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STUDY ON BCN AND BAN RULED SURFACES IN $\mathbb{E}^3$

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Abstract. As a continuation to the study in [8, 12, 15, 17], we construct bi-conservative central normal (BCN) and bi-conservative asymptomatic normal (BAN) ruled surfaces in Euclidean 3-space $\mathbb{E}^3$. For such surfaces, local study is given and some examples are constructed using computer aided geometric design (CAGD).

1. Introduction

The study of bi-conservative and bi-harmonic surfaces is nowadays a very active research subject. Many interesting results on these types have been obtained in the last decade. In 1995 [16], THasanis and T. Vlachos firstly classified bi-conservative hypersurfaces in Euclidean 3-spaces and 4-spaces, where they called such hypersurfaces as H-hypersurfaces. RCaddeo et al. in [7] classified bi-conservative surfaces in the three-dimensional Riemannian space forms. The classification of the results given in [7, 16] showed that bi-conservative hypersurfaces in Riemmannian 3-space forms, and Euclidean 4-spaces must be rotational surfaces (besides the constant mean curvature case). Recently, Chen and Munteanu [8] proved that $\delta(2)$-ideal bi-conservative hypersurfaces in Euclidean space $\mathbb{E}^n$ (arbitrary dimension) is minimal or a spherical hypercylinder.

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In the last few years, from the theory of bi-harmonic submanifolds, arised the study of bi-conservative submanifolds that imposed itself as a very promising and interesting research topic through papers like [12,17]. Closely related to the theory of bi-harmonic submanifolds, the study of bi-conservative submanifolds is a very recent and interesting topic in the field of differential geometry.

In the surfaces theory, it is well-known that a surface is said to be "ruled" if it is generated by a continuously moving of a straight line in the space. Ruled surfaces are one of the simplest objects in geometric modeling. A practical application of ruled surfaces is that they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight [1–5].

More recently, Hamdoon and Omran [15], studied ruled surface generated by the spherical indicatrix of such surface. In this paper, the ruled surfaces generated by the central and the asymptomatic normal for which the tangent of their base curves given as a linear combination of the geodesic frenet trihedron are constructed and obtained. The necessary and sufficient conditions for considered surfaces to be bi-conservative, bi-harmonic, flat, II-flat, harmonic and II-harmonic are obtained using the shape operator of these surfaces. An intrinsic characterization of these bi-conservative surfaces is provided. Finally, some examples are given and plotted using computer aided geometric design.

2. Geometric preliminaries

Let $x : M \to \mathbb{E}^3$ be an isometric immersion of a surface M into $\mathbb{E}^3$. Denote the Levi-Civita connections of M and $\mathbb{E}^3$ by ∇ and $\widetilde{\nabla}$, respectively. Let X and Y denote vector fields tangent to M, and let $\vec{N}$ be a normal vector field. The Gauss and Weingarten formulas are given, respectively, by [9, 10]

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \, \vec{N} = -A_{\vec{N}} \, X, \tag{2.2}$$

where h, A are the second fundamental form and the shape operator, respectively. It is well known that the second fundamental form h and the shape operator A are related by

$$\langle h(X,Y), \vec{N}\rangle = \langle A_{\vec{N}}\ X, Y\rangle. \tag{2.3}$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y)Z, W, \vec{N}\rangle = \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.4}$$

$$(\nabla_X\ A)Y = (\nabla_Y\ A)X, \tag{2.5}$$

where

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[XY]},$$

is the curvature tensor of the Levi-Civita connection on M.

Now, let us recall the following important definitions:

Definition 2.1. [13, 14]. Let $x \subset \mathbb{E}^3$ be a regular surface, and let $\vec{N}$ be a surface normal vector field to x defined in a neighborhood of a point $p \in x$. For each tangent vector $\vec{v}$ to x at p, if we put

$$A_p(v) = -\nabla_v\ \vec{N}(p). \tag{2.6}$$

Then A is called the shape operator of x at p derived from $\vec{N}$. That is, the shape operator A is defined as the negative (directional) derivative of the normal $\vec{N}$ (as a vector valued function on x), see Figure (1).

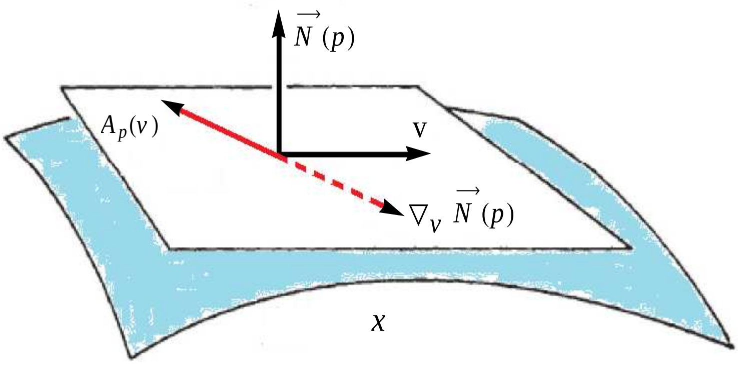

Figure 1. The shape operator of x

Using the coefficients of the first and the second fundamental forms $g_{ij}, h_{ij}, i, j = 1, 2$, respectively, we can easily calculate the shape operator

A of x in the form

$$A = (a_{ij})\,,\ i, j = 1, 2, \tag{2.7}$$

where

$$(a_{ij}) = -\frac{1}{g}\begin{pmatrix} h_{11}g_{22} - h_{12}g_{12} & -h_{11}g_{12} + h_{12}g_{11} \\ h_{12}g_{22} - h_{22}g_{12} & -h_{12}g_{12} + h_{22}g_{11} \end{pmatrix}. \tag{2.8}$$

DEFINITION 2.2. [13, 20]. The Gaussian curvature K of $x \subset \mathbb{E}^3$ is the real-valued function $K\colon x \longrightarrow \mathbb{R}^3$, defined by

$$K = det(A). \tag{2.9}$$

Explicitly, for each point p of x, the Gaussian curvature $K(p)$ of x at p is the determinant of the shape operator A of x at p.

DEFINITION 2.3. [13, 20]. The mean curvature H of $x \subset \mathbb{E}^3$ is the function $H\colon x \longrightarrow \mathbb{R}^3$, defined by

$$H = \frac{1}{2}\,tr(A). \tag{2.10}$$

Explicitly, for each point p of x, the mean curvature $H(p)$ of x at p is the trace of the shape operator A of x at p.

DEFINITION 2.4. [22].

(1): A regular surface for which the mean curvature vanishes identically is called a minimal (harmonic) surface.

(2): A surface is called II-flat if the second Gaussian curvature vanishes identically.

(3): A surface is called II-minimal if the second mean curvature vanishes identically.

DEFINITION 2.5. [11]. A surface x in Euclidean 3-space $\mathbb{E}^3$ is bi-conservative if the mean curvature function H satisfies

$$A(grad\ H) = -Hgrad\ H. \tag{2.11}$$

DEFINITION 2.6. [11]. A surface x in Euclidean 3-space $\mathbb{E}^3$ is said to be bi-harmonic if it satisfies the equation $\Delta^2 x = 0$. According to the well-known Betrami's formula $\Delta x = -2\vec{H}$, the bi-harmonic condition in $\mathbb{E}^3$ is also known as the equation

$$\Delta\vec{H} = 0, \tag{2.12}$$

where Δ is the Laplacian with respect to the first fundamental form of x and is given by

$$\Delta = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial u^i} \Big[\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial u^j} \Big], \tag{2.13}$$

where, (g^{ij}) denotes the associated matrix with the inverse of (g_{ij}).

Using classical notation, we define the second Gaussian curvature K_{II} by [6]
(2.14)

$$K_{II} = \frac{1}{h^2} \left(\begin{vmatrix} \frac{-h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12.1} - \frac{h_{11,2}}{2} \\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12} \\ \frac{h_{22,2}}{2} & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2} \\ \frac{h_{11,2}}{2} & h_{11} & h_{12} \\ \frac{h_{22,1}}{2} & h_{12} & h_{22} \end{vmatrix} \right),$$

where, $h_{ij,l} = \frac{\partial hij}{\partial s^l}$, and $h_{ij,lm} = \frac{\partial^2 hij}{\partial u^l \partial u^m}$, the indices i, j belong to $\{1,2\}$ and the parameters u^1, u^2 are s, v, respectively.

Since Brioschis formulas in Euclidean 3-space $\mathbb{E}^3$, we are able to define the second mean curvature H_{II} of x by replacing the components of the first fundamental form g_{ij} by the components of the second fundamental form h_{ij}, respectively, in Brioschis formula. Consequently, the second mean curvature H_{II} is given by [6]

$$H_{II} = H - \frac{1}{2}\Delta\big(\ln \sqrt{|K|}\,\big), \tag{2.15}$$

where, Δ is the Laplacian with respect to the second fundamental form of x expressed as

$$\Delta = -\frac{1}{\sqrt{|h|}} \sum_{i,j} \frac{\partial}{\partial u^i} \Big[\sqrt{|h|}\, h^{ij} \frac{\partial}{\partial u^j} \Big], \tag{2.16}$$

where, h^{ij} denotes the associated matrix with the inverse of h_{ij}.

3. Geodesic Frenet trihedron of the ruled surface

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation,

$$x : \psi(s,v) = \vec{\alpha}(s) + v\vec{e}(s), \tag{3.1}$$

where $\vec{\alpha}(s)$ represents a space curve which is called the base curve and $\vec{e}(s)$ is a unit vector representing the direction of a straight line and s is the arc-length along the base curve $\vec{\alpha}(s)$.

The vector $\vec{e}(s)$ traces a general space curve (as s varies) on the surface of unit sphere s^2 which it called *spherical indicatrix* of the ruled surface. If we denote the arc-length of $\vec{e}(s)$ as s^*, then

$$s^* = \int_a^b \Big|\frac{d\vec{e}(s)}{ds}\Big| ds. \tag{3.2}$$

The unit normal vector field $\vec{n}$ to the ruled surface (3.1) is

$$\vec{n}(s,v) = \frac{(\frac{d\vec{\alpha}}{ds} + v\frac{d\vec{e}}{ds}) \wedge \vec{e}}{[(\frac{d\vec{\alpha}}{ds} + v\frac{d\vec{e}}{ds})^2 - \langle\frac{d\vec{\alpha}}{ds}, \vec{e}\rangle^2]^{\frac{1}{2}}}, \tag{3.3}$$

The unit normal along a general generator $\vec{l} = \vec{\psi}(s_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called *the asymptotic normal* direction and defined as

$$\vec{g}(s)\Big|_{s=s_0} = \vec{n}(s,v)\Big|_{\substack{s=s_0\\ v\to-\infty}} = \frac{-\frac{d\vec{e}}{ds} \wedge \vec{e}}{\mid \frac{d\vec{e}}{ds} \mid}\Big|_{s=s_0}. \tag{3.4}$$

At v increases to $+\infty$, the unit normal rotates through 180° about $\vec{l}$ and ultimately takes the direction $-\vec{g}$. The point at which $\vec{n}$ has rotated only 90° and is perpendicular to $\vec{g}$ is called the striction point (or central point) on $\vec{l}$. The direction of $\vec{n}$ at this point is denoted by $\vec{t}$ and called *the central normal* of the ruled surface and is given by

$$\vec{t}(s) = \frac{d\vec{e}/ds}{|d\vec{e}/ds|}. \tag{3.5}$$

The Frenet trihedron on a ruled surface can then be defined by the dexterous triplet of vectors $\{\vec{e}, \vec{t}, \vec{g}\}$, where

$$\left.\begin{aligned} \vec{e} =& \textit{spherical indicatrix}, \\ \vec{t} =& \textit{central normal} = \vec{e}' = \frac{d\vec{e}}{ds^*} = \frac{\vec{e_s}}{|\vec{e_s}|}, \\ \vec{g} =& \textit{asymptotic normal} = \vec{e} \wedge \vec{e}' = \frac{\vec{e} \wedge \vec{e_s}}{|\vec{e_s}|}, \end{aligned}\right\} \tag{3.6}$$

where $\vec{e_s} = \frac{d\vec{e}}{ds}$ and $' \equiv \frac{d}{ds^*}$.

Differentiating (3.6) with respect to s^*, we have a set of differential equations similar to the Frenet formula of a space curve, namely

$$\begin{pmatrix} \vec{e}' \\ \vec{t}' \\ \vec{g}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \mu \\ 0 & \mu & 0 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{t} \\ \vec{g} \end{pmatrix} \tag{3.7}$$

where $\mu = \frac{\langle \vec{e}, \vec{e}_s \wedge \vec{e}_{ss} \rangle}{|\vec{e}_s|^3}$ is the geodesic curvature of spherical indicatrix $\vec{e}$. These last equations are called the geodesic Frenet trihedron of the indicatrix $\vec{e}$ for a ruled surface [18, 19, 21].

The striction point on a ruled surface x is the foot of the common normal between two consecutive generators. The set of striction points define *the striction curve* and is given by

$$\vec{c}(s) = \vec{\alpha}(s) - \frac{\langle \vec{\alpha}_s, \vec{e}_s \rangle}{\|\vec{e}_s\|^2} \, \vec{e}(s) \tag{3.8}$$

The parameter of distribution P_e of the ruled surface (3.1) is defined as the limit of the ratio of the shortest distance between the two rulings and their angluded angle which is given by

$$P_e = \frac{\det(\vec{\alpha}', \vec{e}, \vec{e}')}{\langle \vec{e}', \vec{e}' \rangle}. \tag{3.9}$$

If the consecutive generators of a ruled surface intersect, then the surface is said to be developable ($P_e = 0$), otherwise the surface is said to be skew.

Remark 3.1. In this paper, the striction curve of the ruled surface x will be taken as the base curve.

4. BCN ruled surface M

In this section, the ruled surface M which generated by the central normal $\vec{t}$ during the base curve $\vec{c}$ is studied. The necessary and sufficient conditions for M to BCN, harmonic, II-harmonic, II-flat, bi-harmonic and II-bi-harmonic surfaces are constructed and obtained. In this case, for the parametric equation of the surface (3.1), we can define the ruled surface that produced by the central normal $\vec{t}$ during the base curve $\vec{c}$ as follows

$$M : \phi(s,v) = \vec{c}(s) + v\vec{t}(s), \tag{4.1}$$

where the tangent of the base curve of M is given by

$$\vec{c'} = \lambda_1\vec{e} + \lambda_2\vec{t} + \lambda_3\vec{g} \ \in span\{\vec{e},\vec{t},\vec{g}\}, \tag{4.2}$$

with $\langle\vec{c'},\vec{c'}\rangle$=1, implies that $\lambda_1^2+\lambda_2^2+\lambda_3^2 = 1$; where λ_i, i=1,2,3 are scalar functions.

Since the base curve of M is a striction curve, then we can write

$$\langle\vec{c'},\vec{t'}\rangle = \langle\lambda_1\vec{e} + \lambda_2\vec{t} + \lambda_3\vec{g}, \mu\vec{g} - \vec{e}\,\rangle = 0. \tag{4.3}$$

Hence, Eq.(4.2) becomes

$$\vec{c'} = \mu\lambda_3\vec{e} + \lambda_2\vec{t} + \lambda_3\vec{g}, \tag{4.4}$$

That means, the condition for the base curve $\vec{c}$ to be a stiction curve is $\lambda_1 = \mu\lambda_3$, implies that $\mu = const.$, $\lambda_3 \neq 0$.

By calculating the distribution parameter P_t of the ruled surface M, one can get the ruled surface M is developable if and only if λ_3=0; but this is a contradiction, hence the ruled M can not be developable.
Using Eqs. (3.7), (4.1) and (4.4), one can obtain the first fundamental quantities of M as follows

$$g_{11} = \xi_1(\lambda_3^2+v^2)+\lambda_2^2, \quad g_{12} = \lambda_2, \quad g_{22} = 1, \quad g = Det\,(g_{ij}) = \xi_1(\lambda_3^2+v^2). \tag{4.5}$$

where $\xi_1 = 1 + \mu^2$.

Remark 4.1. The only singular point on the ruled surface M is on striction curve (v =0), for which P_t=0.

The unit normal vector field $\vec{n}$ of M is given by

$$\vec{n}(s,v) = \frac{\vec{N}}{\|\vec{N}\|} = \frac{1}{\sqrt{g}}\Big(- (\lambda_3 + v\mu)\vec{e} + (\lambda_3\mu - v)\vec{g}\Big). \tag{4.6}$$

Moreover, the principal normal vector $\vec{e_2}$ of M at the base curve (v=0) is

$$\vec{e_2} = \vec{n}(s,0) = \frac{1}{\sqrt{\xi_1}}\big(- \vec{e} + \mu\vec{g}\big), \ \xi_1 \neq 0. \tag{4.7}$$

From Eqs. (4.4) and (4.7), the binormal vector $\vec{e_3}$ of the curve $\vec{c}$ can be obtained as

$$\vec{e_3} = \frac{1}{\sqrt{\xi_1}}\left(\lambda_2\mu\vec{e} - \lambda_3\xi_1\vec{t} + \lambda_2\vec{g}\right),\ \xi_1 \neq 0. \tag{4.8}$$

From above, we can get the equations that describe the relation between Frenet-Frame $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ of the base curve $\vec{c}$ and the geodesic Frenet trihedron $\{\vec{e}, \vec{t}, \vec{g}\}$ of the indicatrix $\vec{e}$ of M in the form

$$\begin{pmatrix} \vec{e_1} \\ \vec{e_2} \\ \vec{e_3} \end{pmatrix} = \frac{1}{\sqrt{\xi_1}} \begin{pmatrix} \lambda_3\mu & \lambda_2 & \lambda_3 \\ -1 & 0 & \mu \\ \lambda_2\mu & -\lambda_3\xi_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{t} \\ \vec{g} \end{pmatrix}. \tag{4.9}$$

In addition, we can write the second fundamental quantities of M as follows

$$h_{11} = \frac{\lambda_2\lambda_3\xi_1}{\sqrt{g}}, \quad h_{12} = \frac{\lambda_3\xi_1}{\sqrt{g}}, \quad h_{22} = 0, \quad h = Det\,(h_{ij}) = \frac{-\lambda_3^2\xi_1}{\xi_2}, \tag{4.10}$$

where $\xi_2 = \lambda_3^2 + v^2$.

Based on the above results and using Eqs. (2.8), (2.9) and (2.10), one can get the Gaussian K and the mean H curvatures of M as follows

$$K = det(A) = \frac{-\lambda_3^2}{\xi_2^2}. \tag{4.11}$$

Corollary 4.1. *The Gaussian curvature K of the ruled surface M is non positive and K can not be equal to zero along the ruling $\vec{t}$.*

$$H = \frac{1}{2}tr(A) = \frac{-\lambda_2\lambda_3\xi_1}{2\ g^{3/2}}. \tag{4.12}$$

Hence, we get

$$(H_s, H_v) = \left(0,\ \frac{3\lambda_2\lambda_3 v}{2\sqrt{\xi_1}\ \xi_2^{5/2}}\right),\ H_{u^i} = \frac{\partial H}{\partial u^i},\ \{u^i\} = \{s, v\},\ i = 1, 2. \tag{4.13}$$

Thus, one can see that the condition (2.11) can be split into two differential equations as follows

$$a_{11}H_s + a_{12}H_v + HH_s = 0, \tag{4.14}$$

$$a_{21}H_s + a_{22}H_v + HH_v = 0. \tag{4.15}$$

Using (2.8), one can calculate a_{ij} in the following form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = g^{-3/2}\ \lambda_3\xi_1 \begin{pmatrix} 0 & g \\ 1 & -\lambda_2 \end{pmatrix}. \tag{4.16}$$

Solving the Eqs. (4.14) and (4.15), we get the only solution is $\lambda_2 = 0$ and $\lambda_3 \neq 0$, implies that $\vec{c'} \in span\{\vec{e}, \vec{g}\}$. Thus, we can formulate the following theorem

THEOREM 4.1. *The central normal ruled surface M is BCN ruled surface if and only if the tangent of the base curve $\vec{c'} \in span\{\vec{e}, \vec{g}\}$.*

4.1. Harmonic and bi-harmonic properties of BCN ruled surface M. Here, inspired by the concepts the harmonic ($H = 0$) and bi-harmonic ($\Delta\vec{H} = 0$), we can obtain the II-harmonic ($H_{II} = 0$) and II-bi-harmonic ($\Delta H_{II} = 0$) of M.

Using Eqs.(2.14), (4.11) and (4.12), it is easy to calculate the second Gaussian K_{II} and the second mean H_{II} curvatures of M, respectively, as follows

$$K_{II} = \frac{\lambda_2}{2g^{3/2}}\Big(\xi_1(v-\lambda_3)(\lambda_3+v)\Big), \tag{4.17}$$

$$H_{II} = \frac{-\lambda_2}{2g^{3/2}}\Big(-2\lambda_3^2v^2\xi_3 + \lambda_3^4(\mu^2+\xi_3+1) - 3v^4\xi_3\Big), \tag{4.18}$$

where $\xi_3 = \frac{\lambda_2\lambda_3}{\sqrt{\xi_1\xi_2^3}}$.

Using Eqs. (4.6) and (4.12), we can get the mean curvature vector field $\vec{H}$ in the form

$$\vec{H} = H\,\vec{n}(s,v) = \frac{1}{2\ \xi_1\xi_2^2}\Big\{\lambda_2\lambda_3\,(\lambda_3+\mu v)\,,\ 0,\ -\lambda_2\lambda_3\,(\lambda_3\mu - v)\Big\}. \tag{4.19}$$

Using Eqs. (2.13) and (2.16), one can get

(4.20)

$$\Delta\vec{H} = \Big\{\xi_4\big(-4\lambda_3^3+15\mu v^3+24\lambda_3v^2-13\lambda_3^2\mu v\big),\ 0\,,\ \xi_4\big(4\lambda_3^3\mu+15v^3-24\lambda_3\mu v^2-13\lambda_3^2v\big)\Big\},$$

and

(4.21)

$$\Delta H_{II} = \xi_5\Big(\lambda_3^6(3\mu^2+10\xi_6+3)+24\xi_6v^6-54\lambda_3^2\xi_6v^4-\lambda_3^4v^2(15\mu^2+68\xi_6+15)\Big),$$

where $\xi_4 = \frac{\lambda_2^2\lambda_3}{2\xi_1\xi_2^4}$, $\xi_5 = \frac{\lambda_2^2}{2\lambda_3^3\xi_1^{3/2}\xi_2^{7/2}}$ and $\xi_6 = \frac{\lambda_2\lambda_3}{\xi_1^{1/2}\xi_2^{3/2}}$.

From Eqs. (4.12), (4.17), (4.20) and (4.21), one can conclude the following corollary

Corollary 4.2. *The central normal ruled surface M is harmonic, bi-harmonic, II-bi-harmonic and II-flat if and only if the surface M is BCN ruled surface.*

Also, Eq. (4.18) give the following corollary

Corollary 4.3. *The central normal ruled surface M is II-harmonic if and only if one of the following is satisfied*

(i): *The surface M is BCN ruled surface.*

(ii): $\lambda_2 = \pm\frac{\lambda_3^3\ \xi_1^{3/2}\ \xi_1^{1/2}}{\lambda_3^2 - 3v^2},\ \lambda_3^2 \neq 3v^2$.

4.2. Example. Now, we give an example to illustrate our previous investigation in this section.

Example 4.2.1. The elliptic hyperboloid of one sheet is a non-developable BCN ruled surface parameterized by

$$\phi(s,v) = \frac{1}{\sqrt{5}}\Big(\sin(s) - v\cos(s), 1+2v, -\cos(s) - v\sin(s)\Big), \tag{4.22}$$

with λ_2=0 and $\lambda_3 \neq 0$. Short calculations give us the following: $\vec{e} = \frac{1}{2}\big(-\sin(s), 2s, \cos(s)\big)$, $\vec{t} = \frac{\sqrt{5}}{5}\big(-\cos(s), 2, -\sin(s)\big)$, $\vec{g} = \frac{\sqrt{5}}{5}\big(-s\sin(s) - \cos(s), -\frac{1}{2}, s\cos(s) - \sin(s)\big)$, $P_t = \frac{-2}{3}$ and $\mu = \frac{2}{5\sqrt{5}}$, as plotted in Figure (2).

5. BAN ruled surface M^*

In this section, the ruled surface M^* which is generated by the asymptotic normal $\vec{g}$ during the base curve $\vec{c}$ is investigated. The necessary and sufficient conditions for M^* to be BAN, flat, II-flat, harmonic, II-harmonic, bi-harmonic and II-bi-harmonic surfaces are given.
In this case, for the parametric equation of the surface (3.1), we can define the ruled surface that produced by the asymptotic normal $\vec{g}$ during the base curve $\vec{c}$ as follows

$$M^* : \phi^*(s,v) = \vec{c^*}(s) + v\vec{g}(s), \tag{5.1}$$

where the tangent of the base curve of M^* is given by

$$\vec{c^*}' = \lambda_1\vec{e} + \lambda_2\vec{t} + \lambda_3\vec{g}\ \in span\{\vec{e}, \vec{t}, \vec{g}\}, \tag{5.2}$$

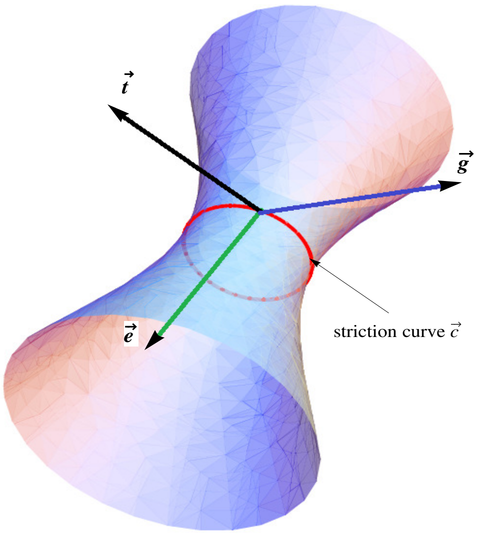

FIGURE 2. Non-developable BCN ruled surface M.

with $\langle \vec{c^{*}}', \vec{c^{*}}' \rangle$=1, implies that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$; where λ_i, i=1,2,3 are scalar functions.
Since the base curve of M^* is a striction curve, then we can write

$$\langle \vec{c^{*}}', \vec{g}' \rangle = \langle \lambda_1 \vec{e} + \lambda_2 \vec{t} + \lambda_3 \vec{g}, \mu \vec{t} \rangle = 0. \tag{5.3}$$

Hence, Eq.(5.2) becomes

$$\vec{c^{*}}' = \lambda_1 \vec{e} + \lambda_3 \vec{g}, \tag{5.4}$$

In view of the distribution parameter P_g of the ruled surface M^*, one can see the ruled surface M^* is developable if and only if $\mu\lambda_1$ =0. Hence, we have the following corollary.

COROLLARY 5.1. *The ruled surface M^* which generated by the asymptotic normal $\vec{g}$ is developable ($P_g = 0$) if and only if one of the following is satisfied*

- **(i):** *The asymptotic normal $\vec{g}$ is tangent of its striction curve $\vec{c}$, i.e, the tangent of the base curve $\vec{c^{*}}'$ and the generator $\vec{g}$ are parallel.*
- **(ii):** *The spherical indicatrix $\vec{e}$ of M^* is a geodesic($\mu = 0$).*

Using Eqs. (3.7) and (5.4), one can obtain the first fundamental quantities of M^* as follows
(5.5)
$$g_{11}^* = 1+v^2\mu^2, \quad g_{12}^* = \lambda_3, \quad g_{22}^* = 1, \quad g^* = Det\left(g_{ij}^*\right) = \lambda_1^2+v^2\mu^2,\ \lambda_1^2 = 1-\lambda_3^2.$$
The discriminant g^* vanishes only if $\lambda_1 = 0$ and the geodesic curvature μ of the spherical indicatrix $\vec{e}$ is equal zero at the same time. As an immediate result we have the following corollary.

Corollary 5.2. *The ruled surface M^* has a singular point if the spherical indicatrix $\vec{e}$ is a geodesic and the base curve is parallel to asymptotic normal $\vec{g}$.*

The the unit normal vector field $\vec{n^*}$ of M^* is given by

$$\vec{n^*}(s,v) = \frac{\vec{N^*}}{\|\vec{N^*}\|} = \frac{1}{\sqrt{g^*}}\left(v\mu\ \vec{e} - \lambda_1\vec{t}\right). \tag{5.6}$$

Moreover, the principal normal vector $\vec{e_2^*}$ of M^* at the base curve (v=0) is

$$\vec{e_2^*} = \vec{n}^*(s,0) = -\vec{t}. \tag{5.7}$$

From Eqs. (5.4) and (5.7), the binormal vector $\vec{e_3^*}$ of the curve $\vec{c^*}$ can be given as

$$\vec{e_3^*} = \vec{e_1^*} \wedge \vec{e_2^*} = \lambda_1\vec{g} - \lambda_3\vec{e}. \tag{5.8}$$

From above, we can get the equations that describe the relation between Frenet-frame $\{\vec{e_1^*}, \vec{e_2^*}, \vec{e_3^*}\}$ of the base curve $\vec{c^*}$ and the geodesic Frenet trihedron $\{\vec{e}, \vec{t}, \vec{g}\}$ of the indicatrix $\vec{e}$ of M^* as in the form

$$\begin{pmatrix} \vec{e_1^*} \\ \vec{e_2^*} \\ \vec{e_3^*} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & -1 & 0 \\ \lambda_3 & 0 & -\lambda_1 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{t} \\ \vec{g} \end{pmatrix}. \tag{5.9}$$

Remark 5.1. For a developable ruled surface M^* with λ_1=0, the relation between Frenet-frame of the base curve $\vec{c^*}$ and the geodesic Frenet trihedron of the indicatrix $\vec{e}$ of M^* is given by

$$\begin{pmatrix} \vec{e_1^*} \\ \vec{e_2^*} \\ \vec{e_3^*} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{t} \\ \vec{g} \end{pmatrix}. \tag{5.10}$$

In addition, we can get the second fundamental quantities of M^* as follows

$$(5.11)\quad \begin{aligned} &h_{11}^* = -\frac{1}{\sqrt{g^*}}\left(\lambda_1\lambda_3\mu + v^2\mu^2 + \lambda_1(\lambda_1 + v\mu')\right), \quad h_{12}^* = -\frac{\lambda_1\mu}{\sqrt{g^*}}, \\ &h_{22}^* = 0, \quad h^* = Det\left(h_{ij}^*\right) = \frac{-\lambda_1^2\mu^2}{g^*}. \end{aligned}$$

Using (2.8) and the same technique as in section 4, one can get the Gaussian curvature K^* of M^* as follows

$$(5.12)\quad K^* = det(A^*) = \frac{-\lambda_1^2\mu^2}{(\lambda_1^2 + v^2\mu^2)^2}.$$

Thus, we have the following theorem

Theorem 5.1. *The Gaussian curvature K^* of the ruled surface M^* is non positive and K^* equal to zero only along the ruling which meet the striction curve at a sigular point* ($P_g = 0,\ v \neq 0$).

Also we can get the mean curvature function H^* of the ruled surface M^* in the form

$$(5.13)\quad H^* = \frac{1}{2}\, tr(A^*) = \frac{\lambda_1\lambda_3\mu - v^2\mu^2 - \lambda_1(\lambda_1 + v\mu')}{2(\lambda_1^2 + v^2\mu^2)^{3/2}}.$$

Thus, one can obtain the following

(5.14)

$$\begin{aligned} H_s^* &= \frac{1}{2(g^*)^{5/2}}\left(\mu'\Big(\lambda_1^3\lambda_3 + v^2\mu\big(-2\lambda_1\lambda_3\mu + v^2\mu^2 + \lambda_1\left(\lambda_1 + 3v\mu'\right)\big)\Big) - \lambda_1 v\mu''\, g^*\right), \\ H_v^* &= \frac{1}{2(g^*)^{5/2}}\Big(-\lambda_1^3\mu' + v^3\mu^4 - 3\lambda_1\lambda_3 v\mu^3 + \lambda_1 v\mu^2\left(\lambda_1 + 2v\mu'\right)\Big). \end{aligned}$$

As an analogously, from Eqs. (4.14) and (4.15), the condition (2.11) can be split into two differential equations as in the form

$$(5.15)\quad a_{11}^* H_s^* + a_{12}^* H_v^* + H^* H_s^* = 0$$

$$(5.16)\quad a_{21}^* H_s^* + a_{22}^* H_v^* + H^* H_v^* = 0$$

Using (2.8), one can get a_{ij} as in the following form

$$(5.17)\quad \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix} = (g^*)^{-3/2}\begin{pmatrix} \xi_7 & \xi_8 \\ \lambda_1\mu & \lambda_1\lambda_3\mu \end{pmatrix},$$

where, $\xi_7 = -v^2\mu^2 - \lambda_1(\lambda_1 + v\mu')$ and $\xi_8 = (\lambda_3 - \lambda_1\mu)(\lambda_1^2 + v^2\mu^2) + v\lambda_1\lambda_3\mu'$.

Hence, Eqs. (5.15) and (5.16) can be expressed, respectively, as in the following form

$$\frac{1}{4(g^*)^4}\sum_{i=0}^{6}\lambda_1^i\,F_i=0, \tag{5.18}$$

$$\frac{1}{4(g^*)^4}\sum_{j=0}^{5}\lambda_1^j\,G_j=0. \tag{5.19}$$

That means, the above equations have been rewritten as a linear combination of λ_1^i which coefficients F_i and G_j are function of the s-variable. In this case, by using mathematica programming, we noticed that the expression is stopped at F_6 and G_5. Therefore, they must be vanished in some s-interval. So, F_i can be given as

$$\begin{aligned}
F_6 &= 2\mu\mu', \quad F_5=-5\mu'\lambda_3+3v\mu''-2v\mu^3,\\
F_4 &= \mu\mu'\lambda_3^2-2v^2\mu^3\mu'-3v^2\mu\mu'-v\mu\mu''\lambda_3+v\mu'(3v\mu''-5\mu'\lambda_3)\\
&\quad +6v\mu^4\lambda_3+2v\mu^2\lambda_3,\\
F_3 &= 6v^3\mu^2\mu''-12v^3\mu\mu'^2-4v^3\mu^5+8v^2\mu^2\mu'\lambda_3-6v\mu^3\lambda_3^2,\\
F_2 &= -4v^4\mu^5\mu'-6v^4\mu^3\mu'-9v^4\mu\mu'^3+3v^4\mu^2\mu'\mu''-\lambda_3v^3\mu^3\mu''\\
&\quad +13\lambda_3v^3\mu^2\mu'^2+6\lambda_3v^3\mu^6+4\lambda_3v^3\mu^4-8\lambda_3^2v^2\mu^3\mu',\\
F_1 &= 3v^5\mu^4\mu''-12v^5\mu^3\mu'^2-2v^5\mu^7+13\lambda_3v^4\mu^4\mu'-6\lambda_3^2v^3\mu^5,\\
F_0 &= -3v^6\mu^5\mu'+2v^5\mu^6\lambda_3,
\end{aligned} \tag{5.20}$$

and G_i can be given as

$$\begin{aligned}
G_5 &= \mu',\\
G_4 &= \mu(2v\mu''-5\lambda_3\mu')+v\mu'^2-v\mu^2,\\
G_3 &= 6v\mu^3\lambda_3-4v^2\mu^2\mu',\\
G_2 &= 2v^3\mu^3\mu''-8v^3\mu^2\mu'^2-2v^3\mu^4+13\lambda_3v^2\mu^3\mu'-9\lambda_3^2v\mu^4,\\
G_1 &= 6v^3\mu^5\lambda_3-5v^4\mu^4\mu',\\
G_0 &= v^5\mu^6.
\end{aligned} \tag{5.21}$$

From F_6, it is worth noting that the only two possibilities for vanishing all the coefficients are:

Case (i): $\mu = 0$, in this case, all the coefficients are vanished identically.

Case (ii): $\mu' = 0$, that is μ=constant $\neq 0$, then we have

$$F_5 = -2v\mu^3,$$

which equals zero if $\mu = 0$, but this is a contradiction. This leads to $G_j = 0$ in (5.19). Thus, we have the proof of the following theorem

THEOREM 5.2. *The asymptotic normal ruled surface M^* is BAN ruled surface if and only if the spherical indicatrix $\vec{e}$ of M^* is geodesic ($\mu = 0$).*

5.1. Harmonic and bi-harmonic properties of BAN ruled surface M^*. Back to Eq. (5.13), we note that the mean curvature of a BAN ruled surface H^* equal zero if the amount in the numerator of Eq. (5.13) vanishes. So that by solving the differential equation in the numerator of Eq. (5.13), we have

(5.22)
$$\mu = \frac{\lambda_1}{2v^2}\left(\sqrt{\lambda_3^2 - 4v^2}\tanh\left(\frac{\sqrt{\lambda_3^2 - 4v^2}\,(s - c_1\lambda_1 v)}{2v}\right) + \lambda_3\right), \; v \neq 0,$$

where c_1 is an arbitrary constant. Then, we have the following corollary

COROLLARY 5.3. *The ruled surface M^* is harmonic if the geodesic curvature of the spherical indicatrix $\vec{e}$ is related by Eq. (5.22).*

Using Eq.(2.14), it is easy to calculate the second Gaussian curvature K^*_{II} of M^* in the form

$$K^*_{II} = \frac{\lambda_1^3\lambda_3\mu - v^4\mu^4 - \lambda_1\lambda_3 v^2\mu^3 - 2\lambda_1^2 v^2\mu^2 - \lambda_1^3\,(\lambda_1 + 2v\mu')}{2\lambda_1^2\,(\lambda_1^2 + v^2\mu^2)^{3/2}} \tag{5.23}$$

REMARK 5.2. The asymptotic normal ruled surface M^* can not be II-flat any way.

Using Eqs.(2.15) and (2.16), one can obtain the second mean curvature H^*_{II} of M^* in the form

(5.24)
$$\begin{aligned}
H^*_{II} =&\frac{\xi_9}{4\lambda_1^4\mu^4\xi_{10}(g^*)^{7/2}}\Bigg(2\lambda_1^4\lambda_3^2\mu^4 - 2\lambda_1^6\mu^2 - 2v^6\mu^8 - 6\lambda_1^2v^4\mu^6 - 6\lambda_1^2\lambda_3^2v^2\mu^6\\
&- 6\lambda_1^4v^2\mu^4 + 2\lambda_1^5v\mu^2\mu'\lambda_1^6\mu'^2 - 2\lambda_1v^5\mu^6\mu' - 4\lambda_1^3v^3\mu^4\mu' + 6\lambda_3\lambda_1^2v^3\mu^5\mu'\\
&+ 2\lambda_1^4v^2\mu^2\mu'^2 - 10\lambda_3\lambda_1^4v\mu^3\mu' + 2\lambda_1^8\mu^4\xi_{10} - 5\lambda_1^2v^4\mu^4\mu'^2 + 4\lambda_1^6v^2\mu^6\xi_{10}\\
&+ 2\lambda_1^2v^4\mu^5\mu'' + 2\lambda_1^4v^4\mu^8\xi_{10} + 2\lambda_1^4v^2\mu^3\mu''\Bigg),
\end{aligned}$$

where $\xi_9 = \lambda_1\lambda_3\mu - v^2\mu^2 - \lambda_1(\lambda_1 + v\mu^{'})$, $\xi_{10} = \frac{\xi_9}{\sqrt{(g^*)^3}}$, $\lambda_1 \neq 0$ *and* $\mu \neq 0$. Consequently, we can get the relation between the mean H^* and the second mean H^*_{II} curvatures of M^* as follows

(5.25)
$$\begin{aligned}
H^*_{II} =&\frac{H^*}{4\lambda_1^4\mu^4\xi_{10}(g^*)^{2}}\Bigg(2\lambda_1^4\lambda_3^2\mu^4 - 2\lambda_1^6\mu^2 - 2v^6\mu^8 - 6\lambda_1^2v^4\mu^6 - 6\lambda_1^2\lambda_3^2v^2\mu^6\\
&- 6\lambda_1^4v^2\mu^4 + 2\lambda_1^5v\mu^2\mu'\lambda_1^6\mu'^2 - 2\lambda_1v^5\mu^6\mu' - 4\lambda_1^3v^3\mu^4\mu' + 6\lambda_3\lambda_1^2v^3\mu^5\mu'\\
&+ 2\lambda_1^4v^2\mu^2\mu'^2 - 10\lambda_3\lambda_1^4v\mu^3\mu' + 2\lambda_1^8\mu^4\xi_{10} - 5\lambda_1^2v^4\mu^4\mu'^2 + 4\lambda_1^6v^2\mu^6\xi_{10}\\
&+ 2\lambda_1^2v^4\mu^5\mu'' + 2\lambda_1^4v^4\mu^8\xi_{10} + 2\lambda_1^4v^2\mu^3\mu''\Bigg).
\end{aligned}$$

Looking at the previous equation, we have only one case for the vanishing of H^*_{II}. The amount between brackets can not be vanished because λ_1 and μ can not be equal zero at the same time. Hence, the only possibility is $H^* = 0$, that is M^* is minimal. Thus, we have the following corollary

Corollary 5.4. *The asymptotic normal ruled surface M^* is II-harmonic if it is harmonic.*

Using Eqs. (5.6) and (5.13), we can get the mean curvature vector field $\vec{H}^*$ of M^* in the form

$$\vec{H}^* = \frac{1}{2(g^*)^2}\Big\{ - v\mu\xi_9,\ \lambda_1\xi_9,\ 0\Big\}. \tag{5.26}$$

Then, by direct computations, one can find the Laplacian operator Δ for $\vec{H}^*$ in the following form

$$\Delta\vec{H}^* = \frac{1}{2\lambda_1(g^*)^4}\left\{\sum_{i=0}^{6}\lambda_1^i F_i \ , \sum_{j=0}^{7}\lambda_1^j G_j \ , \ 0\right\}, \tag{5.27}$$

where F_i are given by

$$\begin{aligned}
F_6 &= 2\mu\,(2v\mu'' - 3\lambda_3\mu') + 2v\mu'^2 + 5v\mu^2,\\
F_5 &= 7v^2\mu^2\mu' - 4\lambda_3 v\mu^3,\\
F_4 &= -v^3\mu^3\mu'' - 17v^3\mu^2\mu'^2 + 40\lambda_3 v^2\mu^3\mu' + v\mu^4(9v^2 - 13\lambda_3^2),\\
F_3 &= 4v^4\mu^4\mu' + 2\lambda_3 v^3\mu^5,\\
F_2 &= 9v^5\mu^4\mu'^2 - 5v^4\mu^5(2\lambda_3\mu' + v\mu'') + 3v^3\mu^6(5\lambda_3^2 + v^2),\\
F_1 &= 6\lambda_3 v^5\mu^7 - 3v^6\mu^6\mu',\\
F_0 &= -v^7\mu^8,
\end{aligned} \tag{5.28}$$

and G_i are given by

$$\begin{aligned}
G_7 &= -2(\mu'' + \mu),\\
G_6 &= 2\lambda_3\mu^2 - 5v\mu\mu'\\
G_5 &= 4\lambda_3^2\mu^3 + 5v^2\mu^2\mu'' + 12v^2\mu\mu'^2 - 34\lambda_3 v\mu^2\mu',\\
G_4 &= -6\lambda_3 v^2\mu^4,\\
G_3 &= 7v^4\mu^4\mu'' - 16v^4\mu^3\mu'^2 + 22\lambda_3 v^3\mu^4\mu' + 6v^2\mu^5(v^2 - 4\lambda_3^2),\\
G_2 &= 5v^5\mu^5\mu' - 8\lambda_3 v^4\mu^6,\\
G_1 &= 4v^6\mu^7,\\
G_0 &= 0.
\end{aligned} \tag{5.29}$$

From the above two Eqs. (5.28) and (5.29), for the two functions F_0 and G_1, one can see that all the components in (5.27) vanish if μ equal zero. Thus we have the following corollary:

Corollary 5.5. *The asymptotic normal ruled surface M^* is biharmonic if the surface M^* is BAN surface.*

Also, by calculating the Laplacian operator Δ for the second mean curvature H^*_{II} function, one can obtain the following form

$$\Delta H^*_{II} = \frac{1}{-2\lambda_1^5\mu^3(g^*)^4}\left(\sum_{i=0}^{10}\lambda_1^i F_i\right). \tag{5.30}$$

After some computations and using the same technique in (5.28), one can get

(5.31)

$$\begin{aligned}
F_{10} =& -4\mu\mu'' + 2\mu^2 + 2\mu^3\mu''\sqrt{g^*} + \mu^4\sqrt{g^*},\\
F_9 =& -12\lambda_3\mu^2\mu'' + 6\lambda_3\mu\mu'^2 + 2\lambda_3\mu^3 + 5v\mu^4\mu'\sqrt{g^*} - 2\lambda_3\mu^5\sqrt{g^*}\\
&+ 4v\mu^{(3)}\mu^2 + v\mu^2\mu' - 22v\mu'^3 + 11v\mu\mu'\mu'',\\
F_8 =& 10\lambda_3^2\mu^4 - 3v^2\mu^5\mu''\sqrt{g^*} + 26\lambda_3 v\mu^5\mu'\sqrt{g^*} - 9v^2\mu^4\mu'^2\sqrt{g^*} + v^2\mu^6\sqrt{g^*}\\
&- 3\lambda_3^2\mu^6\sqrt{g^*} - 7v^2\mu^3\mu'' + 62v^2\mu^2\mu'^2 + 12v^2\mu^4 - 74\lambda_3 v\mu^3\mu',\\
F_7 =& 10\lambda_3^3\mu^5 + 7v^3\mu^{(3)}\mu^4 + 6v^3\mu^4\mu' - 10v^3\mu^2\mu'^3 - 52v^3\mu^3\mu'\mu'' + 49\lambda_3 v^2\mu^4\mu''\\
&+ 95\lambda_3 v^2\mu^3\mu'^2 + 4\lambda_3 v^2\mu^7\sqrt{g^*} + 6\lambda_3 v^2\mu^5 + 2v^3\mu^6\mu'\sqrt{g^*} - 113\lambda_3^2 v\mu^4\mu',\\
F_6 =& -3v^4\mu^5\mu'' + 46v^4\mu^4\mu'^2 + 28v^4\mu^6 + 8\lambda_3 v^3\mu^5\mu' + 15\lambda_3^2 v^2\mu^8\sqrt{g^*}\\
&- 48\lambda_3^2 v^2\mu^6 - 5v^4\mu^7\mu''\sqrt{g^*} + 9v^4\mu^6\mu'^2\sqrt{g^*} - v^4\mu^8\sqrt{g^*}\\
&- 10\lambda_3 v^3\mu^7\mu'\sqrt{g^*},\\
F_5 =& 2v^5\mu^{(3)}\mu^6 + 12v^5\mu^6\mu' + 114v^5\mu^4\mu'^3 - 55v^5\mu^5\mu'\mu'' + 52\lambda_3 v^4\mu^6\mu''\\
&- 226\lambda_3 v^4\mu^5\mu'^2 + 6\lambda_3 v^4\mu^7 + 202\lambda_3^2 v^3\mu^6\mu' - 78\lambda_3^3 v^2\mu^7 - 3v^5\mu^8\mu'\sqrt{g^*}\\
&+ 6\lambda_3 v^4\mu^9\sqrt{g^*},\\
F_4 =& -v^6\mu^7\mu'' - 14v^6\mu^6\mu'^2 + 32v^6\mu^8 + 78\lambda_3 v^5\mu^7\mu' - 46\lambda_3^2 v^4\mu^8 - v^6\mu^{10}\sqrt{g^*},\\
F_3 =& -v^7\mu^{(3)}\mu^8 + 10v^7\mu^8\mu' - 10v^7\mu^6\mu'^3 + 8v^7\mu^7\mu'\mu'' - 9\lambda_3 v^6\mu^8\mu''\\
&+ 21\lambda_3 v^6\mu^7\mu'^2 + 2\lambda_3 v^6\mu^9 - 21\lambda_3^2 v^5\mu^8\mu' + 24\lambda_3^3 v^4\mu^9,\\
F_2 =& -v^8\mu^9\mu'' + 2v^8\mu^8\mu'^2 + 18v^8\mu^{10} - 4\lambda_3 v^7\mu^9\mu' + 12\lambda_3^2 v^6\mu^{10},\\
F_1 =& 3v^9\mu^{10}\mu',\ F_0 = 4v^{10}\mu^{12}.
\end{aligned}$$

From Eq. (5.30), one can see that the amount ΔH^*_{II} becomes undefined at $\mu = 0$.

Then, one can deduce the following remark:

REMARK 5.3. From Eqs. (5.31), it is worth noting that the asymptotic normal ruled surface M^* can not be II-bi-harmonic.

5.2. Examples. Now, we give the following examples to illustrate our previous investigation in this section.

EXAMPLE 5.2.1. The surface

$$\phi_1^*(s,v) = \frac{1}{\sqrt{2}}\Big(-\cos(s) + v\sin(s), \sin(s) - v\cos(s), s+v\Big), \tag{5.32}$$

is a developable BAN ruled surface with $\lambda_1 = \lambda_2$=0 and $\lambda_3 = 1$. Short calculations give us the following
$\vec{e} = \frac{\sqrt{2}}{2}\big(-\sin(s),\ \cos(s),\ 1\big)$, $\vec{t} = \big(-\cos(s),\ -\sin(s),\ 0\big)$, $\vec{g} = \frac{\sqrt{2}}{2}\big(\sin(s),\ -\cos(s),\ 1\big)$,
$P_{\phi_1^*} = 0$ and $\mu = 1$, as plotted in Figure (3).

EXAMPLE 5.2.2. The surface

$$\phi_2^*(s,v) = \frac{1}{\sqrt{2}}\Big(-\cos(s) + v\sin(s), 1+v, \sin(s) + v\cos(s)\Big), \tag{5.33}$$

is a non-developable BAN ruled surface with λ_2 =0, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$. Short calculations give us the following
$\vec{e} = \frac{\sqrt{2}}{2}\big(-\sin(s), 1, -\cos(s)\big)$, $\vec{t} = \big(-\cos(s), 0, \sin(s)\big)$ and $\vec{g} = \frac{\sqrt{2}}{2}\big(\sin(s), 1, \cos(s)\big)$,
$P_{\phi_2^*} = \frac{-1}{2}$ and $\mu = 1$, as plotted in Figure (4).

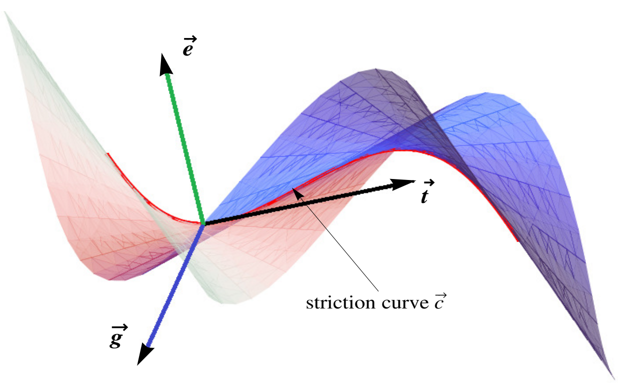

FIGURE 3. Developable BAN ruled surface M^*

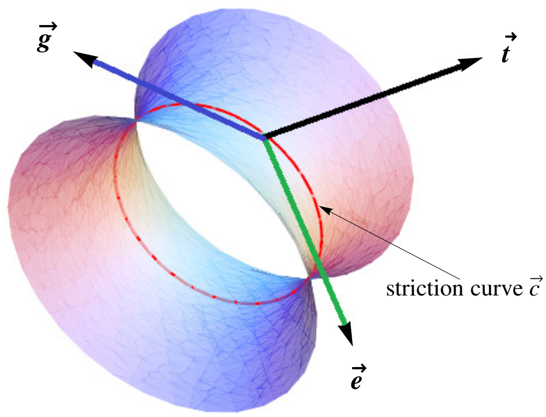

FIGURE 4. Non-developable BAN ruled surface M^*

6. Conclusion

It would be an interesting problem to investigate the classification problems of BCN and BAN ruled surfaces in E^3 using the shape operator of the surfaces M and M^*.

For M, we found that the central normal ruled surface M is BCN ruled surface if and only if the tangent of the base curve $\vec{c'} \in span\{\vec{e}, \vec{g}\}$. We also proved that the surface M is harmonic, II-harmonic, bi-harmonic, II-bi-harmonic and II-flat if and only if the surface M is BCN ruled surface.

In addition, for M^* we proved that the asymptotic normal ruled surface M^* is BAN ruled surface if and only if the spherical indicatrix $\vec{e}$ of M^* is geodesic ($\mu = 0$). Moreover, we demonstrated that the asymptotic normal ruled surface M^* is bi-harmonic surface if the surface M^* is BAN surface. Furthermore, we showed that the asymptotic normal ruled surface M^* is II-harmonic if it is harmonic. Besides that, the relation between the Frenet-frame and the geodesic frame in each case is obtained. Finally, our investigation has been illustrated through examples using computer-aided geometric design.

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