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DETERMINATION OF THE FRICKE FAMILIES

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Abstract. For a positive integer N divisible by 4, let $\mathcal{O}_N^1(\mathbb{Q})$ be the ring of weakly holomorphic modular functions for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients. We present explicit generators of the ring $\mathcal{O}_N^1(\mathbb{Q})$ over $\mathbb{Q}$ in terms of both Fricke functions and Siegel functions, from which we are able to classify all Fricke families of such level N.

1. Introduction

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the complex upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ by fractional linear transformations, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

For a positive integer N, let $\mathcal{F}_N$ be the field of meromorphic modular functions for the principal congruence subgroup $\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2 \pmod{N}\}$ of $\mathrm{SL}_2(\mathbb{Z})$ whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$. It is well known that $\mathcal{F}_1$ is generated over $\mathbb{Q}$ by the elliptic modular function $j(\tau)$, and $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \tag{1}$$

(see §2). For $N \geq 2$, let

$$\mathcal{V}_N = \{\mathbf{v} \in \mathbb{Q}^2 \mid \mathbf{v} \text{ has primitive denominator } N\}.$$

We call a family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ a *Fricke family* of level N, if it satisfies the following three conditions:

(F1) Each $h_{\mathbf{v}}(\tau)$ is weakly holomorphic (that is, holomorphic on $\mathbb{H}$).
(F2) $h_{\mathbf{v}}(\tau)$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$.

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(F3) $h_{\mathbf{v}}(\tau)^\alpha = h_{{}^t\alpha\mathbf{v}}(\tau)$ for all $\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$, where ${}^t\alpha$ means the transpose of α.

There are two important kinds of Fricke families $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ and $\{g_{\mathbf{v}}(\tau)^{12N}\}_{\mathbf{v}}$, one consisting of Fricke functions and the other consisting of $12N$th powers of Siegel functions (see §3). They are building blocks of fields of modular functions and groups of modular units ([7, Chapter 2] and [8, Chapter 6]). Since their special values at imaginary quadratic arguments generate class fields over the corresponding imaginary quadratic fields (see [3], [4] and [8, Chapter 10]), it would be meaningful by themselves and also worth of investigating the structure of Fricke families as a ring.

As far as we understand, there is no known result on constructing all such families. In this paper, we shall first classify all Fricke families of level N, when $N \equiv 0 \pmod 4$ (Theorems 4.3, 6.2 and Corollary 6.4). Furthermore, if we constrain the condition (F1) to

(F1′) every $h_{\mathbf{v}}(\tau)$ is holomorphic on $\mathbb{H}$ except for the set $\{\gamma(\zeta_3), \gamma(\zeta_4) \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$,

then we can also determine all weak families $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ satisfying (F1′), (F2) and (F3) for arbitrary level $N \geq 2$ (Theorem 7.4 and Remark 7.5).

2. Galois actions on functions

In this section, we shall briefly describe the actions of the group

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$$

on the field $\mathcal{F}_N$.

For a positive integer N, the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ has a unique decomposition

$$G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \quad \text{with} \quad G_N = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \mid d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

This group acts on the field $\mathcal{F}_N$ as follows ([9, §6.1–6.2]): Let

$$h(\tau) = \sum_{n \gg -\infty} c_n q^{n/N} \in \mathcal{F}_N \quad (c_n \in \mathbb{Q}(\zeta_N),\ q = e^{2\pi \mathrm{i}\tau}).$$

(A1) The matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right] \in G_N$ acts on $h(\tau)$ as

$$h(\tau)^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right]} = \sum_{n \gg -\infty} c_n^{\sigma_d} q^{n/N},$$

where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ given by $\zeta_N^{\sigma_d} = \zeta_N^d$.

(A2) The matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $h(\tau)$ as

$$h(\tau)^\gamma = (h \circ \widetilde{\gamma})(\tau),$$

where $\widetilde{\gamma}$ is any preimage of the reduction $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ considered as a fractional linear transformation.

Lemma 2.1. *Let* $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ *be a Fricke family of level* $N \geq 2$. *Then* $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ *acts on* $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ *transitively.*

Proof. Note by (F3) that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on the family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$. Let $\mathbf{v} = \begin{bmatrix} a/N \\ b/N \end{bmatrix} \in \mathcal{V}_N$ so that $\gcd(a, b)$ is relatively prime to N. If we take any $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z})$ such that $\det(\alpha)$ is relatively prime to N, then we see by (F3) that

$$h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau)^{\alpha} = h_{{}^t\alpha\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau) = h_{\begin{bmatrix} a/N \\ b/N \end{bmatrix}}(\tau) = h_{\mathbf{v}}(\tau).$$

This implies that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ acts on $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ transitively. □

Remark 2.2. Roughly speaking, this family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ is completely determined by its component $h_{\begin{bmatrix} 1/N \\ 0 \end{bmatrix}}(\tau)$.

3. Fricke and Siegel functions

For a lattice Λ in $\mathbb{C}$, we let

$$g_2(\Lambda) = 60 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^4},\ g_3(\Lambda) = 140 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^6} \text{ and } \Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2.$$

The *elliptic modular function* $j(\tau)$ is defined by

$$j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)} = 1728\left(1 + 27\frac{g_3(\tau)^2}{\Delta(\tau)}\right) \quad (\tau \in \mathbb{H}), \tag{2}$$

where $g_2(\tau) = g_2([\tau, 1])$, $g_3(\tau) = g_3([\tau, 1])$ and $\Delta(\tau) = \Delta([\tau, 1])$. This generates the ring of weakly holomorphic functions in $\mathcal{F}_1$ over $\mathbb{Q}$ ([8, Theorem 2 in Chapter 5]).

The *Weierstrass* $\wp$*-function* relative to Λ is given by

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) \quad (z \in \mathbb{C}).$$

For each $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we define the *Fricke function* $f_{\mathbf{v}}(\tau)$ by

$$f_{\mathbf{v}}(\tau) = -2^7 3^5 \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp_{\mathbf{v}}(\tau) \quad (\tau \in \mathbb{H}), \tag{3}$$

where $\wp_{\mathbf{v}}(\tau) = \wp(v_1\tau + v_2; [\tau, 1])$.

By the *Weierstrass* σ*-function* relative to Λ, we mean the infinite product

$$\sigma(z; \Lambda) = z \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda + (1/2)(z/\lambda)^2} \quad (z \in \mathbb{C}).$$

Taking logarithmic derivative, we achieve the *Weierstrass* ζ*-function* as

$$\zeta(z; \Lambda) = \frac{\sigma'(z; \Lambda)}{\sigma(z; \Lambda)} = \frac{1}{z} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right) \quad (z \in \mathbb{C}).$$

Since $\zeta'(z;\Lambda) = -\wp(z;\Lambda)$ is periodic with respect to Λ, for every $\lambda \in \Lambda$ there is a constant $\eta(\lambda;\Lambda)$ which satisfies

$$\zeta(z+\lambda;\Lambda) - \zeta(z;\Lambda) = \eta(\lambda;\Lambda) \quad (z \in \mathbb{C}).$$

For any $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we then define the *Siegel function* $g_{\mathbf{v}}(\tau)$ by

$$(4) \quad g_{\mathbf{v}}(\tau) = e^{-(v_1\eta(\tau;[\tau,1])+v_2\eta(1;[\tau,1]))(v_1\tau+v_2)/2}\sigma(v_1\tau+v_2;[\tau,1])\eta(\tau)^2 \ (\tau \in \mathbb{H}),$$

where

$$\eta(\tau) = \sqrt{2\pi}\zeta_8 q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad (\tau \in \mathbb{H})$$

is the *Dedekind η-function* which is a 24th root of $\Delta(\tau)$ ([8, Theorem 5 in Chapter 18]). By using the q-product expansion of the Weierstrass σ-function, we get the expression

$$g_{\mathbf{v}}(\tau) = -e^{\pi i v_2(v_1-1)}q^{(1/2)\mathbf{B}_2(v_1)}(1-q^{v_1}e^{2\pi i v_2}) \prod_{n=1}^{\infty}(1-q^{n+v_1}e^{2\pi i v_2})(1-q^{n-v_1}e^{-2\pi i v_2}),$$

where $\mathbf{B}_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial ([8, Chapter 19, §2]). Observe that $g_{\mathbf{v}}(\tau)$ has neither zeros nor poles on $\mathbb{H}$.

Proposition 3.1. *If $N \geq 2$, then $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ and $\{g_{\mathbf{v}}(\tau)^{12N}\}_{\mathbf{v}\in\mathcal{V}_N}$ are Fricke families of level N.*

Proof. See [8, Chapter 6, §2–3] and [7, Proposition 1.3 in Chapter 2]. □

Remark 3.2. We call a function $h(\tau)$ in $\mathcal{F}_N$ a *modular unit* of level $N \geq 1$, if both $h(\tau)$ and $h(\tau)^{-1}$ are integral over $\mathbb{Q}[j(\tau)]$. As is well known, $h(\tau)$ is a modular unit if and only if it has neither zeros nor poles on $\mathbb{H}$ ([7, p. 36] or [2, Proposition 2.3]). Thus $g_{\mathbf{v}}(\tau)^{12N}$ is a modular unit of level N for every $\mathbf{v} \in \mathcal{V}_N$ with $N \geq 2$. Moreover, $g_{\mathbf{v}}(\tau)$ is a modular unit of level $12N^2$ ([7, Theorems 5.2 and 5.3 in Chapter 3]).

For later use, we need the following lemmas.

Lemma 3.3. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$.*

(i) *We have the assertion that $f_{\mathbf{u}}(\tau) = f_{\mathbf{v}}(\tau)$ if and only if $\mathbf{u} \equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$.*
(ii) *If $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$, then we get the relation*

$$(f_{\mathbf{u}}(\tau) - f_{\mathbf{v}}(\tau))^6 = 2^{12}3^6 j(\tau)^2(j(\tau)-1728)^3\frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12}g_{\mathbf{v}}(\tau)^{12}}.$$

Proof. (i) See [1, Lemma 10.4] and definition (3).
(ii) See [8, Theorem 2 in Chapter 18] and definitions (2), (3) and (4). □

Remark 3.4. For $N\geq 2$, let $\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'\in(1/N)\mathbb{Z}^2\setminus\mathbb{Z}^2$ such that $\mathbf{u}\not\equiv\pm\mathbf{v}\,(\mathrm{mod}\ \mathbb{Z}^2)$ and $\mathbf{u}'\not\equiv\pm\mathbf{v}'\ (\mathrm{mod}\ \mathbb{Z}^2)$. Then, the function

$$\frac{f_{\mathbf{u}}(\tau)-f_{\mathbf{v}}(\tau)}{f_{\mathbf{u}'}(\tau)-f_{\mathbf{v}'}(\tau)}=\frac{\wp_{\mathbf{u}}(\tau)-\wp_{\mathbf{v}}(\tau)}{\wp_{\mathbf{u}'}(\tau)-\wp_{\mathbf{v}'}(\tau)}$$

in $\mathcal{F}_N$ has neither zeros nor poles on $\mathbb{H}$ by Lemma 3.3(ii). Thus it is a modular unit of level N by Remark 3.2, called a *Weierstrass unit* of level N.

Lemma 3.5. *Let* $\mathbf{v}\in\mathbb{Q}^2\setminus\mathbb{Z}^2$.

(i) *We have* $g_{-\mathbf{v}}(\tau)=-g_{\mathbf{v}}(\tau)$.
(ii) *If* $\mathbf{s}=\left[\begin{smallmatrix}s_1\\ s_2\end{smallmatrix}\right]\in\mathbb{Z}^2$*, then we get* $g_{\mathbf{v}+\mathbf{s}}(\tau)=(-1)^{s_1s_2+s_1+s_2}e^{-\pi\mathrm{i}(s_1v_2-s_2v_1)}g_{\mathbf{v}}(\tau)$.
(iii) *For each* $\gamma\in\mathrm{SL}_2(\mathbb{Z})$*, we obtain* $(g_{\mathbf{v}}\circ\gamma)(\tau)=\zeta g_{{}^t\gamma\mathbf{v}}(\tau)$ *for some* 12*th root of unity* ζ *depending only on* γ.

Proof. See [6, Proposition 2.4]. □

4. Rings of weakly holomorphic functions

For an integer $N\geq 2$, we denote by Fr_N the set of all Fricke families of level N. Then, Fr_N becomes a ring under the operations

$$\begin{aligned}\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}+\{k_{\mathbf{v}}(\tau)\}_{\mathbf{v}}&=\{(h_{\mathbf{v}}+k_{\mathbf{v}})(\tau)\}_{\mathbf{v}},\\ \{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}\cdot\{k_{\mathbf{v}}(\tau)\}_{\mathbf{v}}&=\{(h_{\mathbf{v}}k_{\mathbf{v}})(\tau)\}_{\mathbf{v}}.\end{aligned}\tag{5}$$

For a positive integer N, let $\mathcal{F}_N^1(\mathbb{Q})$ be the field of meromorphic modular functions for the congruence subgroup

$$\Gamma^1(N)=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\mid\gamma\equiv\begin{bmatrix}1&0\\ *&1\end{bmatrix}\ (\mathrm{mod}\ N)\right\}$$

with rational Fourier coefficients. Further, we let $\mathcal{O}_N^1(\mathbb{Q})$ its subring consisting of weakly holomorphic functions.

Lemma 4.1. *Let* $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}\in\mathrm{Fr}_N$ *with* $N\geq 2$*. Then,* $h_{\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}(\tau)$ *belongs to* $\mathcal{O}_N^1(\mathbb{Q})$.

Proof. For any $\gamma=\left[\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right]\in\Gamma^1(N)$, we see that

$$\begin{aligned}(h_{\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}\circ\gamma)(\tau)&=h_{\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}(\tau)^{\gamma}\quad\text{by (A2)}\\ &=h_{{}^t\gamma\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}(\tau)\quad\text{by (F3)}\\ &=h_{\left[\begin{smallmatrix}a/N\\ b/N\end{smallmatrix}\right]}(\tau)\\ &=h_{\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}(\tau)\quad\text{by the fact } a\equiv 1,\ b\equiv 0\ (\mathrm{mod}\ N)\text{ and (F2).}\end{aligned}$$

Thus $h_{\left[\begin{smallmatrix}1/N\\ 0\end{smallmatrix}\right]}(\tau)$ is modular for $\Gamma^1(N)$.

Now, let $\beta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right] \in G_N$. We get by (F3) and (F2) that

$$h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^\beta = h_{{}^t\beta\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) = h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau),$$

which shows that $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ has rational Fourier coefficients by (A1).

Moreover, since $h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ is weakly holomorphic by (F1), it belongs to $\mathcal{O}_N^1(\mathbb{Q})$. □

Hence we obtain by Lemma 4.1 a ring homomorphism

$$\begin{array}{rccc} \phi_N : & \mathrm{Fr}_N & \to & \mathcal{O}_N^1(\mathbb{Q}) \\ & \{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} & \mapsto & h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau). \end{array} \tag{6}$$

Lemma 4.2. *For $N \geq 2$, let a and b be a pair of integers such that $\gcd(a, b)$ is relatively prime to N. Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\gamma' = \left[\begin{smallmatrix} a & b \\ c' & d' \end{smallmatrix}\right]$ be matrices in $M_2(\mathbb{Z})$ such that $\det(\gamma) \equiv \det(\gamma') \equiv 1 \pmod{N}$. Then, there is a matrix $\delta \in \Gamma^1(N)$ satisfying $\delta\gamma \equiv \gamma' \pmod{N}$.*

Proof. Take $\delta = \left[\begin{smallmatrix} 1 & 0 \\ c'd-cd' & 1 \end{smallmatrix}\right] \in \Gamma^1(N)$. One can then show that

$$\delta\gamma \equiv \begin{bmatrix} a & b \\ c'\det(\gamma) + c(-\det(\gamma') + 1) & d'\det(\gamma) + d(-\det(\gamma') + 1) \end{bmatrix} \equiv \gamma' \pmod{N}$$

due to the fact $\det(\gamma) \equiv \det(\gamma') \equiv 1 \pmod{N}$. □

Theorem 4.3. *If $N \geq 2$, then two rings Fr_N and $\mathcal{O}_N^1(\mathbb{Q})$ are isomorphic via the map ϕ_N stated in (6).*

Proof. Let $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}} \in \ker(\phi)$, and so $\phi_N(\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}) = h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) = 0$. Then we attain by Lemma 2.1 that $h_{\mathbf{v}}(\tau) = 0$ for all $\mathbf{v} \in \mathcal{V}_N$. This shows that ϕ_N is one-to-one.

Now, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. For each $\mathbf{v} = \left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right] \in \mathcal{V}_N$, we take any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ such that $\det(\gamma) \equiv 1 \pmod{N}$, and set $h_{\mathbf{v}}(\tau) = h(\tau)^\gamma$. We first claim that $h_{\mathbf{v}}(\tau)$ is well-defined, independent of the choice of γ. Indeed, if $\gamma' = \left[\begin{smallmatrix} a & b \\ c' & d' \end{smallmatrix}\right]$ is another matrix in $M_2(\mathbb{Z})$ such that $\det(\gamma') \equiv 1 \pmod{N}$, then we see that

$$\begin{aligned} h(\tau)^{\gamma'} &= h(\tau)^{\delta\gamma} \quad \text{for some } \delta \in \Gamma^1(N) \text{ by Lemma 4.2 and (1)} \\ &= h(\tau)^\gamma \quad \text{because } h(\tau) \text{ is modular for } \Gamma^1(N). \end{aligned}$$

Since $h(\tau)$ is weakly holomorphic, so is $h_{\mathbf{v}}(\tau) = h(\tau)^\gamma$ by (A2). Furthermore, $h_{\mathbf{v}}(\tau)$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$ by (1). Let $\alpha = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. We then derive by considering $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ as an element of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ that

$$h_{\mathbf{v}}(\tau)^\alpha = \left(h(\tau)^{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}\right)^{\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]}$$

$$= h(\tau)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{smallmatrix}\right]}$$

$$= \left(h(\tau)^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & \det(\alpha) \end{smallmatrix}\right]}\right)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\alpha)^{-1}(cx+dz) & \det(\alpha)^{-1}(cy+dw) \end{smallmatrix}\right]}$$

$$= h(\tau)^{\left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\alpha)^{-1}(cx+dz) & \det(\alpha)^{-1}(cy+dw) \end{smallmatrix}\right]}$$

since $h(\tau)$ has rational Fourier coefficients

$$= h_{\left[\begin{smallmatrix} (ax+bz)/N \\ (ay+bw)/N \end{smallmatrix}\right]}(\tau)$$

because $\left[\begin{smallmatrix} ax+bz & ay+bw \\ \det(\alpha)^{-1}(cx+dz) & \det(\alpha)^{-1}(cy+dw) \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$

$$= h_{\left[\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}\right]\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau)$$

$$= h_{{}^t\alpha\mathbf{v}}(\tau).$$

Thus the family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}$ satisfies (F3). Lastly, since

$$\phi_N(\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}}) = h_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau),$$

ϕ_N is surjective.

Therefore, we conclude that Fr_N and $\mathcal{O}_N^1(\mathbb{Q})$ are isomorphic via ϕ_N. □

5. Conjugate subgroups of $\mathbf{SL_2}(\mathbb{R})$

For a positive integer N, let

$$\Gamma_1(N) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod N\right\} \quad \text{and} \quad \omega_N = \begin{bmatrix} 1/\sqrt{N} & 0 \\ 0 & \sqrt{N} \end{bmatrix}.$$

Then, we see from the observation

$$\omega_N \begin{bmatrix} a & b \\ c & d \end{bmatrix} \omega_N^{-1} = \begin{bmatrix} a & b/N \\ Nc & d \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

that $\Gamma^1(N)$ and $\Gamma_1(N)$ are conjugate in $\mathrm{SL}_2(\mathbb{R})$, namely,

$$\omega_N \Gamma^1(N) \omega_N^{-1} = \Gamma_1(N). \tag{7}$$

Let $\mathcal{F}_{1,N}(\mathbb{Q})$ be the field of meromorphic modular functions for $\Gamma_1(N)$ with rational Fourier coefficients. One can readily check that the relation (7) gives rise to an isomorphism

$$\begin{array}{ccc} \mathcal{F}_{1,N}(\mathbb{Q}) & \xrightarrow{\sim} & \mathcal{F}_N^1(\mathbb{Q}) \\ h(\tau) = \sum_{n \gg -\infty} c_n q^n & \mapsto & (h \circ \omega_N)(\tau) = h(\tau/N) = \sum_{n \gg -\infty} c_n q^{n/N} \end{array} \tag{8}$$

with inverse map $f(\tau) \mapsto (f \circ \omega_N^{-1})(\tau) = f(N\tau)$. Furthermore, let $\mathcal{O}_{1,N}(\mathbb{Q})$ be the subring of $\mathcal{F}_{1,N}(\mathbb{Q})$ consisting of weakly holomorphic functions. Since the map in (8) preserves weakly holomorphicity, it induces an isomorphism

$$\mathcal{O}_{1,N}(\mathbb{Q}) \xrightarrow{\sim} \mathcal{O}_N^1(\mathbb{Q}). \tag{9}$$

Let $X_1(4)$ be the modular curve corresponding to the congruence subgroup $\Gamma_1(4)$. It is well known that $X_1(4)$ has genus 0 with three inequivalent cusps 0, 1/2 and i∞ ([5, p. 131]). Moreover, the function

$$g_{1,4}(\tau)=\left(\frac{g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(4\tau)}{g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(4\tau)}\right)^8=q^{-1}(1+q)^8\prod_{n=1}^{\infty}\left(\frac{(1-q^{4n+2})(1-q^{4n-2})}{(1-q^{4n+1})(1-q^{4n-1})}\right)^8$$

generates the function field $\mathbb{C}(X_1(4))$ of $X_1(4)$ over $\mathbb{C}$, having values 16, 0 and ∞ at the cusps 0, 1/2 and i∞, respectively ([5, Theorem 3(ii)] and [6, Tables 2 and 3]). Since $g_{1,4}(\tau)$ has rational Fourier coefficients, we deduce by [5, Lemma 4.1]

$$\mathcal{F}_{1,4}(\mathbb{Q})=\mathbb{Q}(g_{1,4}(\tau)). \tag{10}$$

Lemma 5.1. *Let $c\in\mathbb{C}$. Then, $(g_{1,4}(\tau)-c)$ has neither zeros nor poles on $\mathbb{H}$ if and only if $c\in\{0,16\}$.*

Proof. See [2, (4)]. □

Theorem 5.2. *We get the following structures.*

(i) $\mathcal{O}_{1,4}(\mathbb{Q})=\mathbb{Q}[g_{1,4}(\tau),g_{1,4}(\tau)^{-1},(g_{1,4}(\tau)-16)^{-1}]$.

(ii) $\mathcal{O}_4^1(\mathbb{Q})=\mathbb{Q}[g_4^1(\tau),g_4^1(\tau)^{-1},(g_4^1(\tau)-16)^{-1}]$, *where* $g_4^1(\tau)=g_{1,4}(\tau/4)=g_{\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right]}(\tau)^{-8}g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^8$.

Proof. (i) Since $g_{1,4}(\tau)$ and $(g_{1,4}(\tau)-16)$ are modular units in $\mathcal{F}_{1,4}(\mathbb{Q})$ by Lemma 5.1 and (10), we obtain the inclusion $\mathcal{O}_{1,4}(\mathbb{Q})\supseteq\mathbb{Q}[g_{1,4}(\tau),g_{1,4}(\tau)^{-1},(g_{1,4}(\tau)-16)^{-1}]$.

Conversely, let $h(\tau)\in\mathcal{O}_{1,4}(\mathbb{Q})$. By (10), we can express $h(\tau)$ as $h(\tau)=A(g_{1,4}(\tau))/B(g_{1,4}(\tau))$ for some polynomials $A(x),B(x)\in\mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c\in\overline{\mathbb{Q}}$ not equal to 0 or 16. We see by Lemma 5.1 that $g_{1,4}(\tau_0)-c=0$ for some $\tau_0\in\mathbb{H}$, from which we have $B(g_{1,4}(\tau_0))=0$. But, since $A(x)$ is not divisible by $(x-c)$ in $\overline{\mathbb{Q}}[x]$, we achieve $A(g_{1,4}(\tau_0))\neq 0$. This contradicts that $h(\tau)$ is weakly holomorphic. Thus $B(x)$ has no zeros other than 0 and 16, which implies the converse inclusion $\mathcal{O}_{1,4}(\mathbb{Q})\subseteq\mathbb{Q}[g_{1,4}(\tau),g_{1,4}(\tau)^{-1},(g_{1,4}(\tau)-16)^{-1}]$.

(ii) It follows immediately from (i) and the isomorphism given in (9). □

6. Generators for $N\equiv 0 \pmod 4$

Now, we are ready to present explicit generators of the ring $\mathcal{O}_N^1(\mathbb{Q})$ over $\mathbb{Q}$, when $N\equiv 0 \pmod 4$. This amounts to classifying all Fricke families of such level N by Theorem 4.3.

Proposition 6.1. *If $N\geq 2$, then we obtain $\mathcal{F}_N^1(\mathbb{Q})=\mathcal{F}_1(f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau))$.*

Proof. We first recall that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N(\mathbb{Z})/\{\pm I_2\} \simeq G_N \cdot \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}.$$

Observe by (A1) and (A2) that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_N^1(\mathbb{Q})$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \simeq G_N \cdot \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \mid \gamma \equiv \pm \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \pmod{N} \right\}.$$

Let $F = \mathcal{F}_1(f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau))$. Since $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N} \in \mathrm{Fr}_N$ by Proposition 3.1, we have the inclusion $F \subseteq \mathcal{F}_N^1(\mathbb{Q})$ by Lemma 4.1. Suppose that $\alpha = \beta\gamma$ with $\beta \in G_N$ and $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ leaves $f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)$ fixed. We then derive that

$$\begin{aligned}
f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) &= f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{\alpha} \\
&= (f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{\beta})^{\gamma} \\
&= f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)^{\gamma} \quad \text{because } f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) \text{ has rational Fourier coefficients} \\
&= f_{{}^t\gamma\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) \quad \text{by (F2) and (F3) for } \{f_{\mathbf{v}}(\tau)\}_{\mathbf{v}} \\
&= f_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau).
\end{aligned}$$

Thus we get $b \equiv 0 \pmod{N}$ and $a \equiv d \equiv \pm 1 \pmod{N}$ by Lemma 3.3(i) and the fact $\gamma \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$. This yields $F \supseteq \mathcal{F}_N^1(\mathbb{Q})$ by Galois theory. Therefore, we conclude $F = \mathcal{F}_1(f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)) = \mathcal{F}_N^1(\mathbb{Q})$. □

When $N \geq 8$ and $N \equiv 0 \pmod{4}$, we consider a function

$$f_N^1(\tau) = \frac{f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)} \quad (\tau \in \mathbb{H}).$$

It is a modular unit belonging to $\mathcal{O}_N^1(\mathbb{Q})$ by Proposition 3.1, Remark 3.4 and Lemma 4.1.

Theorem 6.2. *If $N \geq 8$ and $N \equiv 0 \pmod{4}$, then we attain*

$$\mathcal{O}_N^1(\mathbb{Q}) = \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)] = \mathbb{Q}[g_4^1(\tau), g_4^1(\tau)^{-1}, (g_4^1(\tau) - 16)^{-1}, f_N^1(\tau)].$$

Proof. It is obvious that $\mathcal{O}_N^1(\mathbb{Q}) \supseteq \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$.

As for the converse inclusion, let $h(\tau) \in \mathcal{O}_N^1(\mathbb{Q})$. Note by Proposition 6.1 and Lemma 4.1 that

$$\mathcal{F}_N^1(\mathbb{Q}) = \mathcal{F}_1(f_{\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau)) = \mathcal{F}_4^1(\mathbb{Q})(f_N^1(\tau)).$$

So, we can express $h = h(\tau)$ as

$$h = c_0 + c_1 f + \cdots + c_{d-1} f^{d-1}, \tag{11}$$

where $f = f_N^1(\tau)$, $d = [\mathcal{F}_N^1(\mathbb{Q}) : \mathcal{F}_4^1(\mathbb{Q})]$ and $c_0, c_1, \ldots, c_{d-1} \in \mathcal{F}_4^1(\mathbb{Q})$. Multiplying both sides of (11) by $1, f, \ldots, f^{d-1}$, respectively, we have a linear system (with unknowns $c_0, c_1, \ldots, c_{d-1}$)

$$\begin{bmatrix} 1 & f & \cdots & f^{d-1} \\ f & f^2 & \cdots & f^d \\ \vdots & \vdots & \ddots & \vdots \\ f^{d-1} & f^d & \cdots & f^{2d-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} h \\ fh \\ \vdots \\ f^{d-1}h \end{bmatrix}.$$

By taking the trace $\mathrm{Tr} = \mathrm{Tr}_{\mathcal{F}_N^1(\mathbb{Q})/\mathcal{F}_4^1(\mathbb{Q})}$ on both sides, we obtain

$$T \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} \mathrm{Tr}(h) \\ \mathrm{Tr}(fh) \\ \vdots \\ \mathrm{Tr}(f^{d-1}h) \end{bmatrix} \quad \text{with} \quad T = \begin{bmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(f) & \cdots & \mathrm{Tr}(f^{d-1}) \\ \mathrm{Tr}(f) & \mathrm{Tr}(f^2) & \cdots & \mathrm{Tr}(f^d) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}(f^{d-1}) & \mathrm{Tr}(f^d) & \cdots & \mathrm{Tr}(f^{2d-2}) \end{bmatrix}.$$

Since every $\mathrm{Tr}(*)$, appeared in the above expression, lies in $\mathcal{O}_4^1(\mathbb{Q})$, we get

$$c_0, c_1, \ldots, c_{d-1} \in \det(T)^{-1}\mathcal{O}_4^1(\mathbb{Q}). \tag{12}$$

If we let $f_1, f_2, \ldots, f_d$ be all the Galois conjugates of f over $\mathcal{F}_4^1(\mathbb{Q})$, then we derive that

$$\begin{aligned} \det(T) &= \begin{vmatrix} \sum_{k=1}^d f_k^0 & \sum_{k=1}^d f_k^1 & \cdots & \sum_{k=1}^d f_k^{d-1} \\ \sum_{k=1}^d f_k^1 & \sum_{k=1}^d f_k^2 & \cdots & \sum_{k=1}^d f_k^d \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^d f_k^{d-1} & \sum_{k=1}^d f_k^d & \cdots & \sum_{k=1}^d f_k^{2d-2} \end{vmatrix} \\ &= \begin{vmatrix} f_1^0 & f_2^0 & \cdots & f_d^0 \\ f_1^1 & f_2^1 & \cdots & f_d^1 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{d-1} & f_2^{d-1} & \cdots & f_d^{d-1} \end{vmatrix} \begin{vmatrix} f_1^0 & f_1^1 & \cdots & f_1^{d-1} \\ f_2^0 & f_2^1 & \cdots & f_2^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_d^0 & f_d^1 & \cdots & f_d^{d-1} \end{vmatrix} \\ &= \prod_{1 \le m < n \le d} (f_m - f_n)^2 \quad \text{by the Vandermonde determinant formula.} \end{aligned}$$

On the other hand, since $f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)$ and $f_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau)$ belong to $\mathcal{F}_4^1(\mathbb{Q})$ by Lemma 4.1, each $(f_m - f_n)$ is of the form

$$\frac{f_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)} - \frac{f_{\left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)} = \frac{f_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}$$

for some $\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right], \left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right] \in \mathcal{V}_N$ such that $\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right] \not\equiv \pm \left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right] \pmod{\mathbb{Z}^2}$ by Lemma 3.3(i). Thus $\det(T)$ is a modular unit in $\mathcal{O}_4^1(\mathbb{Q})$ by Remark 3.4, from which it follows by (11) and (12) that $h(\tau) \in \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$. Therefore we establish the inclusion $\mathcal{O}_N^1(\mathbb{Q}) \subseteq \mathcal{O}_4^1(\mathbb{Q})[f_N^1(\tau)]$, as desired. □

Question 6.3. Whenever $N \not\equiv 0 \pmod 4$, determine whether the ring $\mathcal{O}_N^1(\mathbb{Q})$ is also generated by both Fricke and Siegel functions, or not.

Corollary 6.4. *Let $N \geq 8$ and $N \equiv 0 \pmod 4$. For each $\mathbf{v} = \left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right] \in \mathcal{V}_N$, let*

$$r_{\mathbf{v}}(\tau) = \left(\frac{g_{(N/2)\mathbf{v}}(\tau)}{g_{(N/4)\mathbf{v}}(\tau)}\right)^8 \quad \textit{and} \quad s_{\mathbf{v}}(\tau) = \frac{f_{\mathbf{v}}(\tau) - f_{(N/2)\mathbf{v}}(\tau)}{f_{(N/4)\mathbf{v}}(\tau) - f_{(N/2)\mathbf{v}}(\tau)}.$$

Then, a family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ is a Fricke family of level N if and only if there is a polynomial $P(x,y,z,w) \in \mathbb{Q}[x,y,z,w]$ for which

$$h_{\mathbf{v}}(\tau) = P(r_{\mathbf{v}}(\tau), r_{\mathbf{v}}(\tau)^{-1}, (r_{\mathbf{v}}(\tau)-16)^{-1}, s_{\mathbf{v}}(\tau)) \quad \textit{for all } \mathbf{v} \in \mathcal{V}_N.$$

Proof. For each $\mathbf{v} = \left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right] \in \mathcal{V}_N$, we take any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ and $\widetilde{\gamma} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\det(\gamma) \equiv 1 \pmod N$ and $\widetilde{\gamma} \equiv \pm\gamma \pmod N$. Note that ${}^t\widetilde{\gamma}\mathbf{u} \equiv \pm{}^t\gamma\mathbf{u} \pmod{\mathbb{Z}^2}$ for all $\mathbf{u} \in (1/N)\mathbb{Z}^2$. We then see by (A2) and Lemma 3.5 that

$$\begin{aligned} g_4^1(\tau)^\gamma = (g_4^1 \circ \widetilde{\gamma})(\tau) &= \left(\frac{g_{{}^t\widetilde{\gamma}\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{g_{{}^t\widetilde{\gamma}\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau)}\right)^8 = \left(\frac{g_{{}^t\gamma\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{g_{{}^t\gamma\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau)}\right)^8 \\ &= \left(\frac{g_{\left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix}\right]}(\tau)}{g_{\left[\begin{smallmatrix} a/4 \\ b/4 \end{smallmatrix}\right]}(\tau)}\right)^8 = r_{\mathbf{v}}(\tau). \end{aligned}$$

Furthermore, we get by Proposition 4.1 that

$$f_N^1(\tau)^\gamma = \frac{f_{{}^t\gamma\left[\begin{smallmatrix} 1/N \\ 0 \end{smallmatrix}\right]}(\tau) - f_{{}^t\gamma\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)}{f_{{}^t\gamma\left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix}\right]}(\tau) - f_{{}^t\gamma\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)} = \frac{f_{\left[\begin{smallmatrix} a/N \\ b/N \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix}\right]}(\tau)}{f_{\left[\begin{smallmatrix} a/4 \\ b/4 \end{smallmatrix}\right]}(\tau) - f_{\left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix}\right]}(\tau)} = s_{\mathbf{v}}(\tau).$$

Now, the corollary follows from Theorems 4.3 (with its proof) and 6.2. □

7. Weak Fricke families

Let $\mathbb{H}' = \mathbb{H} \setminus \{\gamma(\zeta_3), \gamma(\zeta_4) \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$. For a positive integer N, we let ${\mathcal{O}_N^1}'(\mathbb{Q})$ be the ring of functions in $\mathcal{F}_N^1(\mathbb{Q})$ which are holomorphic on $\mathbb{H}'$.

Lemma 7.1. *$j(\tau)$ gives to rise a bijection $j(\tau) : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ such that $j(\zeta_3) = 0$ and $j(\zeta_4) = 1728$.*

Proof. See [8, Theorem 4 in Chapter 3]. □

Theorem 7.2. *We have ${\mathcal{O}_1^1}'(\mathbb{Q}) = \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}]$.*

Proof. By Lemma 7.1, we get the inclusion $\mathcal{O}_1^{1'}(\mathbb{Q}) \supseteq \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}]$.

Now, let $h(\tau) \in \mathcal{O}_1^{1'}(\mathbb{Q})$. Since $\mathcal{F}_1^1(\mathbb{Q}) = \mathcal{F}_1 = \mathbb{Q}(j(\tau))$, we may write $h(\tau) = A(j(\tau))/B(j(\tau))$ for some polynomials $A(x), B(x) \in \mathbb{Q}[x]$ which are relatively prime. Suppose that $B(x)$ has a zero $c \in \overline{\mathbb{Q}}$ not equal to 0 or 1728. Since $j(\tau_0) = c$ for some $\tau_0 \in \mathbb{H}'$ by Lemma 7.1, we attain $B(j(\tau_0)) = 0$. But, since $A(x)$ is not divisible by $(x - c)$, we see that $A(j(\tau_0)) \neq 0$, which contradicts that $h(\tau)$ is holomorphic on $\mathbb{H}'$. Thus we conclude that 0 and 1728 are the only possible zeros of $B(x)$, which proves the converse inclusion $\mathcal{O}_1^{1'}(\mathbb{Q}) \subseteq \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}]$. □

Lemma 7.3. *Modular units of level* 1 *are exactly nonzero rational numbers.*

Proof. See [6, Lemma 2.1]. One can also justify by using Lemma 7.1. □

Theorem 7.4. *If $N \geq 2$, then we obtain*

$$\mathcal{O}_N^{1\,\prime}(\mathbb{Q}) = \mathcal{O}_1^{1'}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)] = \mathbb{Q}[j(\tau), j(\tau)^{-1}, (j(\tau) - 1728)^{-1}, f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)].$$

Proof. Since $f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)$ is weakly holomorphic, we get the inclusion $\mathcal{O}_N^{1\,\prime}(\mathbb{Q}) \supseteq \mathcal{O}_1^{1'}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$.

For the converse inclusion, let $h = h(\tau) \in \mathcal{O}_N^{1\,\prime}(\mathbb{Q})$. Since $\mathcal{F}_N^1(\mathbb{Q})$ is generated by $f = f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)$ over $\mathcal{F}_1 = \mathcal{F}_1^1(\mathbb{Q})$ by Proposition 6.1, we can write

$$h = c_0 + c_1 f + \cdots + c_{d-1} f^{d-1}, \tag{13}$$

where $d = [\mathcal{F}_N^1(\mathbb{Q}) : \mathcal{F}_1^1(\mathbb{Q})]$ and $c_0, c_1, \ldots, c_{d-1} \in \mathcal{F}_1^1(\mathbb{Q})$. If $f_1, f_2, \ldots, f_d$ are all the Galois conjugates of f over $\mathcal{F}_1^1(\mathbb{Q})$ and $D = \prod_{1 \leq m,n \leq d}(f_m - f_n)^2$, then one can show that

$$c_0, c_1, \ldots, c_{d-1} \in D^{-1}\mathcal{O}_1^{1'}(\mathbb{Q}) \tag{14}$$

as in the proof of Theorem 6.2. By Lemma 3.3, we see that each $(f_m - f_n)^6$ is of the form

$$(f_m - f_n)^6 = 2^{12}3^6 j(\tau)^2 (j(\tau) - 1728)^3 \frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12} g_{\mathbf{v}}(\tau)^{12}}$$

for some $\mathbf{u}, \mathbf{v} \in \mathcal{V}_N$ such that $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$. It then follows from Lemma 7.3 that

$$D = c j(\tau)^{d(d-1)/3} (j(\tau) - 1728)^{d(d-1)/2} \quad \text{for some nonzero } c \in \mathbb{C}.$$

Now that $D \in \mathcal{F}_1^1(\mathbb{Q}) = \mathbb{Q}(j(\tau))$, we must have $d(d-1)/3 \in \mathbb{Z}$ and $c \in \mathbb{Q}$. Hence we achieve by Theorem 7.2, (13) and (14) that $h(\tau) \in \mathcal{O}_1^{1'}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$. Therefore, the inclusion $\mathcal{O}_N^{1\,\prime}(\mathbb{Q}) \subseteq \mathcal{O}_1^{1'}(\mathbb{Q})[f_{\left[\begin{smallmatrix}1/N\\0\end{smallmatrix}\right]}(\tau)]$ also holds. □

Remark 7.5. For $N \geq 2$, let Fr'_N be the set of *weak* Fricke families of level N, namely, the families $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ satisfying (F1′), (F2) and (F3). It is also a ring under the operations stated in (5). In a similar way to the proof of Theorem 4.3, one can claim that Fr'_N is isomorphic to ${\mathcal{O}_N^1}'(\mathbb{Q})$. Therefore, we deduce by Theorem 7.4 that a family $\{h_{\mathbf{v}}(\tau)\}_{\mathbf{v}\in\mathcal{V}_N}$ of functions in $\mathcal{F}_N$ is a weak Fricke family of level N if and only if there is a polynomial $P(x,y,z,w) \in \mathbb{Q}[x,y,z,w]$ so that

$$h_{\mathbf{v}}(\tau) = P(j(\tau), j(\tau)^{-1}, (j(\tau)-1728)^{-1}, f_{\mathbf{v}}(\tau)) \quad \text{for all } \mathbf{v} \in \mathcal{V}_N.$$

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