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MULTIPLICITY OF SOLUTIONS OF ELLIPTIC SYSTEM USING TWO CRITICAL POINT THEOREM

Hyewon Nam

Abstract. In this paper, we consider the system of three elliptic equations using two critical point theorem. We prove the existence of two solutions for suitable forcing terms, under a condition on the linear part which prevents resonance with eigenvalues of the operator.

1. Introduction

In this work we consider the problem

$$(1)\qquad \begin{cases} -\Delta u = au + bv + (v^+)^{p_1} + f_1 + t\phi_1 & \text{in } \Omega, \\ -\Delta v = bu + av + (u^+)^{p_2} + f_2 + r\phi_1 & \text{in } \Omega, \\ -\Delta w = cw + (w^+)^{p_3} + f_3 + s\phi_1 & \text{in } \Omega, \\ u = v = w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $u^+ = \max\{0, u(x)\}$, $\phi_1 > 0$ is the first eigenfunction of the Laplacian with Dirichlet boundary conditions and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$.

The nonlinearities will be assumed both superlinear and subcritical, that is, $1 < p_1, p_2, p_3 < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

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We may write (1) in vectorial form as

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} (u^+)^{p_1} \\ (v^+)^{p_2} \\ (w^+)^{p_3} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 \text{ in } \Omega, \\ \qquad u = v = w = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where $A = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$; we will assume that A has real eigenvalues $\nu_{i,1} = a + b$, $\nu_{i,2} = a - b$ and $\nu_{i,3} = c$.

Throughout the paper, we will denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_i \leq \ldots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and by $\{\phi_i\}_{i\in\mathbb{N}}$ the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_i\|_{L^2} = 1$ and $\phi_1 > 0$; by $\sigma(-\Delta)$ we will denote the spectrum of the Laplacian, that is, the set $\{\lambda_i : i \in \mathbb{N}\}$.

The results of our study are as follows.

Theorem 1.1. *If A has real eigenvalues that are not $\sigma(-\Delta)$ and f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N \geq 2$ then there exits $(t_0, r_0, s_0) \in \mathbb{R}^3$ such that if*

$$(t, r, s)^T = (t_0, r_0, s_0)^T + (\lambda_1 I - A)(\tau, \rho, \sigma)^T$$

with $\tau, \rho, \sigma < 0$ then a negative solution $(u_{neg}, v_{neg}, w_{neg})$ of (1) exists.

Theorem 1.2. *Let $a - b \notin \sigma(-\Delta)$, $a + b \notin \sigma(-\Delta)$ and $c \notin \sigma(-\Delta)$, f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N \geq 2$ and (t, r, s) as in Theorem 1.1; then there exists a second solution for system (1).*

2. The negative solution

In this section, we will look for negative solutions, in the sense that both components are negative: this is relatively simple since in this case the nonlinear term disappears in (1).

We will need the following.

Lemma 2.1. *If A has real eigenvalues that are not in $\sigma(-\Delta)$ and f_1, f_2, $f_3 \in L^n(\Omega)$ with $n > N$ then there exists a unique solution (u_0, v_0, w_0)*

of the problem

$$\text{(2)} \qquad \begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} & \textit{in } \Omega, \\ u = v = w = 0 & \textit{in } \partial\Omega. \end{cases}$$

Proof. For the matrix A eigenvalue-eigenvector pairs are

$$\nu_{i,1} = a + b, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \nu_{i,2} = a - b, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \nu_{i,3} = c, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hance A is diagonalizable, that is, $X^{-1}AX = D$ where

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} a+b & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = X \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$ then we written the equation (2) as

$$\text{(3)} \qquad \begin{cases} -\Delta\tilde{u} = (a+b)\tilde{u} + \tilde{f}_1 & \text{in } \Omega, \\ -\Delta\tilde{v} = (a-b)\tilde{v} + \tilde{f}_2 & \text{in } \Omega, \\ -\Delta\tilde{w} = c\tilde{w} + \tilde{f}_3 & \text{in } \Omega, \\ \tilde{u} = \tilde{v} = \tilde{w} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix} = X^{-1} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}f_1 + \frac{1}{2}f_2 \\ \frac{1}{2}f_1 - \frac{1}{2}f_2 \\ f_3 \end{bmatrix}.$

Since each real eigenvalue of A is not in $\sigma(-\Delta)$, equation (3) are uniquely solvable.

The hypothesis $f_1, f_2, f_3 \in L^n(\Omega)$ implies that $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3 \in L^n(\Omega)$. By regularity theory and General Sobolev inequalities, $u_0, v_0, w_0 \in W^{2,n}(\Omega) \subseteq C^{1,\alpha}(\bar{\Omega})$ for $\alpha = 1 - \frac{N}{n}$. □

With this result we may obtain the negative solution:

Proof of Theorem 1.1. Let (u_0, v_0, w_0) be the corresponding solution for (2). Assuming that the problem

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \\ w \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 & \text{in } \Omega, \\ u = v = w = 0 & \text{in } \partial\Omega \end{cases}$$

is looking for a solution of the form $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \phi_1$, the coefficients $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ satisfies the condition $(\lambda_1 I - A) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} t \\ r \\ s \end{bmatrix}$. By the superposition principle,

$$(\lambda_1 I - A)^{-1} \begin{bmatrix} t \\ r \\ s \end{bmatrix} \phi_1 + \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \tag{4}$$

is a solution of (1), provided it is nonpositive.

Since $u_0, v_0, w_0 \in C^{1,\alpha}$, we set

$$\alpha_0 = \sup\{\alpha | \alpha\phi_1 + u_0 < 0\},$$
$$\beta_0 = \sup\{\beta | \beta\phi_1 + v_0 < 0\},$$
$$\gamma_0 = \sup\{\gamma | \gamma\phi_1 + w_0 < 0\}.$$

If we set $(t_0, r_0, s_0)^T = (\lambda_1 I - A)(\alpha_0, \beta_0, \gamma_0)^T$ in the condition

$$\begin{bmatrix} t \\ r \\ s \end{bmatrix} = (\lambda_1 I - A) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = (\lambda_1 I - A) \left(\begin{bmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{bmatrix} + \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \\ \gamma - \gamma_0 \end{bmatrix} \right),$$

then $\tau = \alpha - \alpha_0 < 0$, $\rho = \beta - \beta_0 < 0$, $\sigma = \gamma - \gamma_0 < 0$ because $\alpha < \alpha_0$, $\beta < \beta_0$, and $\gamma < \gamma_0$. We get the condition in the claim. □

3. The second solution

We will find the second solution by using a minimax theorem due to Felmer [4].

3.1. The variational structure. We consider the Hilbert space $E = H_0^1 \times H_0^1 \times H_0^1$ equipped with the scalar product

$$\langle (u_1, v_1, w_1), (u_2, v_2, w_2) \rangle_E = \int_\Omega (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + \nabla w_1 \nabla w_2) dx,$$

the related norm $\|(u_1, v_1, w_1)\|_E$ and the bounded symmetric quadratic form

$$\begin{aligned} B((u_1, v_1, w_1), (u_2, v_2, w_2)) &= \int_\Omega (\nabla u_1 \nabla v_2 + \nabla v_1 \nabla u_2 + \nabla w_1 \nabla w_2) dx \\ &\quad - a \int_\Omega (u_1 v_2 + v_1 u_2) dx \\ &\quad - b \int_\Omega (u_1 u_2 + v_1 v_2) dx - c \int_\Omega w_1 w_2 dx. \end{aligned}$$

Let (t, r, s) be as in Theorem 1.1 and $(u_{neg}, v_{neg}, w_{neg})$ be the corresponding negative solution for (1), then we define the functional $F : E \to \mathbb{R}$ for $\mathbf{u} = (u, v, w) \in E$ by

$$F(\mathbf{u}) = \frac{1}{2} B(\mathbf{u}, \mathbf{u}) - H(\mathbf{u}),$$

where

$$\begin{aligned} H(\mathbf{u}) &= \int_\Omega \frac{[(u + u_{neq})^+]^{p_1+1}}{p_1 + 1} dx \\ &\quad + \int_\Omega \frac{[(v + v_{neq})^+]^{p_2+1}}{p_2 + 1} dx + \int_\Omega \frac{[(w + w_{neq})^+]^{p_3+1}}{p_3 + 1} dx. \end{aligned}$$

Then it is simple to see that the functional F is $C^1(E; \mathbb{R})$ and its critical point (u, v, w) are such that $(u + u_{neq}, v + v_{neq}, w + w_{neq})$ are solutions of (1); in particular, the origin is a critical point at level zero and corresponds to the already found negative solution.

In order to find an orthogonal base for E which diagonalizes B, we consider, in a way similar to what was done in [1], the eigenvalue problem

$$B((u, v, w), (\phi, \varphi, \psi)) = \mu \langle (u, v, w), (\phi, \varphi, \psi) \rangle_E, \qquad \forall (\phi, \varphi, \psi) \in E.$$

Let u_i, v_i, and w_i be the Fourier's coefficients for u, v, and w. Then the above eigenvalue problem is summarized as

$$\begin{bmatrix} \mu\lambda_i + b & a - \lambda_i & 0 \\ a - \lambda_i & \mu\lambda_i + b & 0 \\ 0 & 0 & c - \lambda_i + \mu\lambda_i \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (i \in \mathbb{N}), \tag{5}$$

using $(\phi_i, 0, 0)$, $(0, \phi_i, 0)$, $(0, 0, \phi_i)$ as test function.

When the determinant of the above coefficient matrix is zero, we get nontrivial solutions. This is

$$(c - \lambda_i + \mu\lambda_i)(\mu\lambda_i + b)^2 - (c - \lambda_i + \mu\lambda_i)(a - \lambda_i)^2 = 0 \quad (i \in \mathbb{N})$$

and so

$$\mu_{i,1} = \frac{a - b - \lambda_i}{\lambda_i}, \quad \mu_{i,2} = \frac{-a - b + \lambda_i}{\lambda_i}, \quad \mu_{i,3} = \frac{-c + \lambda_i}{\lambda_i} \qquad (i \in \mathbb{N}).$$

From (5) we also get the related eigenvectors

$$\Phi_{i,1} = (\phi_i, -\phi_i, 0), \quad \Phi_{i,2} = (\phi_i, \phi_i, 0), \quad \Phi_{i,3} = (0, 0, \phi_i) \qquad (i \in \mathbb{N}).$$

Because that $\Phi_{i,j}$, $j = 1, 2, 3$ are orthogonal, we normalize to obtain $\Psi_{i,j}$, $j = 1, 2, 3$, that is , $\|\Psi_{i,j}\|_E = 1$:

$$\Psi_{i,1} = \frac{(\phi_i, -\phi_i, 0)}{\sqrt{2\lambda_i}}, \Psi_{i,2} = \frac{(\phi_i, \phi_i, 0)}{\sqrt{2\lambda_i}}, \Psi_{i,3} = \frac{(0, 0, \phi_i)}{\sqrt{\lambda_i}} \quad (i \in \mathbb{N}).$$

With this structure we have

$$\langle \Psi_{i,j}, \Psi_{k,l} \rangle_E = \begin{cases} 1 & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases},$$

$$B(\Psi_{i,j}, \Psi_{k,l}) = \begin{cases} \mu_{i,j} & i = k \text{ and } j = l \\ 0 & i \neq k \text{ or } j \neq l \end{cases},$$

so if we write $(u, v, w) = \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j} \Psi_{i,j}$, we get

$$\|(u, v, w)\|_E^2 = \sum_{i \in \mathbb{N}, j=1,2,3} C_{i,j}^2,$$

$$B((u, v, w), (u, v, w)) = \sum_{i \in \mathbb{N}, j=1,2,3} \mu_{i,j} C_{i,j}^2.$$

In view of this structure we may define

$$E^+ = \overline{\text{span}\{\Psi_{i,j} : \mu_{i,j} > 0, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E^- = \overline{\text{span}\{\Psi_{i,j} : \mu_{i,j} < 0, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E^0 = \text{span}\{\Psi_{i,j} : \mu_{i,j} = 0, i \in \mathbb{N}, j = 1, 2, 3\},$$

and we have

Lemma 3.1. *There exists $\xi^* > 0$ such that*

(6) $$B(\mathbf{u}, \mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|_E^2 \quad \text{for} \quad \mathbf{u} \in E^+$$

(7) $$B(\mathbf{u}, \mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|_E^2 \quad \text{for} \quad \mathbf{u} \in E^-.$$

Moreover, if $a - b \notin \sigma(-\Delta)$, $a + b \notin \sigma(-\Delta)$ *and* $c \notin \sigma(-\Delta)$, *then* $E^0 = \{0\}$.

Proof. The claim is satisfied by setting

$$2\xi^* := \inf\{|\mu_{i,j}| : |\mu_{i,j}| > 0, i \in \mathbb{N}, j = 1, 2, 3\}$$

Since

$$\lim_{i\to\infty} \mu_{i,1} = -1, \qquad \lim_{i\to\infty} \mu_{i,2} = \lim_{i\to\infty} \mu_{i,3} = 1,$$

$2\xi^*$ is strictly positive.

The condition $a - b \notin \sigma(-\Delta)$, $a + b \notin \sigma(-\Delta)$ and $c \notin \sigma(-\Delta)$ implies $\mu_{i,j} \neq 0$ for any $i \in \mathbb{N}$, $j = 1, 2, 3$. □

For later use, we also define $\tilde{n}$ such that for $i \geq \tilde{n}$ we have $a - \lambda_i < b < -a + \lambda_i$, $c < \lambda_i$ and

$$E_h = \overline{\operatorname{span}\{\Psi_{i,j} : i \geq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\}},$$
$$E_l = \operatorname{span}\{\Psi_{i,j} : i \leq \tilde{n}, i \in \mathbb{N}, j = 1, 2, 3\} :$$

we have the following

Lemma 3.2. $(u, v, w) \in E^+ \cap E_h$ *implies* $u = v$ *and* $(u, v, w) \in E^- \cap E_h$ *implies* $u + v = 0$, $w = 0$.

Proof. It follows readily from the fact that for $i \geq \tilde{n}$ we have $\mu_{i,1} < 0$, $\mu_{i,2} > 0$, $\mu_{i,3} > 0$ and that $\Psi_{i,1} = \frac{(\phi_i, -\phi_i, 0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,2} = \frac{(\phi_i, \phi_i, 0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,3} = \frac{(0,0,\phi_i)}{\sqrt{\lambda_i}}$. □

3.2. Estimates for the linking structure. In this section we will prove the estimates we need in order to apply the minimax theorem.

Lemma 3.3. *There exists* $\rho > 0$ *such that*

$$\text{if} \quad \|\mathbf{u}\|_E \leq \rho \quad \text{then} \quad F(\mathbf{u}) \geq 0 \quad \text{for} \quad \mathbf{u} \in E^+.$$

If $\|\mathbf{u}\|_E = \rho$, *then* $F(\mathbf{u}) > 0$.

Proof. Let $\mathbf{u}$ be as above. By the continuous embedding of H_0^1 in L^{p_1+1}, L^{p_2+1}, and L^{p_3+1} we get

$$\begin{aligned}
\int_\Omega \frac{[(u+u_{neg})^+]^{p_1+1}}{p_1+1}dx &\le \int_\Omega \frac{|u|^{p_1+1}}{p_1+1}dx \le C_1\|u\|_{H_0^1}^{p_1+1},\\
\int_\Omega \frac{[(v+v_{neg})^+]^{p_2+1}}{p_2+1}dx &\le \int_\Omega \frac{|v|^{p_2+1}}{p_2+1}dx \le C_2\|v\|_{H_0^1}^{p_2+1},\\
\int_\Omega \frac{[(w+w_{neg})^+]^{p_3+1}}{p_3+1}dx &\le \int_\Omega \frac{|w|^{p_3+1}}{p_3+1}dx \le C_3\|w\|_{H_0^1}^{p_3+1},
\end{aligned}$$

where C_1, C_2 and C_3 are positive constants. By (6) in Lemma 3.1,

$$\frac{1}{2}B(\mathbf{u},\mathbf{u}) \ge \xi^*\|\mathbf{u}\|_E^2 = \xi^*\left(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2+\|w\|_{H_0^1}^2\right).$$

We get

$$\begin{aligned}
F(\mathbf{u}) &\ge \xi^*\left(\|u\|_{H_0^1}^2+\|v\|_{H_0^1}^2+\|w\|_{H_0^1}^2\right)\\
&\quad -C\left(\|u\|_{H_0^1}^{p_1+1}+\|v\|_{H_0^1}^{p_2+1}+\|w\|_{H_0^1}^{p_3+1}\right)\\
&\ge \|u\|_{H_0^1}^2\left(\xi^*-C\rho^{p_1-1}\right)+\|v\|_{H_0^1}^2\left(\xi^*-C\rho^{p_2-1}\right)\\
&\quad +\|w\|_{H_0^1}^2\left(\xi^*-C\rho^{p_3-1}\right)
\end{aligned}$$

where $C=\max\{C_1,C_2,C_3\}$ is a positive number. Since $p_1,p_2,p_3>1$, for $\rho>0$ small enough we obtain $\xi^*-C\rho^{p_j-1}>0$ $j=1,2,3$. Let $C^*=\min\{\xi^*-C\rho^{p_j-1}: j=1,2,3\}>0$, then

$$F(\mathbf{u}) \ge C^*\|\mathbf{u}\|_E^2 \ge 0.$$

If $\|\mathbf{u}\|_E=\rho$, then

$$F(\mathbf{u}) \ge C^*\|\mathbf{u}\|_E^2 = C^*\rho^2 > 0.$$

□

LEMMA 3.4. *There exists* $\mathbf{g}=((g_1,g_1,g_2)\in E^+\cap E_h$ *with* $\|\mathbf{g}\|_E=1$ *and* $\|(g_j)^+\|_{L^\infty}=+\infty$, *for* $j=1,2$.

Proof. Since H_0^1 is not embedded in L^∞ (here is where we need the condition $N\ge 2$), there exists $u_i\in H_0^1$ such that $\|(u_j)^+\|_{L^\infty}=+\infty$, for $j=1,2$; by removing the components of u in the directions of the eigenvectors ϕ_i with $i<\tilde{n}$ we maintain this property since we simply subtract a finite linear combination of regular functions, so we may assume that such components are zero.

Since $\mu_{i,2} > 0$, $\mu_{i,3} > 0$ and $\Psi_{i,2} = \frac{(\phi_i,\phi_i,0)}{\sqrt{2\lambda_i}}$, $\Psi_{i,3} = \frac{(0,0,\phi_i)}{\sqrt{\lambda_i}}$, for $i \geq \tilde{n}$, we have that $(u_1, u_1, u_2) \in E^+ \cap E_h$.

Finally, we obtain $\|(g_1, g_1, g_2)\|_E = 1$ by a suitable rescaling of (u_1, u_1, u_2). □

LEMMA 3.5. *Let* $\mathbf{g} = (g_1, g_1, g_2)$ *as in the lemma above. Then there exist* $R, \theta > 0$ *with* $R\theta > \rho$ *such that* $F(\mathbf{u}) \leq 0$ *for*

(a) $\mathbf{u} \in E^-$,
(b) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E = R$, $0 \leq \tau \leq \theta R$,
(c) $\mathbf{u} = \mathbf{w} + \tau\mathbf{g}$; $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E \leq R$, $\tau = \theta R$.

Proof. (a) Let $\mathbf{u} \in E^-$. By (7) in Lemma 3.1,

$$F(\mathbf{u}) \leq \frac{1}{2}B(\mathbf{u}, \mathbf{u}) \leq -\xi^* \|\mathbf{u}\|_E^2 \leq 0.$$

(b) Let $\mathbf{w} \in E^-$ with $\|\mathbf{w}\|_E = R$ and $0 \leq \tau \leq \theta R$. Observe that $\mathbf{g}$ is orthogonal to $\mathbf{w}$, that is, $\langle \mathbf{w}, \mathbf{g} \rangle_E = 0 = B(\mathbf{w}, \mathbf{g})$; then we estimate, by using (7) in Lemma 3.1,

$$\begin{aligned} F(\mathbf{u}) &\leq \frac{1}{2}B(\mathbf{u}, \mathbf{u}) = \frac{1}{2}B(\mathbf{w} + \tau\mathbf{g}, \mathbf{w} + \tau\mathbf{g}) = \frac{1}{2}B(\mathbf{w}, \mathbf{w}) + \frac{1}{2}\tau^2 B(\mathbf{g}, \mathbf{g}) \\ &\leq -\xi^* \|\mathbf{w}\|_E^2 + \frac{1}{2}\tau^2 B(\mathbf{g}, \mathbf{g}) = R^2 \left(-\xi^* + \frac{1}{2}\left(\frac{\tau}{R}\right)^2 B(\mathbf{g}, \mathbf{g}) \right) \\ &\leq R^2 \left(-\xi^* + \frac{1}{2}\theta^2 B(\mathbf{g}, \mathbf{g}) \right) \end{aligned}$$

Since $\|\mathbf{g}\|_E = 1$, $B(\mathbf{g}, \mathbf{g}) \geq 2\xi^* > 0$(by (6) in Lemma 3.1) and then $0 < \frac{2\xi^*}{B(\mathbf{g},\mathbf{g})}$. By fixing $0 < \theta < \sqrt{\frac{2\xi^*}{B(\mathbf{g},\mathbf{g})}}$, such that last term is negative, the claim (b) is proved.

(c) Consider now $\|\mathbf{w}\|_E \leq R$, $\tau = \theta R$, and let

$$P_l \mathbf{w} = (\sigma_1, \sigma_2, \sigma_3), \qquad P_h \mathbf{w} = (\delta_1, \delta_2, \delta_3)$$

where P_l and P_h are the orthogonal projections onto E_l and E_h, respectively. In this way, $P_h \mathbf{w} \in E^- \cap E_h$ and then it is of the form $P_h \mathbf{w} = (\delta_1, -\delta_1, 0)$, by Lemma 3.2.

Write now

$$\begin{aligned}\int_\Omega \left[(u+u_{neg})^+\right]^{p_1+1} dx &= \int_\Omega \left[(\sigma_1+\delta_1+\theta R g_1+u_{neg})^+\right]^{p_1+1} dx \\ &= R^{p_1+1}\int_\Omega \left[\left(\frac{\sigma_1+\delta_1+u_{neg}}{R}+\theta g_1\right)^+\right]^{p_1+1} dx \qquad (8)\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[(v+v_{neg})^+\right]^{p_2+1} dx &= \int_\Omega \left[(\sigma_2-\delta_1+\theta R g_1+v_{neg})^+\right]^{p_2+1} dx \\ &= R^{p_2+1}\int_\Omega \left[\left(\frac{\sigma_2-\delta_1+v_{neg}}{R}+\theta g_1\right)^+\right]^{p_2+1} dx \qquad (9)\end{aligned}$$

$$\begin{aligned}\int_\Omega \left[(w+w_{neg})^+\right]^{p_3+1} dx &= \int_\Omega \left[(\sigma_3+\theta R g_2+w_{neg})^+\right]^{p_3+1} dx \\ &= R^{p_3+1}\int_\Omega \left[\left(\frac{\sigma_3+w_{neg}}{R}+\theta g_2\right)^+\right]^{p_3+1} dx \qquad (10)\end{aligned}$$

Since u_{neq}, v_{neq}, and w_{neq} are fixed and bounded, and $\sigma_1, \sigma_2, \sigma_3$ are linear combinations of a finite number of eigenvectors of the Laplacian(because of $P_l\mathbf{w} \in E_l$), there exists a constant C such that

$$|u_{neq}|, |v_{neq}|, |w_{neq}| < \frac{C}{2} \quad \text{and} \quad |\sigma_1|, |\sigma_2|, |\sigma_3| < \frac{C}{2},$$

so, for $R > 1$,

$$\frac{|\sigma_1+u_{neq}|}{R}, \frac{|\sigma_2+v_{neq}|}{R}, \frac{|\sigma_3+w_{neq}|}{R} < C.$$

Moreover, since $\mathbf{g}$ and θ have already been fixed and $\|(g_j)^+\|_{L^\infty} = \infty$, for $j = 1, 2$, we know that

$$\Omega^* = \{x \in \Omega : \theta g_1(x) > C+1 \quad \text{and} \quad \theta g_2(x) > C+1\}$$

has positive measure; we observe that $\|(g_1)^+\|_{L^\infty} = \infty$ implies

$$\max\{\theta g_1 + \delta_1/R\} > C+1 \quad \text{and} \quad \max\{\theta g_1 - \delta_1/R\} > C+1$$

for any bounded function δ_1 and any $R > 1$: then $\Omega^* \subset \Omega_1^* \cup \Omega_2^* \cup \Omega_3^*$, where

$$\begin{aligned}\Omega_1^* &= \{x \in \Omega : \theta g_1 + \delta_1/R > C + 1\},\\ \Omega_2^* &= \{x \in \Omega : \theta g_1 - \delta_1/R > C + 1\},\\ \Omega_3^* &= \{x \in \Omega : \theta g_2 > C + 1\}\end{aligned}$$

(observe that both Ω_i^* $(i = 1, 2)$ depend on $\mathbf{w}$ and R, but Ω^* does not).

Then $|\Omega_1^*| \geq |\Omega^*|/3$ or $|\Omega_2^*| \geq |\Omega^*|/3$ or $|\Omega_3^*| \geq |\Omega^*|/3$ and, as a consequence, for any $\mathbf{w}$ as assumed and $R > 1$, one of the following three cases hold:

(i) Let $|\Omega_1^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_1^*$,

$$\frac{\sigma_1 + \delta_1 + u_{neq}}{R} + \theta g_1 > 1,$$

since $\theta g_1 + \delta_1/R > C + 1$ and $-C < \sigma_1/R + u_{neq}/R < C$.
We conclude from (8) that

$$\begin{aligned} H(\mathbf{u}) &\geq \int_\Omega \frac{[(u + u_{neg})^+]^{p_1+1}}{p_1 + 1} dx \\ &= \frac{R^{p_1+1}}{p_1 + 1} \int_\Omega \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g_1 \right)^+ \right]^{p_1+1} dx \\ &\geq \frac{R^{p_1+1}}{p_1 + 1} \int_{\Omega_1^*} \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g_1 \right)^+ \right]^{p_1+1} dx \\ &\geq \frac{R^{p_1+1}}{p_1 + 1} |\Omega_1^*| \geq \frac{|\Omega^*| R^{p_1+1}}{3(p_1 + 1)} \end{aligned}$$

(ii) Let $|\Omega_2^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_2^*$,

$$\frac{\sigma_2 - \delta_1 + v_{neq}}{R} + \theta g_1 > 1,$$

since $\theta g_1 - \delta_1/R > C + 1$ and $-C < \sigma_2/R + v_{neq}/R < C$.

We conclude from (9) that

$$
\begin{aligned}
H(\mathbf{u}) &\geq \int_\Omega \frac{[(v+v_{neg})^+]^{p_2+1}}{p_2+1} dx \\
&= \frac{R^{p_2+1}}{p_2+1} \int_\Omega \left[\left(\frac{\sigma_2 - \delta_1 + v_{neg}}{R} + \theta g_1 \right)^+ \right]^{p_2+1} dx \\
&\geq \frac{R^{p_2+1}}{p_2+1} \int_{\Omega_2^*} \left[\left(\frac{\sigma_2 - \delta_1 + v_{neg}}{R} + \theta g_1 \right)^+ \right]^{p_2+1} dx \\
&\geq \frac{R^{p_2+1}}{p_2+1} |\Omega_2^*| \geq \frac{|\Omega^*| R^{p_2+1}}{3(p_2+1)}
\end{aligned}
$$

(iii) Let $|\Omega_3^*| \geq |\Omega^*|/3$.
For any $x \in \Omega_3^*$,

$$
\frac{\sigma_3 + w_{neq}}{R} + \theta g_2 > 1,
$$

since $\theta g_2 > C+1$ and $-C < \sigma_3/R + w_{neq}/R < C$.
We conclude from (10) that

$$
\begin{aligned}
H(\mathbf{u}) &\geq \int_\Omega \frac{[(w+w_{neg})^+]^{p_3+1}}{p_3+1} dx \\
&= \frac{R^{p_3+1}}{p_3+1} \int_\Omega \left[\left(\frac{\sigma_3 + w_{neg}}{R} + \theta g_2 \right)^+ \right]^{p_3+1} dx \\
&\geq \frac{R^{p_3+1}}{p_3+1} \int_{\Omega_3^*} \left[\left(\frac{\sigma_3 + w_{neg}}{R} + \theta g_2 \right)^+ \right]^{p_3+1} dx \\
&\geq \frac{R^{p_3+1}}{p_3+1} |\Omega_3^*| \geq \frac{|\Omega^*| R^{p_3+1}}{3(p_3+1)}
\end{aligned}
$$

Let $\tilde{C} = \min\{\frac{|\Omega^*|}{3(p_i+1)} : i = 1,2,3\}$ then $\tilde{C} > 0$ does not depend on R and $\mathbf{w}$. And we conclude that $H(\mathbf{u}) \geq \tilde{C} R^{\min\{p_1,p_2,p_3\}+1}$.

Finally, by estimating the first terms as in point (b), we get

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{w}+\theta R\mathbf{g}, \mathbf{w}+\theta R\mathbf{g}) - H(\mathbf{u}) \\ &\le \frac{1}{2}B(\mathbf{w}+\theta R\mathbf{g}, \mathbf{w}+\theta R\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}+1} \\ &\le -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\theta^2R^2B(\mathbf{g},\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}+1} \\ &\le R^2\left(\frac{1}{2}\theta^2B(\mathbf{g},\mathbf{g}) - \tilde{C}R^{\min\{p_1,p_2,p_3\}-1}\right) : \end{aligned}$$

since $p_1, p_2, p_3 > 1$, we may choose $R > 1$(and also $R > \rho/\theta$) large enough to make the last expression negative; this concludes the proof of the claim (c). □

For $\mathbf{g} = ((g_1, g_1, g_2)$, θ and R in the lemma above, we set

$$\begin{aligned} S &= \{\mathbf{u} : \mathbf{u} \in E^+, \|\mathbf{u}\|_E \le \rho\}, \\ Q &= \{\mathbf{u} = \mathbf{w} + \tau\mathbf{g} : \mathbf{w} \in E^-, \|\mathbf{w}\|_E \le R, 0 \le \tau \le \theta R\}. \end{aligned}$$

Lemma 3.6. *We have*

$$\sup_Q F < +\infty.$$

Proof. By estimating the first terms as in point (b) in Lemma 3.5,

$$\begin{aligned} F(\mathbf{u}) &\le \frac{1}{2}B(\mathbf{u},\mathbf{u}) = \frac{1}{2}B(\mathbf{w}+\tau\mathbf{g}, \mathbf{w}+\tau\mathbf{g}) \\ &= \frac{1}{2}B(\mathbf{w},\mathbf{w}) + \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \\ &\le -\xi^*\|\mathbf{w}\|_E^2 + \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \\ &\le \frac{1}{2}\tau^2B(\mathbf{g},\mathbf{g}) \le \frac{1}{2}\theta^2R^2B(\mathbf{g},\mathbf{g}) < +\infty. \end{aligned}$$

□

3.3. The PS conditions. In this section we will prove that the PS condition holds, which was required for the application of the minimax theorem.

Lemma 3.7. *(PS condition). Under the considered hypotheses, the functional F satisfies the PS condition, that is, let ϵ_n be a sequence of* *positive reals converging to zero and* $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subseteq E$ *be such that*

$$|F(\mathbf{u}_n)| \leq T, \tag{11}$$

$$|F'(\mathbf{u}_n)[\phi,\varphi,\psi]| \leq \epsilon_n \|(\phi,\varphi,\psi)\|_E \qquad \forall(\phi,\varphi,\psi) \in E, \tag{12}$$

then $\{\mathbf{u}_n\}$ *admits a convergent subsequence.*

Proof. First, we want to prove that $\|\mathbf{u}_n\|_E$ is bounded: so we consider for the sake of contradiction a subsequence such that $\|\mathbf{u}_n\|_E \to \infty$ and we define

$$(U_n, V_n, W_n) = \frac{1}{\|\mathbf{u}_n\|_E}(u_n, v_n, w_n),$$

so that (up to a further subsequence) $(U_n, V_n, W_n) \to (U, V, W)$ weakly in E.

Applying the definition of the functional F,

$$\begin{aligned} F(\mathbf{u}_n) \;&=\; \frac{1}{2}B(\mathbf{u}_n,\mathbf{u}_n) - \int_\Omega \frac{[(u_n+u_{neq})^+]^{p_1+1}}{p_1+1}dx \\ &\quad - \int_\Omega \frac{[(v_n+v_{neq})^+]^{p_2+1}}{p_2+1}dx - \int_\Omega \frac{[(w_n+w_{neq})^+]^{p_3+1}}{p_3+1}dx \end{aligned}$$

and

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n \;&=\; B(\mathbf{u}_n,\mathbf{u}_n) - \int_\Omega [(u_n+u_{neq})^+]^{p_1} u_n dx \\ &\quad - \int_\Omega [(v_n+v_{neq})^+]^{p_2} v_n dx - \int_\Omega [(w_n+w_{neq})^+]^{p_3} w_n dx. \end{aligned}$$

Now observe that

$$\begin{aligned} \int_\Omega [(u_n+u_{neq})^+]^{p_1} u_n dx \;&=\; \int_\Omega [(u_n+u_{neq})^+]^{p_1+1} dx \\ &\quad + \int_\Omega [(u_n+u_{neq})^+]^{p_1} (-u_{neq}) dx \end{aligned}$$

(and an analogous relation holds for the term in v_n and w_n); then,

$$\begin{aligned} F'(\mathbf{u}_n)\mathbf{u}_n \;&= B(\mathbf{u}_n,\mathbf{u}_n) \\ &\quad - \int_\Omega [(u_n+u_{neq})^+]^{p_1+1} dx - \int_\Omega [(u_n+u_{neq})^+]^{p_1} (-u_{neq}) dx \\ &\quad - \int_\Omega [(v_n+v_{neq})^+]^{p_2+1} dx - \int_\Omega [(v_n+v_{neq})^+]^{p_2} (-v_{neq}) dx \\ &\quad - \int_\Omega [(w_n+w_{neq})^+]^{p_3+1} dx - \int_\Omega [(w_n+w_{neq})^+]^{p_3} (-w_{neq}) dx \end{aligned}$$

and by considering $F(\mathbf{u}_n) - \frac{1}{2}F(\mathbf{u}_n)\mathbf{u}_n$, we get

$$\begin{aligned}
&\kappa_1 \int_\Omega [(u_n + u_{neq})^+]^{p_1+1} dx + \frac{1}{2}\int_\Omega [(u_n + u_{neq})^+]^{p_1}(-u_{neq})dx \\
&\quad + \kappa_2 \int_\Omega [(v_n + v_{neq})^+]^{p_2+1} dx + \frac{1}{2}\int_\Omega [(v_n + v_{neq})^+]^{p_2}(-v_{neq})dx \\
&\quad + \kappa_3 \int_\Omega [(w_n + w_{neq})^+]^{p_3+1} dx + + \frac{1}{2}\int_\Omega [(w_n + w_{neq})^+]^{p_3}(-w_{neq})dx \\
&\le T + \frac{1}{2}\epsilon_n \|\mathbf{u}_n\|_E;
\end{aligned}$$

(where $\kappa_i = \frac{1}{2} - \frac{1}{p_i+1}$) by observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(u_n + u_{neq})^+]^{p_1+1} dx \to 0, \tag{13}$$

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(v_n + v_{neq})^+]^{p_2+1} dx \to 0, \tag{14}$$

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega [(w_n + w_{neq})^+]^{p_3+1} dx \to 0. \tag{15}$$

For any $(\phi, \varphi, \psi) \in E$ with $\|(\phi, \varphi, \psi)\|_E = 1$, we get

$$\begin{aligned}
&\frac{1}{\|\mathbf{u}_n\|_E} F'(\mathbf{u}_n)[\phi, \varphi, \psi] \\
&\quad = B((U_n, V_n, W_n), (\phi, \varphi, \psi)) - \int_\Omega \frac{[(u_n + u_{neq})^+]^{p_1}}{\|\mathbf{u}_n\|_E}\phi dx \\
&\qquad - \int_\Omega \frac{[(v_n + v_{neq})^+]^{p_2}}{\|\mathbf{u}_n\|_E}\varphi dx - \int_\Omega \frac{[(w_n + w_{neq})^+]^{p_3}}{\|\mathbf{u}_n\|_E}\psi dx
\end{aligned}$$

From (12) we get

$$\frac{1}{\|\mathbf{u}_n\|_E} F'(\mathbf{u}_n)[\phi, \varphi, \psi] \to 0$$

which, by using the weak convergence of (U_n, V_n, W_n) and (13), (14), (15), implies that

$$B((U, V, W), (\phi, \varphi, \psi)) = 0.$$

This means that (U, V, W) is a solution of

$$-\Delta(U, V, W)^T = A(U, V, W)^T,$$

where $A = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$. Since for the matrix B eigenvalue-eigenvector pairs are

$$\nu_{i,1} = a + b, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \nu_{i,2} = a - b, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \nu_{i,3} = c, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(U, V, W) is the unique solution and then it is zero if real eigenvalues of A are not in $\sigma(-\Delta)$.

This gives rise to a contradiction since by definition we have

$$\|(U, V, W)\|_E = 1.$$

We conclude that $\|\mathbf{u}_n\|_E$ is bounded.

It is now simple to see that $\mathbf{u}_n$ admits a convergent subsequence. In fact, up to a subsequence, $(u_n, v_n, w_n) \to (u, v, w)$ weakly in E, then we calculate the inner product of $(u_n, v_n, w_n) - (u, v, w)$ and $\Psi_{i,j}$ to obtain that the convergence is in fact strong. □

3.4. The second solution through the minimax theorem. Now, we will prove the main theorem.

PROPOSITION 3.1. *There exists a critical point $\mathbf{u} \in E$ for the functional F with $F(\mathbf{u}) > 0$(and then $\mathbf{u} \neq (0, 0, 0)$, so that it is a second solution).*

Proof. By using the estimates in Lemma 3.3, Lemma 3.5, Lemma 3.6 and the PS condition in Lamma 3.7., there exists $0 < \rho < R$ such that

$$\sup_{\partial Q} F \leq 0 < \inf_{\partial S} F,$$

and

$$\sup_{Q} F < +\infty, \qquad \inf_{S} F \geq 0 > -\infty.$$

By the two critical point theorem in [2], F has at least two critical values c_1 and c_2

$$\inf_{S} F \leq c_1 \leq \sup_{\partial Q} F < \inf_{\partial S} F \leq c_2 \leq \sup_{Q} F.$$

Since $\inf_S F \geq 0$ and $\sup_{\partial Q} F \leq 0$, $\inf_S F = c_1 = \sup_{\partial Q} F = 0$. and $c_2 > 0$. □

Finally, we may conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Theorem 1.1 and Proposition 3.1 imply Theorem 1.2.

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Hyewon Nam
Department of General Education
Namseoul University
Cheonan 31020, Korea
E-mail: hwnam@nsu.ac.kr