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THE GRADIENT RECOVERY FOR FINITE VOLUME ELEMENT METHOD ON QUADRILATERAL MESHES

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Abstract. We consider the finite volume element method for elliptic problems using isoparametric bilinear elements on quadrilateral meshes. A gradient recovery method is presented by using the patch interpolation technique. Based on some superclose estimates, we prove that the recovered gradient $R(\nabla u_h)$ possesses the superconvergence: $\|\nabla u - R(\nabla u_h)\| = O(h^2)\|u\|_3$. Finally, some numerical examples are provided to illustrate our theoretical analysis.

1. Introduction

The derivative (gradient) recovery techniques are postprocess techniques that reconstruct the derivative from the discrete solution to achieve better derivative approximation, for example, to obtain the superconvergent result. In the 1960s, the derivative recovery technique had been used to compute the derivative (stress) via the C^0-finite elements [7]. In particular, the simple averaging or weighted averaging methods were employed by engineers for linear finite elements [16]. Subsequently, new derivative recovery techniques have been developed, for example, the L_2-projection post-processing technique [6, 14], the well known Zienkiewicz-Zhu's patch recovery technique (SPR) [22], the interpolation postprocess technique [10], the polynomial preserving recovery technique (PPR) [13] and the derivative patch interpolation recovery technique [19], and so on. However, all these recovery techniques were presented for the finite element methods on triangle or rectangular meshes.

Finite volume element (FVE) method is a discrete technique for solving partial differential equations. In general, it represents the conservation of an interest quantity, such as mass, momentum, or energy in fluid mechanics, so that it can be expected to simulate corresponding physical phenomena more

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effectively. Readers are referred to the monograph [8] for general presentation of the FVE method and to [1, 2, 3, 4, 9, 11, 12, 15, 17, 18, 20] and the references therein for details.

Compared with FVE methods on triangle and rectangular meshes, less works can be found for FVE methods on quadrilateral meshes. We know that finite element methods on quadrilateral meshes usually have better accuracy than that on triangle meshes, and quadrilateral meshes are more flexible than rectangular meshes in handing complicated domain geometries. So quadrilateral meshes are also used frequently in practical applications. For isoparametric bilinear FVE method solving elliptic problems on quadrilateral meshes, Schmidt, Li et al. [9, 15] first give the optimal H^1-error estimate; Then, Lv and Li [11] further obtain the optimal L_2-error estimate if the quadrilateral mesh is h^2-uniform; Recently, Lv and Li [12] also derive a superconvergence result in the average gradient form.

In this paper, we study the isoparametric bilinear FVE method to solve the following elliptic problem on quadrilateral meshes,

$$\begin{cases} -div(A\nabla u) + c\,u = f, \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a polygonal domain with boundary $\partial\Omega$, coefficient matrix $A = (a_{ij})_{2\times 2}$. Our main goal is to present a gradient recovery method for the FVE solution u_h on quadrilateral meshes by using the patch interpolation technique [19]. This recovery method can provide a better approximation to the gradient of the exact solution. In fact, based on some superclose estimates, we prove that the recovered gradient $Q(\nabla u_h)$ possesses the superconvergence:

$$\|\nabla u - Q(\nabla u_h)\| = O(h^2)\|u\|_3. \tag{1.2}$$

This paper is organized as follows. In Section 2, we introduce the FVE scheme and some related results. Section 3 is devoted to deriving some superclose estimates for the interpolation function. In Section 4, we present the gradient recovery method and give its superconvergence analysis. Finally, in Section 5, numerical experiments are provided to illustrate our theoretical analysis.

We shall use the standard notation for the Sobolev space $W^{m,p}(D)$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$. In order to simplify the notations, we set $W^{m,2}(D) = H^m(D)$, $\|\cdot\|_{m,2,D} = \|\cdot\|_{m,D}$, and when $D = \Omega$ we skip the index D. Furthermore, notations $(\ ,\ )$ and $\|\cdot\|$ denote the inner product and norm in space $L_2(\Omega)$, respectively. We use letter C to represent a generic positive constant, independent of the mesh size h.

2. Finite volume element method on quadrilateral meshes

2.1. Partition and isoparametric bilinear transformation

Let $T_h = \bigcup\{K\}$ be a convex quadrilateral mesh partition of domain Ω so that $\overline{\Omega} = \bigcup_{K \in T_h}\{\overline{K}\}$, where $h = \max h_K$, h_K is the diameter of element K. We assume that partition T_h is regular, that is, all the inner angles of any element in T_h are uniformly bounded away from 0 and π, and there exists a positive constant $\gamma > 0$ such that

$$h_K/\rho_K \le \gamma, \ \forall K \in T_h, \tag{2.1}$$

where ρ_K denotes the diameter of the biggest ball included in K.

The following *strong regular* and h^2-*uniform* mesh conditions will be used in our analysis.

Definition 2.1. A regular quadrilateral partition T_h is called strongly regular if for any element $K = \Box P_1P_2P_3P_4$ in T_h, it holds (see Fig.1)

$$|\overrightarrow{P_1P_2} + \overrightarrow{P_3P_4}| + |\overrightarrow{P_1P_4} + \overrightarrow{P_3P_2}| \le Ch_K^2. \tag{2.2}$$

Furthermore, a strongly regular quadrilateral partition T_h is called h^2-uniform, if for any two adjacent elements K and K', it holds (see Fig. 1)

$$|\overrightarrow{P_1P_2} + \overrightarrow{P_1P_6}| \le Ch_K^2. \tag{2.3}$$

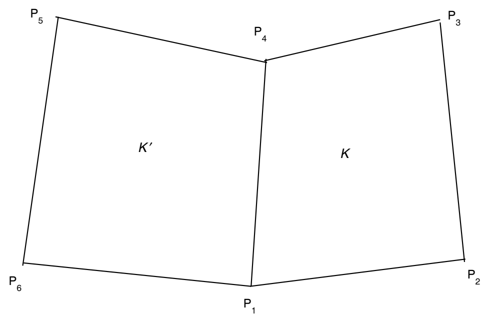

FIGURE 1. Two adjacent elements K and K'.

Let $l = |P_1P_4|$ be the length of the common edge of K and K' (see Fig. 1). Since

$$h_K = \frac{h_K}{\rho_K}\frac{\rho_K}{l}\frac{l}{h_{K'}}h_{K'} \le \gamma h_{K'},$$

then h_K in (2.3) may be replaced by $h_{K'}$.

It is well known that a FVE method usually concerns two mesh partitions: a basic partition T_h and its dual partition T_h^*. We here form the dual partition T_h^* in the following way. For each element $K \in T_h$, we connect the center of K to the midpoints of its edges by straight lines. Then, for each nodal point

P in T_h, there exists a polygonal region $K_P^* = \Box G_1G_2G_3G_4$ surrounding P, K_P^* is called the dual element at point P, and T_h^* is the union of all such dual elements, see Fig. 2.

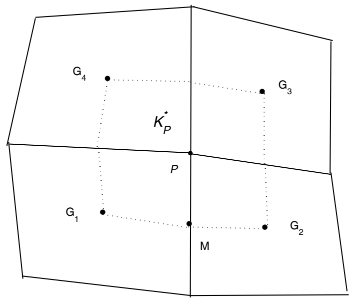

FIGURE 2. The dual element K_P^* surrounding point P.

Let $\widehat{K} = [0,1] \times [0,1]$ be the reference element in the $\hat{\mathbf{x}} = (\xi, \eta)$ plane. Then, for each convex quadrilateral $K = \Box P_1P_2P_3P_4$, there exists an invertible isoparametric bilinear mapping $F_K : (\xi, \eta) \in \widehat{K} \to \mathbf{x} = (x, y) \in K$ such that (see Fig. 3)

$$\mathbf{x} = P_1 + (P_2 - P_1)\xi + (P_4 - P_1)\eta + (P_1 - P_2 + P_3 - P_4)\xi\eta. \tag{2.4}$$

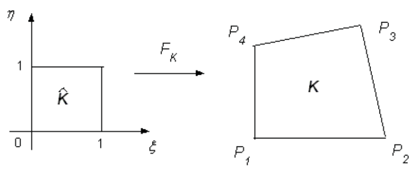

FIGURE 3. The isoparametric bilinear transformation.

Let $P_i = (x_i, y_i)$. Transformation (2.4) also can be expressed as follows.

$$x = x_1 + a_1\xi + a_2\eta + a_3\xi\eta, \tag{2.5}$$

$$y = y_1 + b_1\xi + b_2\eta + b_3\xi\eta, \tag{2.6}$$

where

$$a_1 = x_2 - x_1,\ a_2 = x_4 - x_1,\ a_3 = x_1 - x_2 + x_3 - x_4,$$
$$b_1 = y_2 - y_1,\ b_2 = y_4 - y_1,\ b_3 = y_1 - y_2 + y_3 - y_4.$$

Denote the Jacobi matrix of mapping F_K by $\mathcal{J}_K$ and the determinant of $\mathcal{J}_K$ by J_K, then we have

$$\mathcal{J}_K = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} a_1 + a_3\eta & a_2 + a_3\xi \\ b_1 + b_3\eta & b_2 + b_3\xi \end{pmatrix}.$$

Furthermore, by the differentiation law of inverse function we have

$$\mathcal{J}_K^{-1} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \frac{1}{J_K} \begin{pmatrix} b_2 + b_3\xi & -(a_2 + a_3)\xi \\ -(b_1 + b_3\eta) & a_1 + a_3\eta \end{pmatrix}. \tag{2.7}$$

Under mesh conditions (2.2) and (2.3), by straightforward computation, one can derive the following results, see [9, 11].

Lemma 2.1. *Assume that partition T_h is strongly regular, then*

$$|J_K|_{0,\infty} = O(h_K^2),\ |\mathcal{J}_K|_{0,\infty} = O(h_K),\ |\mathcal{J}_K^{-1}|_{0,\infty} = O(h_K^{-1}), \tag{2.8}$$

$$|\mathcal{J}_K^{-1}|_{m,\infty} \le C h_K^{-1},\ m = 1, 2, 3, \tag{2.9}$$

$$|\mathcal{J}_K^{-1}(\xi,\eta) - \mathcal{J}_K^{-1}(\xi',\eta')|_{0,\infty} \le C,\ \forall\, (\xi,\eta), (\xi',\eta') \in \widehat{K}. \tag{2.10}$$

Furthermore, if partition T_h is h^2-uniform, then for any two adjacent elements K and K',

$$|\mathcal{J}_K^{-1}(\xi,\eta) - \mathcal{J}_{K'}^{-1}(\xi',\eta')|_{0,\infty} \le C,\ \forall\, (\xi,\eta) \in \widehat{K}, (\xi',\eta') \in \widehat{K}. \tag{2.11}$$

Let function $\hat{u}(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)) = u \circ F_K(\xi,\eta)$, where $u \circ F_K$ denotes the compound function of $u(x,y)$ and the mapping $F_K(\xi,\eta)$. From Lemma 2.1 and noting that

$$\widehat{\nabla}\hat{u} = \mathcal{J}_K^T \nabla u,\quad \nabla u = \mathcal{J}_K^{-T} \widehat{\nabla}\hat{u},$$

we can derive the following estimates

$$|\hat{u}|_{m,p,\widehat{K}} \le C h_K^{m-\frac{2}{p}} \|u\|_{m,p,K},\ m = 0, 1, 2, 3,\ 1 \le p \le \infty, \tag{2.12}$$

$$|u|_{m,p,K} \le C h_K^{-m+\frac{2}{p}} \|\hat{u}\|_{m,p,\widehat{K}},\ m = 0, 1, 2, 3,\ 1 \le p \le \infty. \tag{2.13}$$

2.2. Finite volume element scheme

Consider problem (1.1). As usual, we assume that there exist positive constants C_1 and C_2 such that

$$C_1 \mathbf{z}^T \mathbf{z} \le \mathbf{z}^T A(x,y) \mathbf{z} \le C_2 \mathbf{z}^T \mathbf{z},\ \forall\, \mathbf{z} \in R^2,\ (x,y) \in \Omega, \tag{2.14}$$

We further assume that $A \in [W^{1,\infty}(\Omega)]^{2\times 2}$, $c \in L_\infty(\Omega)$ and $c \ge 0$.

Associated with partition T_h and T_h^*, we introduce the trial function space U_h and test function space V_h, respectively,

$$U_h = \{\, u_h \in C^0(\overline{\Omega}) :\ u_h|_K = P_{\widehat{K}} \circ F_K^{-1},\ P_{\widehat{K}} \in Q_{11}(\widehat{K}),\ \forall\, K \in T_h,\ u_h|_{\partial\Omega} = 0 \,\},$$

$$V_h = \{\, v_h \in L_2(\Omega) :\ v_h|_{K_P^*} = \text{constant},\ \forall\, P \in N_h,\ v_h|_{K_P^*} = 0,\ \forall\, P \in \partial\Omega \},$$

where $Q_{11}(\widehat{K})$ is the set of all bilinear polynomials on $\widehat{K}$ and N_h is the set of all mesh points of T_h.

Let u be the solution of problem (1.1). Using the Green formula, we obtain

$$-\int_{\partial K_P^*} n\cdot(A\nabla u)vds+\int_{K_P^*} cuv=\int_{K_P^*} fv,\quad K_P^*\in T_h^*,\ v\in V_h, \tag{2.15}$$

where n is the outward unit normal vector on the boundary concerned. According to weak formula (2.15), we set the bilinear form $a_h(u,v)$ by

$$a_h(u,v)=\sum_{K_P^*\in T_h^*}\Big(-\int_{\partial K_P^*} n\cdot(A\nabla u)vds+\int_{K_P^*} cuv\Big),\ u\in H^2(\Omega),\ v\in V_h, \tag{2.16}$$

and define the FVE approximation to problem (1.1): Find $u_h\in U_h$ such that

$$a_h(u_h,v_h)=(f,v_h),\ \forall\, v_h\in V_h. \tag{2.17}$$

Let $\Pi_h^*: U_h\to V_h$ be the interpolation operator defined by

$$\Pi_h^* v_h=\sum_{P\in N_h} v_h(P)\chi_{_P},\ \forall\, v_h\in U_h,$$

where $\chi_{_P}$ is the characteristic function of the dual element K_P^*. Obviously, Π_h^* is a one to one mapping from U_h onto V_h. Then, we obtain the equivalent scheme of (2.17): Find $u_h\in U_h$ such that

$$a_h(u_h,\Pi_h^* v_h)=(f,\Pi_h^* v_h),\ \forall\, v_h\in U_h, \tag{2.18}$$

which is the FVE scheme actually used in our argument. From (2.15) and (2.18), we can derive the error equation

$$a_h(u-u_h,\Pi_h^* v_h)=0,\ \forall\, v_h\in U_h. \tag{2.19}$$

Let $\Pi_h u\in U_h$ be the usual isoparametric bilinear interpolation of continuous function u. In our analysis, the following approximation property and trace inequality will be used frequently, see [5]. For $1<p\le\infty$, we have

$$\|u-\Pi_h u\|_{m,p,K}\le Ch_K^{2-m}\|u\|_{2,p,K},\ m=0,1,2, \tag{2.20}$$

$$\|u\|_{0,p,\partial K}\le Ch_K^{-\frac{1}{p}}\big(\|u\|_{0,p,K}+h_K\|\nabla u\|_{0,p,K}\big),\ u\in W^{1,p}(K). \tag{2.21}$$

Furthermore, the following two lemmas hold.

Lemma 2.2 ([8])**.** *Let $\widehat{\Pi}_h^*\hat v_h=\widehat{\Pi_h^* v}_h$. Then for $v_h\in U_h$, we have*

$$\int_{\widehat K}(\hat v_h-\widehat{\Pi}_h^*\hat v_h)d\widehat K=0,\quad \int_{\widehat\tau}(\hat v_h-\widehat{\Pi}_h^*\hat v_h)ds=0, \tag{2.22}$$

$$\|v_h-\Pi_h^* v_h\|_{0,q,K}\le Ch_K\|v_h\|_{1,q,K},\ 1\le q\le\infty, \tag{2.23}$$

where $\widehat\tau\subset\partial\widehat K$ be any one edge of the reference element $\widehat K$.

Lemma 2.3 ([9, 15])**.** *Let partition T_h be strongly regular, u and u_h be the solutions of problems* (1.1) *and* (2.18)*, respectively. Then, we have*

$$a_h(v_h,\Pi_h^* v_h)\ge C\|v_h\|_1^2,\ \forall\, v_h\in U_h,$$
$$\|u-u_h\|_1\le Ch\|u\|_2.$$

3. Superclose estimate for the interpolation approximation

The superclose estimate of interpolation function usually provides a useful analysis tool in the study of superconvergence of finite element method [10, 21]. In this section, we establish some superclose estimates for the finite volume element method.

Let w^c be the piecewise constant approximation of function w on T_h,

$$w^c|_K = \frac{1}{|K|}\int_K w,\ K \in T_h; \|w - w^c\|_{0,p,K} \le Ch_K|w|_{1,p,K},\ 1 \le p \le \infty. \tag{3.1}$$

Lemma 3.1. *Let partition T_h be strongly regular and $\mathbf{a}^c$ be a constant vector, $u \in W^{3,p}(\Omega)$. Then, for $v \in U_h$, we have*

$$\begin{aligned} &|(\mathbf{a}^c \cdot \nabla(u - \Pi_h u)_x, \Pi_h^* v - v)_K| + |(\mathbf{a}^c \cdot \nabla(u - \Pi_h u)_y, \Pi_h^* v - v)_K| \\ &\le Ch_K^2\|u\|_{3,p,K}\|v_h\|_{1,q,K},\ 2 \le p, q \le \infty,\ 1/p + 1/q = 1. \end{aligned} \tag{3.2}$$

Proof. Let $w = u - \Pi_h u$ and $e_h = \Pi_h^* v - v$. Using the isoparametric bilinear transformation, we have

$$\begin{aligned} &(\mathbf{a}^c \cdot \nabla(u - \Pi_h u)_x, \Pi_h^* v - v)_K = \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}\widehat{\nabla}(\widehat{w}_\xi \xi_x + \widehat{w}_\eta \eta_x)\hat{e}_h J_K d\widehat{K} \\ &= \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(\widehat{\nabla}\widehat{w}_\xi \xi_x + \widehat{\nabla}\widehat{w}_\eta \eta_x)\hat{e}_h J_K \\ &\quad + \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(\widehat{w}_\xi \widehat{\nabla}\xi_x + \widehat{w}_\eta \widehat{\nabla}\eta_x)\hat{e}_h J_K \\ &= E_1 + E_2. \end{aligned} \tag{3.3}$$

From (2.7) we obtain

$$\begin{aligned} E_1 &= \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(\widehat{\nabla}\widehat{w}_\xi(b_2 + b_3\xi) - \widehat{\nabla}\widehat{w}_\eta(b_1 + b_3\eta))\hat{e}_h \\ &= \int_{\widehat{K}} \mathbf{a}^c \cdot (\mathcal{J}_K^{-T} - \mathcal{J}_K^{-T}(0,0))(\widehat{\nabla}\widehat{w}_\xi(b_2 + b_3\xi) - \widehat{\nabla}\widehat{w}_\eta(b_1 + b_3\eta))\hat{e}_h \\ &\quad + \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(0,0)(\widehat{\nabla}\widehat{w}_\xi b_2 - \widehat{\nabla}\widehat{w}_\eta b_1)\hat{e}_h \\ &\quad + \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(0,0)(\widehat{\nabla}\widehat{w}_\xi b_3\xi - \widehat{\nabla}\widehat{w}_\eta b_3\eta)\hat{e}_h = F_1 + F_2 + F_3. \end{aligned} \tag{3.4}$$

It follows from Lemma 2.1 and condition (2.2) that

$$|\mathcal{J}_K^{-T}|_\infty \le Ch_K^{-1},\ |\mathcal{J}_K^{-T} - \mathcal{J}_K^{-T}(0,0)|_\infty \le C,\ b_1 = b_2 = O(h_K),\ b_3 = O(h_K^2),$$

therefore, we have from (2.12), (2.20) and (2.23) that

$$\begin{aligned} F_1 + F_3 &\le Ch_K|\widehat{w}|_{2,p,\widehat{K}}\|\hat{e}_h\|_{0,q,\widehat{K}} \le Ch_K h_K^{2-\frac{2}{p}}\|w\|_{2,p,K}h_K^{-\frac{2}{q}}\|e_h\|_{0,q,K} \\ &\le Ch_K^2\|u\|_{2,p,K}\|v\|_{1,q,K}. \end{aligned}$$

For F_2, noting that $\widehat{\nabla}(\widehat{\Pi}_h\hat{u})_\xi$ and $\widehat{\nabla}(\widehat{\Pi_h}\hat{u})_\eta$ are constants, then from (2.22), (3.1) and (2.12), we have

$$\begin{aligned}F_2 &= \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(0,0)(\widehat{\nabla}\hat{u}_\xi b_2 - \widehat{\nabla}\hat{u}_\eta b_1)\hat{e}_h \\ &= \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(0,0)(\widehat{\nabla}\hat{u}_\xi b_2 - (\widehat{\nabla}\hat{u}_\xi)^c b_2 - \widehat{\nabla}\hat{u}_\eta b_1 + (\widehat{\nabla}\hat{u}_\eta)^c b_1)\hat{e}_h \\ &\le C|\mathcal{J}_K^{-T}|_\infty(|b_1| + b_2|)|\hat{u}|_{3,p,\widehat{K}}\|\hat{e}_h\|_{0,q,\widehat{K}} \\ &\le Ch_K^{3-\frac{2}{p}}\|u\|_{3,p,K}h_K^{-\frac{2}{q}}\|e_h\|_{0,q,K} \le Ch_K^2\|u\|_{3,p,K}\|v\|_{1,q,K}.\end{aligned}$$

Combining estimates $F_1 \sim F_3$, we obtain from (3.4) that

$$E_1 \le Ch_K^2\|u\|_{3,p,K}\|v\|_{1,q,K}.$$

Next, we estimate E_2. Since $|\mathcal{J}_K^{-T}| = O(h_K^{-1})$, $|J_K| = O(h_K^2)$ and (see (2.9))

$$|\widehat{\nabla}\xi_x| \le |\xi_{xx}x_\xi + \xi_{xy}y_\xi| + |\xi_{xx}x_\eta + \xi_{xy}y_\eta| \le Ch_K(|\xi_{xx}| + |\xi_{xy}|) \le C,$$

then, we have from (2.12), (2.20) and (2.23) that

$$\begin{aligned}E_2 &= \int_{\widehat{K}} \mathbf{a}^c \cdot \mathcal{J}_K^{-T}(\widehat{w}_\xi\widehat{\nabla}\xi_x + \widehat{w}_\eta\widehat{\nabla}\eta_x)\hat{e}_h J_K \le Ch_K|\widehat{w}|_{1,p,\widehat{K}}\|\hat{e}_h\|_{0,q,\widehat{K}} \\ &\le Ch_K h_K^{1-\frac{2}{p}}\|u - \Pi_h u\|_{1,p,K}h_K^{-\frac{2}{q}}\|e_h\|_{0,q,K} \le Ch_K^2\|u\|_{2,p,K}\|v\|_{1,q,K}.\end{aligned}$$

The proof is completed by substituting estimates E_1 and E_2 into (3.4). □

Let $\mathcal{E}_h^0$ be the union of all interior edges of elements in T_h.

Lemma 3.2. *Let partition T_h be h^2-uniform and matrix $A_M|_\tau$ be constant on each $\tau \in \mathcal{E}_h^0$, $u \in W^{3,p}(\Omega)$. Then, we have for $2 \le p, q \le \infty$, $1/p + 1/q = 1$,*

$$\Big|\sum_{K\in T_h}\int_{\partial K} n\cdot A_M\nabla(u - \Pi_h u)(\Pi_h^* v - v)ds\Big| \le Ch^2\|u\|_{3,p}\|v\|_{1,q},\ v \in U_h.$$

Proof. Let $w = u - \Pi_h u$ and $e_h = \Pi_h^* v - v$. We need to estimate

$$\begin{aligned}&\sum_{K\in T_h}\int_{\partial K} n\cdot A_M\nabla w e_h ds \\ &= \sum_{K\in T_h}\sum_{\tau\in\partial K\backslash\partial\Omega}\int_\tau n\cdot A_M\nabla w e_h ds = \sum_{K\in T_h}\sum_{\tau\in\partial K\backslash\partial\Omega}F(\tau) \\ &= \sum_{\tau\in\mathcal{E}_h^0}\big(F(\tau\cap\partial K) + F(\tau\cap\partial K')\big),\end{aligned} \tag{3.5}$$

where K and K' be two adjacent elements with the common edge τ.

Let quadrilateral element $K = \Box P_1P_2P_3P_4$, $\tau \in \partial K$ be an edge of K, for example, $\tau = P_1P_4$, see Fig. 1. On edge τ ($\hat{\tau} = \{\,\xi = 0,\ 0 \le \eta \le 1\}$), we have from (2.5)-(2.6) that

$$ds = \sqrt{(dx)^2 + (dy)^2}\,|_{\xi=0} = \sqrt{x_\eta^2 + y_\eta^2}\,d\eta|_{\xi=0} = |P_1P_4|d\eta.$$

Therefore, we can write

$$F(\tau) = \int_\tau n \cdot A_M \nabla w e_h ds = \int_0^1 |P_1P_4|\hat{n} \cdot A_M \mathcal{J}_K^{-T}(\hat{\tau}) \widehat{\nabla} \widehat{w} \hat{e}_h d\eta. \tag{3.6}$$

Let $K' = \Box P_6P_1P_4P_5$ be an adjacent element of K with the common edge $\tau = \partial K \cap \partial K' = P_1P_4$ and $\tau' = \tau \cap \partial K'$ (see Fig. 1). Since the outward normal vector $n'|_{\tau'} = -n|_\tau$, then we have

$$\begin{aligned} F(\tau \cap \partial K) + F(\tau \cap \partial K') = & \int_0^1 |P_1P_4|\hat{n} \cdot A_M \mathcal{J}_K^{-T}(\hat{\tau}) \widehat{\nabla} \widehat{w}(\hat{\tau}) \hat{e}_h d\eta \\ & - \int_0^1 |P_1P_4|\hat{n} \cdot A_M \mathcal{J}_{K'}^{-T}(\hat{\tau}') \widehat{\nabla} \widehat{w}(\hat{\tau}') \hat{e}_h d\eta. \end{aligned}$$

Set $\mathbf{a}_h = |P_1P_4|\hat{n} \cdot A_M = O(h_K)$. Noting that e_h is continuous across edge $\tau = \tau'$ (excepting at the midpoint of τ), we obtain

$$\begin{aligned} & F(\tau \cap \partial K) + F(\tau \cap \partial K') \\ = & \int_0^1 \mathbf{a}_h \cdot (\mathcal{J}_K^{-T}(\hat{\tau}) - \mathcal{J}_K^{-1}(0,0)) \widehat{\nabla} \widehat{w}(\hat{\tau}) \hat{e}_h d\eta \\ & + \int_0^1 \mathbf{a}_h \cdot \mathcal{J}_K^{-T}(0,0)(\widehat{\nabla} \widehat{w}(\hat{\tau}) - \widehat{\nabla} \widehat{w}(\hat{\tau}')) \hat{e}_h d\eta \\ & + \int_0^1 \mathbf{a}_h \cdot (\mathcal{J}_K^{-T}(0,0) - \mathcal{J}_{K'}^{-T}(\hat{\tau}')) \widehat{\nabla} \widehat{w}(\hat{\tau}')) \hat{e}_h d\eta \\ = & S_1 + S_2 + S_3. \end{aligned} \tag{3.7}$$

Using Lemma 2.1, trace inequality (2.21) and the finite element inverse inequality, we obtain

$$\begin{aligned} S_1 + S_3 \le & Ch_K(\|\widehat{\nabla}\widehat{w}\|_{0,p,\widehat{K}} + |\widehat{\nabla}\widehat{w}|_{1,p,\widehat{K}})\|\hat{e}_h\|_{0,q,\widehat{K}} \\ & + Ch_{K'}(\|\widehat{\nabla}\widehat{w}\|_{0,p,\widehat{K}'} + |\widehat{\nabla}\widehat{w}|_{1,p,\widehat{K}'})\|\hat{e}_h\|_{0,q,\widehat{K}'} \\ \le & Ch_K(h_K^{1-\frac{2}{p}}\|w\|_{1,p,K} + h_K^{2-\frac{2}{p}}\|w\|_{2,p,K})h_K^{-\frac{2}{q}}\|e_h\|_{0,q,K} \\ & + Ch_{K'}(h_{K'}^{1-\frac{2}{p}}\|w\|_{1,p,K'} + h_{K'}^{2-\frac{2}{p}}\|w\|_{2,p,K'})h_{K'}^{-\frac{2}{q}}\|e_h\|_{0,q,K'} \\ \le & Ch^2\|u\|_{2,p,K\cup K'}\|v\|_{1,q,K\cup K'}. \end{aligned}$$

Next, we estimate S_2. Let $\Pi_h|_K = \Pi_K$, $\Pi_h|_{K'} = \Pi_{K'}$. Since ∇u is continuous across edge $\tau = \tau'$, we have

$$\widehat{\nabla}\widehat{w}(\hat{\tau}) - \widehat{\nabla}\widehat{w}(\hat{\tau}') = \widehat{\nabla}\widehat{\Pi}_{\widehat{K}}\hat{u}(\hat{\tau}) - \widehat{\nabla}\widehat{\Pi}_{\widehat{K}'}\hat{u}(\hat{\tau}'). \tag{3.8}$$

Noting that the bilinear interpolation $\widehat{\Pi}_{\widehat{K}}\hat{u}$ can be written as

$$\widehat{\Pi}_{\widehat{K}}\hat{u} = u_1 + (u_2 - u_1)\xi + (u_4 - u_1)\eta + (u_1 - u_2 + u_3 - u_4)\xi\eta,\ (\xi,\eta) \in \widehat{K},$$

where $u_i = u(P_i)$, and

$$(u_1 - u_2 + u_3 - u_4) = -\int_0^1 \hat{u}_\xi(\xi,0)d\xi + \int_0^1 \hat{u}_\xi(\xi,1)d\xi$$
$$= \int_0^1\int_0^1 \hat{u}_{\xi\eta}d\xi d\eta = \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \hat{u}_{\xi\eta}d\widehat{K} = (\hat{u}_{\xi\eta})^c(\widehat{K}),$$

then, on edge $\hat{\tau} = \hat{\tau}' = \{\xi = 0,\ 0 \le \eta \le 1\}$, we have

$$\begin{aligned}&\widehat{\nabla}\widehat{\Pi}_{\widehat{K}}\hat{u}(\hat{\tau}) - \widehat{\nabla}\widehat{\Pi}_{\widehat{K}'}\hat{u}(\hat{\tau}')\\ &= (u_2 - u_1 + (\hat{u}_{\xi\eta})^c(\widehat{K})\eta, u_4 - u_1)^T - (u_1 - u_6 + (\hat{u}_{\xi\eta})^c(\widehat{K}')\eta, u_5 - u_6)^T\\ &= (u_2 - 2u_1 + u_6, u_4 - u_1 - u_5 + u_6)^T + ((\hat{u}_{\xi\eta})^c(\widehat{K}) - (\hat{u}_{\xi\eta})^c(\widehat{K}'))\eta\boldsymbol{\varepsilon},\end{aligned}$$

where vector $\boldsymbol{\varepsilon} = (1,0)^T$. Now, from (3.8) and (2.22) we obtain

$$\begin{aligned}S_2 &= \int_0^1 \mathbf{a}_h \cdot \mathcal{J}_K^{-T}(0,0)((\hat{u}_{\xi\eta})^c(\widehat{K}) - (\hat{u}_{\xi\eta})^c(\widehat{K}'))\eta\boldsymbol{\varepsilon}\hat{e}_h d\eta\\ &= \int_0^1 \mathbf{a}_h \cdot \mathcal{J}_K^{-T}(0,0)((\hat{u}_{\xi\eta})^c(\widehat{K}) - \hat{u}_{\xi\eta} + \hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^c(\widehat{K}'))\eta\boldsymbol{\varepsilon}\hat{e}_h d\eta.\end{aligned}$$

It follows from Lemma 2.1, trace inequality (2.21) and the finite element inverse inequality that

$$\begin{aligned}S_2 &\le C(\|\hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^c\|_{0,p,\widehat{K}} + |\hat{u}_{\xi\eta}|_{1,p,\widehat{K}})\|\hat{e}_h\|_{0,q,\widehat{K}}\\ &\quad + C(\|\hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^c\|_{0,p,\widehat{K}'} + |\hat{u}_{\xi\eta}|_{1,p,\widehat{K}'})\|\hat{e}_h\|_{0,q,\widehat{K}'}\\ &\le C|\hat{u}|_{3,p,\widehat{K}}\|\hat{e}_h\|_{0,q,\widehat{K}} + C|\hat{u}|_{3,p,\widehat{K}'}\|\hat{e}_h\|_{0,q,\widehat{K}'}\\ &\le Ch_K^{3-\frac{2}{p}}\|u\|_{3,p,K}h_K^{-\frac{2}{q}}\|e_h\|_{0,q,K} + Ch_{K'}^{3-\frac{2}{p}}\|u\|_{3,p,K'}h_{K'}^{-\frac{2}{q}}\|e_h\|_{0,q,K'}\\ &\le Ch^2\|u\|_{3,p,K\cup K'}\|v\|_{1,q,K\cup K'}.\end{aligned}$$

Substituting estimates $S_1 \sim S_3$ into (3.7), it yields

$$F(\tau\cap\partial K) + F(\tau\cap\partial K') \le Ch^2\|u\|_{3,p,K\cup K'}\|v\|_{1,q,K\cup K'}. \tag{3.9}$$

The proof is completed by combining (3.9) with (3.5). □

We known that the conventional bilinear form of finite element method for problem (1.1) reads as

$$a(u,v) = \int_\Omega A\nabla u\cdot\nabla v + cuv. \tag{3.10}$$

Under strongly regular mesh condition, the following interpolation weak estimate has been established [21] for $2 \le p \le \infty$, $1/p + 1/q = 1$,

$$|a(u - \Pi_h u, v)| \le Ch^2\|u\|_{3,p}\|v\|_{1,q},\ \forall\, v \in U_h. \tag{3.11}$$

Below we will prove that estimate (3.11) also holds for the bilinear form of the FVE method $a_h(u, \Pi_h^* v)$. In order to use the known result (3.11) in our argument, we need to give the difference between bilinear forms $a(u,v)$ and $a_h(u,\Pi_h^* v)$.

Let $U_h + H^2(\Omega) = \{w : w = u_h + u, u_h \in U_h, u \in H^2(\Omega)\}$ be the algebraic sum space.

Lemma 3.3. *It holds for $w \in U_h + H^2(\Omega)$ and $v \in U_h$ that*

$$a_h(w,\Pi_h^* v) - a(w,v) = \sum_{K\in T_h} \int_{\partial K} n \cdot A\nabla w(\Pi_h^* v - v)ds + \sum_{K\in T_h} (-div(A\nabla w) + cw, \Pi_h^* v - v)_K. \tag{3.12}$$

Proof. Using the integration by parts, we obtain

$$\int_K A\nabla w \cdot \nabla v = -\int_K div(A\nabla w)v + \int_{\partial K} n \cdot (A\nabla w) v ds,$$

and (see Fig. 2)

$$\sum_{K\in T_h} \int_K div(A\nabla w)\Pi_h^* v = \sum_{K\in T_h} \sum_{K_P^*\in T_h^*} \int_{K_P^*\cap K} div(A\nabla w)\Pi_h^* v$$
$$= \sum_{K\in T_h} \int_{\partial K} n \cdot (A\nabla w)\Pi_h^* v ds + \sum_{K_P^*\in T_h^*} \int_{\partial K_P^*} n \cdot (A\nabla w)\Pi_h^* v ds.$$

Then, from the definitions of $a(w,v)$ and $a_h(w,\Pi_h^* v)$ (see (2.16) and (3.10)), the desired estimate is derived. □

Theorem 3.1. *Let partition T_h be h^2-uniform, $u \in W^{3,p}(\Omega)$. Then, we have for $2 \le p, q \le \infty$, $1/p + 1/q = 1$ that*

$$|a_h(u - \Pi_h u, \Pi_h^* v)| \le Ch^2 \|u\|_{3,p} \|v\|_{1,q}, \ \forall\, v \in U_h\,. \tag{3.13}$$

Proof. Denote by A_M the value of matrix A at the centroid of edge $\tau \subset \partial K$. Then, it follows from Lemma 3.3 that

$$\begin{aligned} & a_h(u - \Pi_h u, \Pi_h^* v) - a(u - \Pi_h u, v) \\ = & \sum_{K\in T_h} \int_{\partial K} n \cdot (A - A_M)\nabla(u - \Pi_h u)(\Pi_h^* v - v)ds \\ & + \sum_{K\in T_h} \int_{\partial K} n \cdot (A_M \nabla(u - \Pi_h u))(\Pi_h^* v - v)ds \\ & + \sum_{K\in T_h} (-div(A\nabla(u - \Pi_h u)), \Pi_h^* v - v)_K \\ & + \sum_{K\in T_h} (c(u - \Pi_h u), \Pi_h^* v - v)_K \\ = & R_1 + R_2 + R_3 + R_4. \end{aligned} \tag{3.14}$$

Using trace inequality (2.21) and the approximation properties, we have

$$\begin{aligned} R_1 &\le C \sum_{K\in T_h} h_K |A|_{1,\infty} \|\nabla(u-\Pi_h u)\|_{0,p,\partial K} \|v-\Pi_h^* v\|_{0,q,\partial K} \\ &\le Ch^2\|u\|_{2,p}\|v\|_{1,q}, \\ R_4 &\le C \sum_{K\in T_h} \|u-\Pi_h u\|_{1,p,K}\|v-\Pi_h^* v\|_{0,q,K} \le Ch^2\|u\|_{2,p}\|v\|_{1,q}. \end{aligned}$$

Next, it follows from Lemma 3.2 that

$$R_2 \le Ch^2\|u\|_{3,p}\|v\|_{1,q}.$$

Now, we need to estimate R_3. Set $A = (\mathbf{a}_1, \mathbf{a}_2)$. Since

$$div(A\nabla w) = (div\,\mathbf{a}_1, div\,\mathbf{a}_2)\cdot\nabla w + \mathbf{a}_1\cdot\nabla w_x + \mathbf{a}_2\cdot\nabla w_y, \tag{3.15}$$

then we have

$$\begin{aligned} R_3 = &\sum_{K\in T_h} (-(div\,\mathbf{a}_1, div\,\mathbf{a}_2)\cdot\nabla(u-\Pi_h u), \Pi_h^* v - v)_K \\ &- \sum_{K\in T_h} ((\mathbf{a}_1-\mathbf{a}_1^c)\cdot\nabla(u-\Pi_h u)_x + (\mathbf{a}_2-\mathbf{a}_2^c)\cdot\nabla(u-\Pi_h u)_y, \Pi_h^* v - v)_K \\ &- \sum_{K\in T_h} (\mathbf{a}_1^c\cdot\nabla(u-\Pi_h u)_x + \mathbf{a}_2^c\cdot\nabla(u-\Pi_h u)_y, \Pi_h^* v - v)_K. \end{aligned}$$

Using Lemma 3.1 and the approximation properties, it yields

$$R_3 \le Ch^2\|u\|_{3,p}\|v\|_{1,q}.$$

Substituting estimates $R_1 \sim R_4$ into (3.14), the proof is completed. □

From Theorem 3.1, we immediately obtain the following superclose result.

$$\|\Pi_h u - u_h\|_1 \le Ch^2\|u\|_3. \tag{3.16}$$

In fact, from Lemma 2.3, error equation (2.19) and the interpolation weak estimate (3.13), we obtain

$$\begin{aligned} C\|u_h-\Pi_h u\|_1^2 &\le a_h(u_h-\Pi_h u, \Pi_h^*(u_h-\Pi_h u)) \\ &= a_h(u-\Pi_h u, \Pi_h^*(u_h-\Pi_h u)) \le Ch^2\|u\|_3\|u_h-\Pi_h u\|_1. \end{aligned}$$

This gives estimate (3.16)

Remark 3.1. We note that Lv and Li in [12] derived estimate (3.13) for $p = q = 2$ by a lengthy and complex argument. For this estimate, we here give an alternative and more accessible argument, in particular, our result holds true for $2 \le p, q \le \infty$. Such result will be useful in the study of $W^{1,p}$-superconvergence, see [10, 21].

4. Gradient recovery formula and superconvergence

In this section, we will construct the gradient recovery formula for the isoparametric bilinear element and make the superconvergence estimate for the recovered gradient.

Let P be an interior mesh point and elements K_1, K_2, K_3, K_4 take point P as a vertex. Denote by $S_p = \bigcup_{i=1}^{4} K_i$ the patch recovery domain at point P, see Fig. 4. Let $F_{K_i} : \widehat{K}_i \to K_i$ be the isoparametric bilinear mapping and $\widehat{S}_p = \bigcup_{i=1}^{4} \widehat{K}_i$ be the reference patch recovery domain, see Fig. 4.

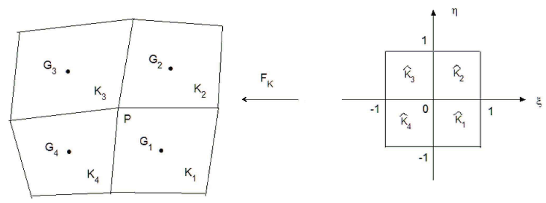

FIGURE 4. The patch recovery domain S_p around point P.

We first give the gradient recovery formula on the reference domain $\widehat{S}_p$.

Let $\widehat{G}_i$ be the centroid of $\widehat{K}_i$ ($G_i = F_{K_i}(\widehat{G}_i)$ is the centroid of element K_i) and further let $\widehat{\varphi}_i \in Q_{11}(\widehat{S}_P)$ be the basis function corresponding to $\widehat{G}_i$ such that $\widehat{\varphi}_i(\widehat{G}_j) = \delta_{ij}$. Then, functions $\{\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4\}$ form the base of bilinear polynomial space $Q_{11}(\widehat{S}_P)$. Denote the piecewise smooth function space by

$$W_h(\widehat{S}_P) = \{\, \widehat{w} : \widehat{w}|_{\widehat{K}_i} = \text{polynomial},\ \widehat{K}_i \in \widehat{S}_p \,\}.$$

Now, let derivative operator $\widehat{D} = \widehat{D}_\xi$ or $\widehat{D}_\eta$. For $\widehat{w} \in W_h(\widehat{S}_P)$, we define the derivative recovery operator $\widehat{Q} :\ \widehat{D}\widehat{w} \to \widehat{Q}(\widehat{D}\widehat{w}) \in Q_{11}(\widehat{S}_P)$ such that

$$\widehat{Q}(\widehat{D}\widehat{w}) = \sum_{i=1}^{4} \widehat{D}\widehat{w}(\widehat{G}_i)\widehat{\varphi}_i(\xi,\eta),\ (\xi,\eta) \in \widehat{S}_P,\ \widehat{w} \in W_h(\widehat{S}_P). \tag{4.1}$$

It is easy to see that $\widehat{Q}(\widehat{D}\widehat{w})$ is the bilinear interpolation of $\widehat{D}\widehat{w}$ on $\widehat{S}_P$ with the interpolation nodes $\{\widehat{G}_1, \widehat{G}_2, \widehat{G}_3, \widehat{G}_4\}$. For $\hat{w} \in W_h(\widehat{S}_P)$, function $\widehat{D}\hat{w}$ may be discontinuous on $\widehat{S}_P$, while $\widehat{Q}(\widehat{D}\hat{w})$ is a bilinear polynomial on $\widehat{S}_P$.

From (4.1), we also obtain the gradient recovery formula on $\widehat{S}_p$:

$$\widehat{Q}(\widehat{\nabla}\widehat{w}) = (\widehat{Q}(\widehat{D}_\xi\widehat{w}), \widehat{Q}(\widehat{D}_\eta\widehat{w}))^T = \sum_{i=1}^{4} \widehat{\nabla}\widehat{w}(\widehat{G}_i)\widehat{\varphi}_i(\xi,\eta),\ (\xi,\eta) \in \widehat{S}_P. \tag{4.2}$$

Lemma 4.1. *Let $\widehat{G}_0 = (\xi_0, \eta_0)$ be the centroid of $\widehat{K}$. Then*

$$\widehat{\nabla}(\hat{u} - \widehat{\Pi}_h \hat{u})(\widehat{G}_0) = 0,\ \forall\, \hat{u} \in P_2(\widehat{K}), \tag{4.3}$$

where $P_2(S)$ represents the set of all quadratic polynomials on set S.

Proof. Let $(\xi_0, \eta_0) = ((\xi_1 + \xi_2)/2, (\eta_1 + \eta_2)/2)$ be the centroid of element $\widehat{K} = (\xi_1, \xi_2) \times (\eta_1, \eta_2)$. Further let I_ξ and I_η be the linear interpolation operators with respect to variable ξ and η, respectively, such that $I_\xi \hat{u}(\xi_i, \eta) = \hat{u}(\xi_i, \eta)$ and $I_\eta \hat{u}(\xi, \eta_i) = \hat{u}(\xi, \eta_i)$, $i = 1, 2$. Then, the bilinear interpolation operator $\widehat{\Pi}_h$ can be represented as $\widehat{\Pi}_h = I_\eta I_\xi$. It follows from the Taylor expansion that for $\hat{u} \in P_2(\widehat{K})$,

$$\hat{u} - I_\xi \hat{u} = \frac{1}{2}(\xi - \xi_1)(\xi - \xi_2)\hat{u}_{\xi\xi},\ \ \hat{u} - I_\eta \hat{u} = \frac{1}{2}(\eta - \eta_1)(\eta - \eta_2)\hat{u}_{\eta\eta}.$$

Then, we have (note that $\hat{u}_{\xi\xi} = \text{constant}$)

$$\begin{aligned}\hat{u} - \widehat{\Pi}_h \hat{u} &= \hat{u} - I_\eta \hat{u} + I_\eta(\hat{u} - I_\xi \hat{u})\\ &= \frac{1}{2}(\eta - \eta_1)(\eta - \eta_2)\hat{u}_{\eta\eta} + I_\eta\big(\frac{1}{2}(\xi - \xi_1)(\xi - \xi_2)\hat{u}_{\xi\xi}\big)\\ &= \frac{1}{2}(\eta - \eta_1)(\eta - \eta_2)\hat{u}_{\eta\eta} + \frac{1}{2}(\xi - \xi_1)(\xi - \xi_2)\hat{u}_{\xi\xi}.\end{aligned}$$

It yields

$$(\hat{u} - \widehat{\Pi}_h \hat{u})_\xi = (\xi - \frac{\xi_1 + \xi_2}{2})\hat{u}_{\xi\xi},\ (\hat{u} - \widehat{\Pi}_h \hat{u})_\eta = (\eta - \frac{\eta_1 + \eta_2}{2})\hat{u}_{\eta\eta},\ \forall\, \hat{u} \in P_2(\widehat{K}).$$

The proof is completed. ☐

From Lemma 4.1 we see that $G_i = F_{K_i}(\widehat{G}_i)$ is the Gauss point, that is, for any $u = \hat{u} \circ F_{K_i}$, $\hat{u} \in P_2(\widehat{K}_i)$, it holds

$$\nabla(u - \Pi_h u)(G_i) = \mathcal{J}_{K_i}^{-T} \widehat{\nabla}(\hat{u} - \widehat{\Pi}_h \hat{u})(\widehat{G}_i) = 0. \tag{4.4}$$

Lemma 4.2. *The following properties hold true for operator $\widehat{Q}$*

$$\widehat{D}\hat{u} = \widehat{Q}(\widehat{D}\hat{u}),\ \forall\, \hat{u} \in P_2(\widehat{S}_P), \tag{4.5}$$

$$\widehat{D}\hat{u} = \widehat{Q}(\widehat{D}\widehat{\Pi}_h \hat{u}),\ \forall\, \hat{u} \in P_2(\widehat{S}_P). \tag{4.6}$$

Proof. First, note that $\widehat{D}\hat{u} \in Q_{11}(\widehat{S}_P)$ if $\hat{u} \in P_2(\widehat{S}_P)$. Then, equality (4.5) comes from (4.1) and the uniqueness of interpolation polynomial. Moreover, it follows from (4.3) that $\widehat{D}\hat{u}(\widehat{G}_i) = \widehat{D}\widehat{\Pi}_h \hat{u}(\widehat{G}_i)$. Thus, using (4.1) we obtain $\widehat{Q}(\widehat{D}\hat{u}) = \widehat{Q}(\widehat{D}\widehat{\Pi}_h \hat{u})$. Together with (4.5), we complete the proof. ☐

Lemma 4.3. *Recovery operator $\widehat{Q}$ is bounded and*

$$\|\widehat{Q}(\widehat{D}\hat{w})\|_{0,\widehat{S}_P} \le \|\widehat{Q}\|\, \|\widehat{D}\hat{w}\|_{0,\widehat{S}_P},\ \forall\, \hat{w} \in W_h(\widehat{S}_P), \tag{4.7}$$

where $\|\widehat{Q}\|$ represents the bound of operator $\widehat{Q}$. Furthermore, we have

$$\|\widehat{D}\hat{u} - \widehat{Q}(\widehat{D}\widehat{\Pi}_h \hat{u})\|_{0,\widehat{S}_P} \le C|\hat{u}|_{3,\widehat{S}_P},\ \hat{u} \in H^3(\widehat{S}_P). \tag{4.8}$$

Proof. First, it is easy to see that

$$\|\widehat{\varphi}_i\|_{0,\widehat{K}} \le (meas(\widehat{K}))^{\frac{1}{2}} \le 1,\ \widehat{K} \in \widehat{S}_P,\ i = 1, \ldots, 4. \tag{4.9}$$

Hence, it follows from (4.1) and the inverse inequality that

$$\begin{aligned}\|\widehat{Q}(\widehat{D}\hat{w})\|_{0,\widehat{S}_P} &= \Big(\sum_{\widehat{K}\in\widehat{S}_P} \|\widehat{Q}(\widehat{D}\hat{w})\|^2_{0,\widehat{K}}\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{\widehat{K}\in\widehat{S}_P} \Big(\sum_{i=1}^{4} |\widehat{D}\hat{w}(\widehat{G}_i)|\, \|\widehat{\varphi}_i\|_{0,\widehat{K}}\Big)^2\Big)^{\frac{1}{2}} \\ &\le C\Big(\sum_{\widehat{K}\in\widehat{S}_P} \|\widehat{D}\hat{w}\|^2_{0,\widehat{K}}\Big)^{\frac{1}{2}} = C\|\widehat{D}\hat{w}\|_{0,\widehat{S}_P}.\end{aligned}$$

Estimate (4.7) is derived. Now, for given $\hat{v} \in L_2(\widehat{S}_P)$, we introduce the linear functional

$$F(\widehat{u}) = (\widehat{D}\hat{u} - \widehat{Q}(\widehat{D}\widehat{\Pi}_h\hat{u}), \hat{v})_{\widehat{S}_P},\ \hat{u} \in H^3(\widehat{S}_P). \tag{4.10}$$

Using (4.7) and the boundness of interpolation operator $\widehat{\Pi}_h$, we obtain

$$\begin{aligned}|F(\hat{u})| &\le \|\widehat{D}\hat{u} - \widehat{Q}(\widehat{D}\widehat{\Pi}_h\hat{u})\|_{0,\widehat{S}_P}\|\hat{v}\|_{0,\widehat{S}_P} \\ &\le C\|\hat{u}\|_{3,\widehat{S}_P}\|\hat{v}\|_{0,\widehat{S}_P} + \|\widehat{Q}\|\Big(\sum_{\widehat{K}\in\widehat{S}_P} \|\widehat{D}\widehat{\Pi}_h\hat{u}\|^2_{0,\widehat{K}}\Big)^{\frac{1}{2}}\|\hat{v}\|_{0,\widehat{S}_P} \\ &\le C\|\hat{u}\|_{3,\widehat{S}_P}\|\hat{v}\|_{0,\widehat{S}_P}.\end{aligned}$$

Hence, F is a linear bounded functional on $H^3(\widehat{S}_P)$. Moreover, it follows from Lemma 4.2 that

$$F(\hat{u}) = 0,\ \forall\, \hat{u} \in P_2(\widehat{S}_P).$$

Thus, using the Bramble-Hilbert Lemma, we derive

$$|F(\hat{u})| \le C|\hat{u}|_{3,\widehat{S}_P}\|\hat{v}\|_{0,\widehat{S}_P},\ \forall\, \hat{v} \in L_2(\widehat{S}_P),$$

Together with (4.10), it yields

$$\|\widehat{D}\hat{u} - \widehat{Q}(\widehat{D}\widehat{\Pi}_h\hat{u})\|_{0,\widehat{S}_P} \le C|\hat{u}|_{3,\widehat{S}_P}.$$

This gives estimate (4.8). □

Now, for $u_h \in U_h$, we define its recovery gradient on the actual patch domain S_P by the formula:

$$Q(\nabla u_h)(x,y) = \mathcal{J}_{K_j}^{-T}\widehat{Q}(\widehat{\nabla}\hat{u}_h),\ (x,y) \in K_j,\ K_j \in S_P, \tag{4.11}$$

where $\widehat{Q}(\widehat{\nabla}\hat{u}_h)$ is given by (4.2). Since

$$Q(\nabla u_h)(G_j) = \mathcal{J}_{K_j}^{-T}\widehat{Q}(\widehat{\nabla}\hat{u}_h)(\widehat{G}_j) = \mathcal{J}_{K_j}^{-T}\widehat{\nabla}\hat{u}_h(\widehat{G}_j) = \nabla u_h(G_j),$$

therefore, $Q(\nabla u_h)$ is in fact an interpolation function of ∇u_h on patch domain S_P with the Gauss points $\{G_i\}$ as the interpolation nodes.

Theorem 4.1. *Let quadrilateral partition T_h be h^2-uniform, u and u_h be the solutions of problems* (1.1) *and* (2.18)*, respectively, $u \in H^3(\Omega)$. Then, the recovered gradient $Q(\nabla u_h)$ satisfies the local estimate*

$$\|\nabla u - Q(\nabla u_h)\|_{0,S_P} \le Ch^2\|u\|_{3,S_P} + C\|\nabla(\Pi_h u - u_h)\|_{0,S_P}. \tag{4.12}$$

Proof. From (4.11), Lemma 2.1 and Lemma 4.3, we obtain

$$\begin{aligned}
\|\nabla u - Q(\nabla u_h)\|_{0,S_P} &= \Big(\sum_{j=1}^{4}\int_{\widehat{K}_j} |\mathcal{J}_{K_j}^{-T}(\widehat{\nabla}\hat{u}_h - \widehat{Q}(\widehat{\nabla}\hat{u}_h))|^2 J_{K_j} d\widehat{K}_j\Big)^{\frac{1}{2}}\\
&\le C\Big(\sum_{j=1}^{4}\int_{\widehat{K}_j} |\widehat{\nabla}\hat{u} - \widehat{Q}(\widehat{\nabla}\hat{u}_h)|^2 d\widehat{K}_j\Big)^{\frac{1}{2}}\\
&= C\|\widehat{\nabla}\hat{u} - \widehat{Q}(\widehat{\nabla}\hat{u}_h)\|_{0,\widehat{S}_P}\\
&\le \|\widehat{\nabla}\hat{u} - \widehat{Q}(\widehat{\nabla}\widehat{\Pi}_h\hat{u})\|_{0,\widehat{S}_P} + \|\widehat{Q}(\widehat{\nabla}\widehat{\Pi}_h\hat{u} - \widehat{\nabla}\hat{u}_h)\|_{0,\widehat{S}_P}\\
&\le C|\hat{u}|_{3,\widehat{S}_P} + C\|\widehat{\nabla}(\widehat{\Pi}_h\hat{u} - \hat{u}_h)\|_{0,\widehat{S}_P}\\
&\le Ch^2\|u\|_{3,S_P} + C\|\nabla(\Pi_h u - u_h)\|_{0,S_P}.
\end{aligned}$$

This gives estimate (4.12). □

The gradient recovery formula (4.11) is local, below we expand this formula to the whole domain Ω.

We will define the global recovery formula element-wise. For an interior node P_j, let S_{P_j} be the patch recovery domain around point P_j and $Q_j = Q$ the gradient recovery operator on S_{P_j} defined by (4.11). Note that for a given element K, there are four patch domains $\{S_{P_j}\}$ covering K (or less than four if K is a boundary element). In order to balance the values of $Q_j(\nabla u_h)|_{K\cap S_{P_j}}$ for different S_{P_j} (or Q_j), we define the global recovery operator Q_Ω by the average formula,

$$Q_\Omega(\nabla u_h)|_K = \frac{1}{N_K}\sum_{S_{P_j}\cap K\neq\emptyset} Q_j(\nabla u_h)|_{S_{P_j}\cap K},\ K \in T_h, \tag{4.13}$$

where $N_K \le 4$ is the total number of elements in set $\{S_{P_j} : S_{P_j}\bigcap K \neq \emptyset\}$.

Theorem 4.2. *Under the conditions of Theorem 4.1, the following superconvergence estimate holds*

$$\|\nabla u - Q_\Omega(\nabla u_h)\| \le Ch^2\|u\|_3. \tag{4.14}$$

Proof. It follows from (4.12) that

$$\|\nabla u - Q_\Omega(\nabla u_h)\|^2 = \sum_{K\in T_h}\|\nabla u - Q_\Omega(\nabla u_h)\|_{0,K}^2$$

$$
\begin{aligned}
&\le \sum_{K\in T_h} \| \frac{1}{N_K} \sum_{S_{P_j}\cap K\neq\emptyset} (\nabla u - Q_j(\nabla u_h))\|^2_{0,S_{P_j}\cap K} \\
&\le \sum_{K\in T_h} \max_{S_{P_j}\cap K\neq\emptyset} \|\nabla u - Q_j(\nabla u_h)\|^2_{0,S_{P_j}} \\
&\le \sum_{K\in T_h} \max_{S_{P_j}\cap K\neq\emptyset} \Big(Ch^4\|u\|^2_{3,S_{P_j}} + C\|\nabla(\Pi_h u - u_h)\|^2_{0,S_{P_j}} \Big) \\
&\le C\big(h^4\|u\|_3^2 + \|\nabla(\Pi_h u - u_h)\|^2\big).
\end{aligned}
$$

We complete the proof by using estimate (3.16). □

5. Numerical example

In this section, we will present some numerical results to illustrate our theoretical analysis.

Let us consider problem (1.1) with the data:

$$
A(x,y) = \begin{pmatrix} e^{2x} + y^3 + 1 & e^{xy} \\ e^{xy} & e^{2y} + x^3 + 1 \end{pmatrix}, \quad c(x,y) = 1 + xy.
$$

We take $\Omega = [0,1]^2$ and the exact solution $u(x,y) = 2\sin(2\pi x)\sin(3\pi y)$.

We present numerical results using sequence of meshes $\{T_i\}$. This quadrilateral mesh sequence is generated in the following way. We first make an original quadrilateral mesh T_1 with mesh size $h = h_1$. Then we connect the midpoints of each edge of elements in T_i $(i \ge 1)$ to obtain the refined mesh T_{i+1} which has half mesh size of T_i, see Fig. 5. It is easy to see that such bisection quadrilateral meshes must be h^2-uniform. Denote by $e(T_i)$ the error on mesh T_i in the corresponding norm, then the numerical convergence rate r is computed by the formula $r = \ln(\,e(T_i)/e(T_{i+1})\,)/\ln 2$. The numerical results are given in Table 5.1. We see that an $O(h^2)$ order of convergence rate is achieved for the recovered gradient as the theoretical prediction.

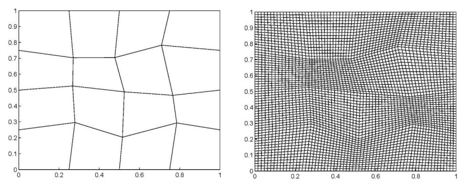

FIGURE 5. Left: original mesh T_1; Right: refined mesh T_5 obtained by bisection partition.

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TABLE 5.1. Convergence rate and error estimator.

	$\|\nabla u - Q_\Omega \nabla u_h\|$		$\|\nabla u - \nabla u_h\|$	
mesh size	error	rate	error	rate
$h_1 = 0.373$	0.3621	-	0.6424	-
$h_1/2$	0.9166e-1	1.982	3.2276e-1	0.993
$h_1/4$	0.2314e-1	1.986	1.6194e-1	0.995
$h_1/8$	0.5821e-2	1.991	0.8108e-1	0.998
$h_1/16$	1.4621e-3	1.993	4.0513e-2	1.001
$h_1/32$	0.3670e-3	1.994	2.0257e-2	1.000

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