{n-1}|}{2|a_{n-1}| + n|a_n|(1+k^2)} \max_{|z|=1} |P(z)|.$$

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