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FEKETE-SZEGÖ INEQUALITIES FOR A SUBCLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. In this paper, by means of the Sălăgean operator, we introduce a new subclass $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to this class, we consider Fekete-Szegö inequalities.

1. Introduction

Let $\mathbb{R}=(-\infty,\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N}:=\{1,2,3,\ldots\}=\mathbb{N}_0\backslash\{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z:z\in\mathbb{C}\quad\text{and}\quad|z|<1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z)\prec g(z)\qquad(z\in\mathbb{U}),$$

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if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $f \in \mathcal{A}$, Sălăgean [11] introduced the following operator:

$$D^0 f(z) = f(z), \tag{2}$$

$$D^1 f(z) = z f'(z) := Df(z), \tag{3}$$

$$\vdots$$

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad (n \in \mathbb{N}). \tag{4}$$

If f is given by (1), then from (3) and (4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0), \tag{5}$$

with $D^n f(0) = 0$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [9] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{6}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class Σ, see [14] (see also [3]). In fact, the aforecited work of Srivastava *et al.* [14] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [10], and others (see, for example, [1, 2, 4, 5, 6, 7, 13, 15]).

Let φ be an analytic and univalent function with positive real part on $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \tag{7}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function φ satisfies the above conditions unless otherwise stated.

Now we define the following subclass of function class Σ.

Definition 1.1. *Let $m, n \in \mathbb{N}_0\,; m > n$ and $\gamma \in \mathbb{C}\backslash\{0\}$. A function $f \in \Sigma$ given by* (1) *is said to be in the class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) \prec \varphi(z) \tag{8}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)} - 1\right) \prec \varphi(w), \tag{9}$$

where $z, w \in \mathbb{U}$ and the function $g = f^{-1}$ is defined by (6).

In particular, we set $\mathcal{B}_{\Sigma}^{m,n}(1;\varphi) := \mathcal{B}_{\Sigma}^{m,n}(\varphi)$.

Remark 1. (**i**) If we set $m = 1$ and $n = 0$, then the class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\gamma;\varphi)$ consists of bi-starlike functions of complex order γ of Ma-Minda type.

(**ii**) If we set $m = 2$ and $n = 1$, then the class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ reduces to the class $\mathcal{C}_{\Sigma}(\gamma;\varphi)$ consists of bi-convex functions of complex order γ of Ma-Minda type.

The classes $\mathcal{S}_{\Sigma}^{*}(\gamma;\varphi)$ and $\mathcal{C}_{\Sigma}(\gamma;\varphi)$ are introduced and studied by Deniz [8]. In particular, we get $\mathcal{S}_{\Sigma}^{*}(1;\varphi) = \mathcal{ST}_{\Sigma}(\varphi)$ and $\mathcal{C}_{\Sigma}(1;\varphi) = \mathcal{CV}_{\Sigma}(\varphi)$ (see [17]).

Remark 2. (**i**) For $0 < \alpha \leq 1$, if we let

$$\varphi(z) := \varphi_\alpha(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$$

in Definition 1.1, then we have a new class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\alpha)$ which consists of functions satisfying the conditions

$$\left|\arg\left\{1+\frac{1}{\gamma}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right\}\right|<\frac{\alpha\pi}{2}$$

and

$$\left|\arg\left\{1+\frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)}-1\right)\right\}\right|<\frac{\alpha\pi}{2}.$$

For $\gamma=1$, we get the class $\mathcal{B}_{\Sigma}^{m,n}(1;\alpha)=\mathcal{H}_{\Sigma}^{m,n}(\alpha)$ introduced and studied by Şeker [12]. For $m=1$, $n=0$ and $\gamma=1$, the class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\alpha)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}[\alpha]$ of strongly bi-starlike functions of order α.

(**ii**) For $0\leq\beta<1$, if we let

$$\varphi(z):=\varphi_{\beta}(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots$$

in Definition 1.1, then we have a new class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\beta)$ which consists of functions satisfying the conditions

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{D^m f(z)}{D^n f(z)}-1\right)\right\}>\beta$$

and

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)}-1\right)\right\}>\beta.$$

For $\gamma=1$, we get the class $\mathcal{B}_{\Sigma}^{m,n}(1;\beta)=\mathcal{H}_{\Sigma}^{m,n}(\beta)$ introduced and studied by Şeker [12]. The class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\beta)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\beta)$ of bi-starlike functions of order β for $m=1$, $n=0$ and $\gamma=1$; also reduces to the class $\mathcal{C}_{\Sigma}(\beta)$ of bi-convex functions of order β for $m=2$, $n=1$ and $\gamma=1$.

The classes $\mathcal{S}_{\Sigma}^{*}[\alpha]$, $\mathcal{S}_{\Sigma}^{*}(\beta)$ and $\mathcal{C}_{\Sigma}(\beta)$ are introduced and studied by Brannan and Taha [3].

In order to derive our main results, we need the following lemmas.

Lemma 1.2. [9] *Let* $p(z)=1+c_1z+c_2z^2+\cdots\in\mathcal{P}$, *where* $\mathcal{P}$ *is the family of all functions* p, *analytic in* $\mathbb{U}$, *for which* $\Re p(z)>0$. *Then* $|c_k|\leq 2$ *for* $k=1,2,\ldots$.

Lemma 1.3. [16] *Let* $k,l\in\mathbb{R}$ *and* $z_1,z_2\in\mathbb{C}$. *If* $|z_1|<R$ *and* $|z_2|<R$, *then*

$$|(k+l)z_1+(k-l)z_2|\leq\begin{cases}2R|k|, & |k|\geq|l|\\[2ex] 2R|l| & |k|\leq|l|\end{cases}.$$

In this paper, we obtain the Fekete-Szegö inequalities for functions belong to the class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ as well as its special classes.

2. Main Results

Theorem 2.1. *Let $m,n\in\mathbb{N}_0\,;m>n$ and $\gamma\in\mathbb{C}\backslash\{0\}$. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ and $\mu\in\mathbb{R}$. Then*

(10)
$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{|\gamma|B_1}{3^m-3^n} & , \quad |\mu-1|\le\psi\\ \frac{|\mu-1||\gamma|^2B_1^3}{\left|[(3^m-3^n)-2^n(2^m-2^n)]\gamma B_1^2+(2^m-2^n)^2(B_1-B_2)\right|} & , \quad |\mu-1|\ge\psi\end{cases},$$

where

$$\psi=\left|1-\frac{2^n\left(2^m-2^n\right)}{3^m-3^n}+\frac{\left(2^m-2^n\right)^2\left(B_1-B_2\right)}{\left(3^m-3^n\right)\gamma B_1^2}\right|. \tag{11}$$

Proof. Let $f\in\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$ and $g=f^{-1}$. Then there exist analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, such that

$$1+\frac{1}{\gamma}\left(\frac{D^mf(z)}{D^nf(z)}-1\right)=\varphi(u(z)) \tag{12}$$

and

$$1+\frac{1}{\gamma}\left(\frac{D^mg(w)}{D^ng(w)}-1\right)=\varphi(v(w)). \tag{13}$$

Define the functions p and q by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots \qquad (z\in\mathbb{U})$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+q_1w+q_2w^2+\cdots \qquad (w\in\mathbb{U})$$

or equivalently

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right) \qquad (z\in\mathbb{U}) \tag{14}$$

and

(15)
$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left(q_1w+\left(q_2-\frac{q_1^2}{2}\right)w^2+\cdots\right) \qquad (w\in\mathbb{U}).$$

Using (14) and (15) in (12) and (13), we obtain

$$\frac{2^m-2^n}{\gamma}a_2 = \frac{1}{2}B_1p_1 \tag{16}$$

$$\frac{(3^m-3^n)a_3-2^n(2^m-2^n)a_2^2}{\gamma} = \frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2 \tag{17}$$

$$-\frac{2^m-2^n}{\gamma}a_2 = \frac{1}{2}B_1q_1 \tag{18}$$

$$\frac{(3^m-3^n)\left(2a_2^2-a_3\right)-2^n(2^m-2^n)a_2^2}{\gamma} = \frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2. \tag{19}$$

From (16) and (18), we find that

$$p_1=-q_1 \tag{20}$$

and

$$2\frac{(2^m-2^n)^2}{\gamma^2}a_2^2=\frac{1}{4}B_1^2\left(p_1^2+q_1^2\right). \tag{21}$$

Also summing (17) and (19) results in

$$\frac{2}{\gamma}\left[(3^m-3^n)-2^n\left(2^m-2^n\right)\right]a_2^2=\frac{1}{2}B_1\left(p_2+q_2\right)-\frac{1}{4}\left(B_1-B_2\right)\left(p_1^2+q_1^2\right).$$

Combining this with (21) leads to

$$a_2^2=\frac{\gamma^2B_1^3\left(p_2+q_2\right)}{4\left\{\left[(3^m-3^n)-2^n\left(2^m-2^n\right)\right]\gamma B_1^2+\left(2^m-2^n\right)^2\left(B_1-B_2\right)\right\}}. \tag{22}$$

On the other hand, by subtracting (19) from (17) and a computation using (20) finally lead to

$$a_3=a_2^2+\frac{\gamma B_1}{4\left(3^m-3^n\right)}\left(p_2-q_2\right). \tag{23}$$

From (22) and (23) it follows that

$$a_3-\mu a_2^2=\frac{\gamma B_1}{4}\left[\left(h(\mu)+\frac{1}{3^m-3^n}\right)p_2+\left(h(\mu)-\frac{1}{3^m-3^n}\right)q_2\right],$$

where

$$h(\mu)=\frac{(1-\mu)\gamma B_1^2}{\left[(3^m-3^n)-2^n\left(2^m-2^n\right)\right]\gamma B_1^2+\left(2^m-2^n\right)^2\left(B_1-B_2\right)}.$$

Since all B_j $(j \in \mathbb{N})$ are real and $B_1 > 0$, by Lemma 1.2 and Lemma 1.3, we conclude that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|\gamma| B_1}{3^m - 3^n} & , \quad 0 \leq |h(\mu)| \leq \frac{1}{3^m - 3^n} \\ \\ |\gamma| B_1 |h(\mu)| & , \quad |h(\mu)| \geq \frac{1}{3^m - 3^n} \end{cases},$$

which completes the proof. □

Letting $m = 1$, $n = 0$ and $\gamma = 1$ in Theorem 2.1, then we have the following consequence.

Corollary 2.2. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{ST}_\Sigma(\varphi)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1}{2} & , \quad |\mu - 1| \leq \frac{1}{2}\left|1 + \frac{B_1 - B_2}{B_1^2}\right| \\ \\ \frac{|\mu-1| B_1^3}{\left|B_1^2 + B_1 - B_2\right|} & , \quad |\mu - 1| \geq \frac{1}{2}\left|1 + \frac{B_1 - B_2}{B_1^2}\right| \end{cases}.$$

Remark 3. If we choose $\mu = 0$ in the above corollary, then we get [17, Corollary 11].

Letting $m = 2$, $n = 1$ and $\gamma = 1$ in Theorem 2.1, then we have the following consequence.

Corollary 2.3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{CV}_\Sigma(\varphi)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1}{6} & , \quad |\mu - 1| \leq \frac{1}{3}\left|1 + \frac{2(B_1 - B_2)}{B_1^2}\right| \\ \\ \frac{|\mu-1| B_1^3}{2\left|B_1^2 + 2(B_1 - B_2)\right|} & , \quad |\mu - 1| \geq \frac{1}{3}\left|1 + \frac{2(B_1 - B_2)}{B_1^2}\right| \end{cases}.$$

Remark 4. If we choose $\mu = 0$ in the above corollary, then we get [17, Corollary 18].

If we take $\mu = 1$ and $\mu = 0$ in Theorem 2.1, then we have the following Corollary 2.4 and Corollary 2.5, respectively.

Corollary 2.4. *Let $m, n \in \mathbb{N}_0\,; m > n$ and $\gamma \in \mathbb{C}\backslash\{0\}$. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the function class $\mathcal{B}_\Sigma^{m,n}(\gamma;\varphi)$. Then*

$$\left|a_3 - a_2^2\right| \leq \frac{|\gamma| B_1}{3^m - 3^n}.$$

Corollary 2.5. *Let $m,n\in\mathbb{N}_0\,;m>n$ and $\gamma\in\mathbb{C}\backslash\{0\}$. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\varphi)$. Then*

$$|a_3|\leq\begin{cases}\frac{|\gamma|B_1}{3^m-3^n} & ,\quad 1\leq\psi\\ \frac{|\gamma|^2B_1^3}{\left|[(3^m-3^n)-2^n(2^m-2^n)]\gamma B_1^2+(2^m-2^n)^2(B_1-B_2)\right|} & ,\quad 1\geq\psi\end{cases},$$

where ψ is defined by (11).

If we take $m=1$ and $n=0$ in Corollary 2.4 and Corollary 2.5, then we have the following consequence.

Corollary 2.6. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{S}_{\Sigma}^{*}(\gamma;\varphi)$. Then*

$$\left|a_3-a_2^2\right|\leq\frac{|\gamma|\,B_1}{2}$$

and

$$|a_3|\leq\begin{cases}\frac{|\gamma|B_1}{2} & ,\quad 1\leq\frac{1}{2}\left|1+\frac{B_1-B_2}{\gamma B_1^2}\right|\\ \frac{|\gamma|^2B_1^3}{\left|\gamma B_1^2+B_1-B_2\right|} & ,\quad 1\geq\frac{1}{2}\left|1+\frac{B_1-B_2}{\gamma B_1^2}\right|\end{cases}.$$

If we take $m=2$ and $n=1$ in Corollary 2.4 and Corollary 2.5, then we have the following consequence.

Corollary 2.7. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{C}_{\Sigma}(\gamma;\varphi)$. Then*

$$\left|a_3-a_2^2\right|\leq\frac{|\gamma|\,B_1}{6}$$

and

$$|a_3|\leq\begin{cases}\frac{|\gamma|B_1}{6} & ,\quad 1\leq\frac{1}{3}\left|1+\frac{2(B_1-B_2)}{\gamma B_1^2}\right|\\ \frac{|\gamma|^2B_1^3}{2\left|\gamma B_1^2+2(B_1-B_2)\right|} & ,\quad 1\geq\frac{1}{3}\left|1+\frac{2(B_1-B_2)}{\gamma B_1^2}\right|\end{cases}.$$

If we consider the functions φ_α and φ_β, defined in Remark 2 in Theorem 2.1, then we get the following Corollary 2.8 and Corollary 2.9, respectively.

Corollary 2.8. *Let $m,n \in \mathbb{N}_0\,; m > n$ and $\gamma \in \mathbb{C}\backslash\{0\}$. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\alpha)$ $(0 < \alpha \leq 1)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2|\gamma|\alpha}{3^m - 3^n} & , \quad |\mu - 1| \leq \sigma \\ \\ \frac{4|\mu-1||\gamma|^2\alpha^2}{\left|2[(3^m-3^n)-2^n(2^m-2^n)]\gamma\alpha + (2^m-2^n)^2(1-\alpha)\right|} & , \quad |\mu - 1| \geq \sigma \end{cases},$$

where

$$\sigma = \left|1 - \frac{2^n\left(2^m - 2^n\right)}{\left(3^m - 3^n\right)} + \frac{\left(2^m - 2^n\right)^2\left(1-\alpha\right)}{2\left(3^m - 3^n\right)\gamma\alpha}\right|.$$

Corollary 2.9. *Let $m,n \in \mathbb{N}_0\,; m > n$ and $\gamma \in \mathbb{C}\backslash\{0\}$. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{B}_{\Sigma}^{m,n}(\gamma;\beta)$ $(0 \leq \beta < 1)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2|\gamma|(1-\beta)}{3^m - 3^n} & , \quad |\mu - 1| \leq \left|1 - \frac{2^n(2^m-2^n)}{3^m-3^n}\right| \\ \\ \frac{2|\mu-1||\gamma|(1-\beta)}{|(3^m-3^n)-2^n(2^m-2^n)|} & , \quad |\mu - 1| \geq \left|1 - \frac{2^n(2^m-2^n)}{3^m-3^n}\right| \end{cases}.$$

Taking $m = 1$, $n = 0$ and $\gamma = 1$ in Corollary 2.9, we get the following consequence.

Corollary 2.10. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{S}_{\Sigma}^{*}(\beta)$ $(0 \leq \beta < 1)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 1 - \beta & , \quad \mu \in \left[\frac{1}{2}, \frac{3}{2}\right] \\ \\ 2\left|\mu - 1\right|\left(1 - \beta\right) & , \quad \mu \in \left(-\infty, \frac{1}{2}\right] \cup \left[\frac{3}{2}, \infty\right) \end{cases}.$$

Taking $m = 2$, $n = 1$ and $\gamma = 1$ in Corollary 2.9, we get the following consequence.

Corollary 2.11. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{C}_{\Sigma}(\beta)$ $(0 \leq \beta < 1)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1-\beta}{3} & , \quad \mu \in \left[\frac{2}{3}, \frac{4}{3}\right] \\ \\ \left|\mu - 1\right|\left(1 - \beta\right) & , \quad \mu \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{4}{3}, \infty\right) \end{cases}.$$

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