Honam Mathematical J. **39** (2017), No. 4, pp. 569–574
https://doi.org/10.5831/HMJ.2017.39.4.569

ENERGETIC SUBSETS OF BE-ALGEBRAS

Young Bae Jun and Sun Shin Ahn*

Abstract. The notion of I_{BE}-energetic subsets in BE-algebras is introduced, and several properties are investigated. Characterizations of I_{BE}-energetic subsets are discussed, and conditions for a subset to be an I_{BE}-energetic subset are provided.

1. Introduction

As a generalization of a BCK-algebra, the notion of BE-algebras has been introduced by H. S. Kim and Y. H. Kim in [5]. The study of BE-algebras has been continued in papers [1], [2], [3], [6] and [7]. Jun et al. [4] have introduced the notions of S-energetic subsets and I-energetic subsets in BCK/BCI-algebras, and investigated several properties.

In this paper, we introduce the notion of I_{BE}-energetic subsets in BE-algebras, and investigate several properties. We consider characterizations of I_{BE}-energetic subsets and provide conditions for a subset to be an I_{BE}-energetic subset.

2. Preliminaries

We display basic notions on BE-algebras. We refer the reader to the papers [2, 5] for further information regarding BE-algebras.

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms

Received July 6, 2017. Accepted October 22, 2017.
2010 Mathematics Subject Classification. 06F35, 03G25.
Key words and phrases. BE-algebra, I_{BE}-energetic subset.
*Corresponding author. Tel: +82 2 2260 3410, Fax: +82 2 2266 3409.

hold:

(1) $(\forall x \in X)\,(x * x = 1),$

(2) $(\forall x \in X)\,(x * 1 = 1),$

(3) $(\forall x \in X)\,(1 * x = x),$

(4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)).$ (exchange)

We say that 1 is the *unit* of X.

A nonempty subset I of a BE-algebra X is called an *ideal* of X if it satisfies

(5) $X * I \subseteq I,$

(6) $(\forall x \in X)\,(\forall a, b \in I)\,((a * (b * x)) * x \in I)$

where $X * I = \{x * a \mid x \in X,\ a \in I\}$.

A BE-algebra X is said to be *transitive* if it satisfies: for all $x, y, z \in X$, $(y * z) * ((x * y) * (x * z)) = 1$. A BE-algebra X is said to be *self distributive* if it satisfies: for all $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$.

3. Energetic subsets

In what follows, let X denote a BE-algebra unless otherwise specified.

Definition 3.1. *A nonempty subset A of X is said to be I_{BE}-energetic if it satisfies*

(7) $(\forall a, b, x \in X)\,((a * (b * x)) * x \in A \ \Rightarrow\ \{a, b\} \cap A \neq \emptyset)\,.$

Example 3.2. *Let $X = \{1, a, b, c, d, 0\}$ be a set with the following Cayley table:*

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra (see [2]). It is routine to verify that $A := \{0, c, d\}$ is an I_{BE}-energetic subset of X. But $B := \{0, b, c, d\}$ is not an I_{BE}-energetic subset of X since $(a * (a * b)) * b = b \in B$ but $\{a, a\} \cap B = \emptyset$.

Example 3.3. *Let $X = \{1, a, b, c, d\}$ be a set with the following Cayley table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra (see [2]). It is routine to verify that $A := \{b, c, d\}$ is an I_{BE}-energetic subset of X.

Proposition 3.4. *Let A be a nonempty subset of X which does not contain the unit. If A is I_{BE}-energetic, then*

$$(8) \qquad (\forall a, x \in X)\,((a * x) * x \in A \;\Rightarrow\; a \in A)\,.$$

Proof. Assume that $(a*x)*x \in A$ for all $a, x \in X$. Then $(a*(1*x))*x = (a*x)*x \in A$ by (3), which implies from (7) that $\{a, 1\} \cap A \neq \emptyset$. Since $1 \notin A$, it follows that $a \in A$. □

Proposition 3.5. *For every I_{BE}-energetic subset A of X, if A does not contain the unit, then*

$$(9) \qquad (\forall a, x \in X)\,(a * x = 1,\; x \in A \;\Rightarrow\; a \in A)\,.$$

Proof. Assume that $1 \notin A$ and let $a, x \in X$ be such that $a*x = 1$ and $x \in A$. Then $(a*x)*x = 1*x = x \in A$, and so $a \in A$ by Proposition 3.4. □

Proposition 3.6. *Let A be a nonempty subset of X which does not contain the unit. If A satisfies the following condition:*

$$(10) \qquad (\forall x, y, z \in X)\,(x * z \in A \;\Rightarrow\; \{y,\; x * (y * z)\} \cap A \neq \emptyset)\,,$$

then the condition (9) is valid.

Proof. Let $a, x \in X$ be such that $a * x = 1$ and $x \in A$. Since $1*x = x \in A$, it follows from (10) that $\{a, 1\} \cap A = \{a, 1*(a*x)\} \cap A \neq \emptyset$. Hence $a \in A$ since $1 \notin A$. □

Theorem 3.7. *Let A be an I_{BE}-energetic subset of X which does not contain the unit. If A satisfies:*

$$(11) \qquad (\forall a, x \in X)\,(x * a \in A \;\Rightarrow\; a \in A)\,,$$

then $X \setminus A$ is an ideal of X.

Proof. We know from (11) that $a \in X \setminus A \ \Rightarrow\ x * a \in X \setminus A$ for all $a, x \in X$, that is, $X * X \setminus A \subseteq X \setminus A$. Let $a, b \in X \setminus A$. If $(a*(b*x))*x \in A$ for some $x \in X$, then $\{a, b\} \cap A \neq \emptyset$ by (7), which implies that $a \in A$ or $b \in A$. This is a contradiction, and so $(a*(b*x))*x \in X \setminus A$. Therefore $X \setminus A$ is an ideal of X. □

Theorem 3.8. *Let I be a nonempty subset of X satisfying the condition* (6). *Then $A := X \setminus I$ is an I_{BE}-energetic subset of X.*

Proof. Let $a, b, x \in X$ be such that $(a * (b * x)) * x \in A$. Assume that $\{a, b\} \cap A = \emptyset$. Then $a \notin A$ and $b \notin A$, and so $a, b \in I$. It follows from (6) that $(a * (b * x)) * x \in I$ for all $x \in X$. This is a contradiction, and therefore $\{a, b\} \cap A \neq \emptyset$. Hence $A := X \setminus I$ is an I_{BE}-energetic subset of X. □

Theorem 3.8 shows that X can be partitioned by a subset satisfying the condition (6) and an I_{BE}-energetic subset.

Theorem 3.9. *Let A be a nonempty subset of a transitive BE-algebra X which does not contain the unit. Then A is I_{BE}-energetic if and only if A satisfies the condition* (10).

Proof. Assume that A is I_{BE}-energetic and let $x * z \in A$ for $x, z \in X$. If $\{y,\ x * (y * z)\} \cap A = \emptyset$ for some $y \in X$, then $y \in X \setminus A$ and $x*(y*z) \in X \setminus A$. It follows from Proposition 3.4 that $(y*z)*z \in X \setminus A$. By the transitivity of X, we have $((y*z)*z)*((x*(y*z))*(x*z)) = 1$, and so

$$(((y * z) * z) * ((x * (y * z)) * (x * z))) * (x * z) = 1 * (x * z) = x * z \in A.$$

Hence $\{(y * z) * z,\ x * (y * z)\} \cap A \neq \emptyset$ by (7), and so $(y * z) * z \in A$ or $x * (y * z) \in A$. This is a contradiction, and so $\{y,\ x * (y * z)\} \cap A \neq \emptyset$.

Conversely, suppose that A satisfies the condition (10). Let $a, b, x \in X$ be such that $(a*(b*x))*x \in A$. Then $\{b,\ (a*(b*x))*(b*x)\} \cap A \neq \emptyset$ by (10), and so $b \in A$ or $(a * (b * x)) * (b * x) \in A$. If $b \in A$, then clearly $\{a, b\} \cap A \neq \emptyset$. If $(a * (b * x)) * (b * x) \in A$, then $(b * (a * x)) * (b * x) = (a * (b * x)) * (b * x) \in A$ by (4). The transitivity of X induces $((a*x)*x)*((b*(a*x))*(b*x)) = 1$. Hence $(a*x)*x \in A$ by Proposition 3.6, and thus

$$\{a, 1\} \cap A = \{a, (a * x) * (a * x)\} \cap A \neq \emptyset$$

by (1) and (10). Since $1 \notin A$, it follows that $a \in A$ and so that $\{a, b\} \cap A \neq \emptyset$. Therefore A is an I_{BE}-energetic subset of X. □

Note that Theorem 3.9 also holds in a self distributive BE-algerba since every self distributive BE-algerba is transitive.

Theorem 3.10. *If A and B are I_{BE}-energetic subsets of X, then $A \cap B$ is also an I_{BE}-energetic subset of X.*

Proof. Let $(a*(b*x))*x \in A \cap B$ for $a, b, x \in X$. Then $(a*(b*x))*x \in A$ and $(a*(b*x))*x \in B$. It follows that $\{a, b\} \cap A \neq \emptyset$ and $\{a, b\} \cap B \neq \emptyset$. Hence $\{a, b\} \cap (A \cap B) = (\{a, b\} \cap A) \cap (\{a, b\} \cap B) \neq \emptyset$, and therefore $A \cap B$ is an I_{BE}-energetic subset of X. □

For any $u, v \in X$, we consider sets

$$X_u^v := \{z \in X \mid u * (v * z) = 1\} \text{ and } A_u^v := X \setminus X_u^v.$$

Obviously, $u, v \notin A_u^v$, $A_u^v = A_v^u$ and A_u^v does not contain the unit. We know that A_u^v may not be I_{BE}-energetic as seen in the following example.

Example 3.11. *Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ in Example 3.2. We know that $A_c^d = \{0, b\}$ and it is not I_{BE}-energetic since $(a * (a * b)) * b = b \in A_c^d$ but $\{a, a\} \cap A_c^d = \emptyset$.*

We consider conditions for the set A_u^v to be I_{BE}-energetic.

Theorem 3.12. *If X is a self distributive BE-algebra, then A_u^v is I_{BE}-energetic for all $u, v \in X$.*

Proof. Let $(a * (b * x)) * x \in A_u^v$ for $a, b, x \in X$. Assume that $\{a, b\} \cap A_u^v = \emptyset$. Then $a \notin A_u^v$ and $b \notin A_u^v$, which imply that $u * (v * a) = 1 = u * (v * b)$. Using (3) and the self distributivity of X, we have

$$\begin{aligned} u * (v * ((a * (b * x)) * x)) &= ((u * (v * a)) * (u * (v * (b * x)))) * (u * (v * x)) \\ &= (u * (v * (b * x))) * (u * (v * x)) \\ &= ((u * (v * b)) * (u * (v * x))) * (u * (v * x)) = 1, \end{aligned}$$

and so $(a * (b * x)) * x \notin A_u^v$. This is a contradiction, and therefore $\{a, b\} \cap A_u^v \neq \emptyset$. Hence A_u^v is an I_{BE}-energetic subset of X for all $u, v \in X$. □

References

[1] S. S. Ahn, Y. H. Kim and J. M. Ko, *Filters in commutative BE-algerbas,* Commun. Korean Math. Soc. **27** (2012), no. 2, 233–242.

[2] S. S. Ahn and K. S. So, *On ideals and upper sets in BE-algerbas,* Sci. Math. Jpn. **68** (2008), no. 2, 279–285.

[3] R. A. Borzooei, A. B. Saeid, A. Rezaei, A. Radfar and R. Ameri, *On pseudo BE-algerbas,* Discussions Math. Gen. Alg. Appl. **33** (2013), 95–108.

[4] Y. B. Jun, S. S. Ahn and E. H. Roh, *Energetic subsets and permeable values with applications in BCK/BCI-algebras,* Appl. Math. Sci. **7** (2013), no. 89, 4425–4438.
[5] H. S. Kim and Y. H. Kim, *On BE-algerbas,* Sci. Math. Jpn. **66** (2007), no. 1, 113–116.
[6] H. S. Kim and K. J. Lee, *Extended upper sets in BE-algerbas,* Bull. Malays. Math. Sci. Soc. **34** (2011), no. 3, 511–520.
[7] B. L. Meng, *On filters in BE-algerbas,* Sci. Math. Jpn. **71** (2010), no. 2, 201–207.

Young Bae Jun
Department of Mathematics Education, Gyeongsang National University
Jinju, 52828, Korea.
E-mail: skywine@gmail.com

Sun Shin Ahn
Department of Mathematics Education, Dongguk University,
Seoul 04620, Korea.
E-mail: sunshine@dongguk.edu