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NORMALITY CRITERIA FOR A FAMILY OF HOLOMORPHIC FUNCTIONS CONCERNING THE TOTAL DERIVATIVE IN SEVERAL COMPLEX VARIABLES

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Abstract. In this paper, we investigate a family of holomorphic functions in several complex variables concerning the total derivative (or called radial derivative), and obtain some well-known normality criteria such as the Miranda's theorem, the Marty's theorem and results on the Hayman's conjectures in several complex variables. A high-dimension version of the famous Zalcman's lemma for normal families is also given.

1. Introduction and main results

Let $\mathbb{C}$ be the open complex plane and G be a domain of $\mathbb{C}$. A family $\mathcal{F}$ of meromorphic functions on G is said to be normal, in the sense of Montel, if every sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of G to function f which is meromorphic or identically ∞, the convergence being with respect to the spherical metric $d\sigma = |dw|/(1+|w|^2)$. The family $\mathcal{F}$ is said to be normal at a point $z_0 \in G$, if there exists a neighborhood of z_0 in which $\mathcal{F}$ is normal. It is well known that $\mathcal{F}$ is normal in G if and only if it is normal at every point of G. At the beginning of the 20th century, P. Montel introduced the concept of normal families and built the theory of normal families. One major study of normal families theory is to seek normality criteria. Corresponding to the famous Picard's theorem which says that a nonconstant entire function can omit at most one value, Montel obtained the following result called later as the Montel's theorem (see [23]): *Let $\mathcal{F}$ be a family of holomorphic functions in $G \subseteq \mathbb{C}$. If $f(z) \neq 0$, $f(z) \neq 1$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.* Soon after, Montel raised an important question: *Whether does the conclusion still hold that if the condition given that $f(z) \neq 1$ changed into $f^{(k)}(z) \neq 1$?* In 1935, Miranda obtained the following theorem (see [22]) to answer the Montel's question.

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Theorem 1.1 ([22]). *Let $\mathcal{F}$ be a family of holomorphic functions in a domain $G \subseteq \mathbb{C}$, and k be a positive integer. If $f(z) \neq 0, f^{(k)}(z) \neq 1$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.*

Bloch had ever conjectured that a family of holomorphic (meromorphic) functions which have a common property P in a domain G will in general be a normal family if P reduces a holomorphic (meromorphic) function in the complex plane $\mathbb{C}$ to a constant. Unfortunately, the Bloch's principle is not universally true. However, the point of departure for the normality criteria provides a well way to study normal family. In the work of [2, 4, 7, 13], some Picard-type results with respect to differential polynomials are showed as follows: Let f be a meromorphic function in the plane $\mathbb{C}$, and $a(a \neq 0)$ be a finite complex number, n be a positive integer. If either $f^n f'(z) \neq a(n \geq 1)$, or $f'(z) - f^n(z) \neq a(n \geq 5)$, then $f(z)$ is a constant. Hayman had ever conjectured that

A. Let $\mathcal{F}$ be a family of holomorphic functions in $G \subseteq \mathbb{C}$ and n be a positive integer. If $f'(z)f^n(z) \neq 1$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G;

B. Let $\mathcal{F}$ be a family of meromorphic functions in $G \subseteq \mathbb{C}$ and $n \geq 5$ be a positive integer. If $f'(z) - af^n(z) \neq b(a \neq 0)$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.

For the conjecture A, many significant contributions along this line have been made, see [33] for $n \geq 2$ and in 1982, Oshkin [25] confirmed that the normality of $\mathcal{F}$ holds for $n = 1$. Furthermore, many authors considered normality criteria for the case of meromorphic functions corresponding to the conjecture A, and proved that it is also true. For $n \geq 5$, see [33], and the result of $n \geq 3$ had been obtained in [12], for $n = 2$, see [26], and $n = 1$ was also confirmed in [2, 4, 36].

For the conjecture B, S. Y. Li [16] proved the case of $n \geq 5$, and see [2, 4, 27, 36] for $n = 3, n = 4$. If $\mathcal{F}$ is a family of holomorphic functions, Drasin [6] proved that $\mathcal{F}$ is normal for $n \geq 3$, and the result of $n = 2$ can be found in [34]. The famous Zaclman's lemma [35] plays a key role in the proofs of the conjectures A and B.

In 1931, F. Marty proved a well-known normality criterion.

Theorem 1.2 ([21, 30]). *A family $\mathcal{F}$ of meromorphic functions in a domain $G \subset \mathbb{C}$ is normal in G if and only if $\left\{ f^\sharp(z) = \frac{|f'(z)|}{1+|f(z)|^2} : f \in \mathcal{F} \right\}$ is locally uniformly bounded in G.*

Following the Marty's theorem, some significant generalizations along this line have been made. For examples, in 1985, Royden [28] obtained: *Let $\mathcal{F}$ be a family of meromorphic functions on $G \subset \mathbb{C}$ with the property that for each compact set $K \subset G$, there is a monotone increasing function h_K such that $|f'(z)| \leq h_K(|f(z)|)$ for all $f \in \mathcal{F}$ and all $z \in K$. Then $\mathcal{F}$ is normal.* In 1986, Li and Xie considered the higher order derivatives and generalized the Marty's theorem as follows.

Theorem 1.3 ([17]). *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $G \subset \mathbb{C}$ such that each $f \in \mathcal{F}$ has zeros only of multiplicities $\geq k$ $(k \in \mathbb{N})$. Then the family $\mathcal{F}$ is normal in G if and only if*

$$\left\{ \frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G.

Some authors considered the Marty's theorem under the condition of $\frac{|f^{(k)}(z)|}{1+|f(z)|^{\alpha}} \geq \varepsilon$ $(\varepsilon > 0)$ instead of $f^{\sharp}(z)$. For examples, in 2010, J. Grahl and S. Nevo proved for the case of $k = 1, \alpha = 2$ (see [9]). In 2011, Liu, Nevo and Pang[19] generalized the result for the case of $k = 1$, $\alpha > 1$. Chen, Nevo and Pang obtained the corresponding result for all $k \in \mathbb{N}$, $\alpha > 1$ (see [5]). See also a very recent paper [10].

In the case of higher dimension, a family $\mathcal{F}$ of holomorphic mappings of a domain G in $\mathbb{C}^m$ into complex projective space $P^N(\mathbb{C})$ is said to be normal on G if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of G to a holomorphic mapping of G into $P^N(\mathbb{C})$. Bloch [3], Fujimoto [8] and Green [11] established the Picard-type theorem for holomorphic mappings from $\mathbb{C}^m$ into $P^N(\mathbb{C})$. Nochka [24] extended the results [3, 8, 11] to the case of finite intersection multiplicity. Tu [31] gave some normality criteria for families of holomorphic mappings of several complex variables into $P^N(\mathbb{C})$ for fixed hyperplanes related to Nochka's results. Bargmann, Bonk, Hinkkanen and Martin [1] introduced a new idea related to the Montel's theorem and proved a normality criterion for families of meromorphic functions omitting three continuous functions. However, as far as we known, there are very little papers to consider the derivative for normal families, except that Tu and Zhang [32] studied this topic using the definition of $\sum_{j=1}^{m} e_j f_{z^j}$ where $\|e\| = \sqrt{\sum_{j=1}^{m} |e_j|^2} = 1$.

Throughout the rest of the paper, we use the following notations:

$$\begin{aligned}
& z = (z^1, z^2, \ldots, z^m) \in \mathbb{C}^m, \quad z^j \in \mathbb{C}(j = 1, 2, \ldots, m); \\
& \mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{C}^m, \quad \|z - z_0\| = \sqrt{\sum_{j=1}^{m} |z^j - z_0^j|^2}; \\
& \Delta_\delta(z_0) := \{z \in \mathbb{C}^m : \|z - z_0\| < \delta\}.
\end{aligned}$$

In 2004, L. Jin introduced the following definition of the total derivative for entire functions in several complex variables, and obtained some Picard-type theorems concerning the total derivative as follows. As far as we know, the total derivative of f is also called the radial derivative of f at z (see for example [29, 37]), and the k-th order total derivative is also called iterated radial derivative (see for example [14, 18]).

Definition ([15]). Let f be an entire function on $\mathbb{C}^m$, $z = (z^1, z^2, \ldots, z^m) \in \mathbb{C}^m$, the total derivative Df of f is defined by

$$Df(z) = \sum_{j=1}^{m} z^j f_{z^j}(z),$$

where f_{z^j} is the partial derivative of f with respect to z^j $(j = 1, 2, \ldots, m)$. The k-th order total derivative $D^k f$ of f is defined by

$$D^k f = D(D^{k-1} f)$$

inductively.

Theorem 1.4 ([15, Theorems 1.1, 1.2]). *Let f be an entire function on $\mathbb{C}^m$, let a and $b(\neq 0)$ be two distinct complex numbers, and let k be a positive integer. If either*

$$f(z) \neq a,\ D^k f(z) \neq b,$$

or

$$f^k(z) \cdot Df(z) \neq b,\ k \geq 2,$$

then $f(z)$ is constant.

In [20], F. Lü investigated the Picard type theorems for meromorphic functions concerning the total derivative in several complex variables.

According to the Bloch's principle, one may ask whether there exist some normal criteria for a family of holomorphic functions with respect to Theorem 1.4 concerning the total derivative? In this paper, we mainly consider this and will extend the Miranda's theorem, the Marty's theorem, the results on Hayman's conjectures to several complex variables. The first one is an extension of the Miranda's theorem (Theorem 1.1).

Theorem 1.5. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G of $\mathbb{C}^m$, and k be a positive integer. If $f(z) \neq 0$ and $D^k f(z) \neq 1$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.*

Next, we extend the Marty's theorem, and obtain a theorem concerning the total derivative as follows, in which a necessary condition for the case $\alpha > 1$ is given, and a sufficient condition for the case $\alpha = 2$ are also given. Owing to the definition of the total derivative, we successfully take a special way of the domain $\Delta(z_0)$ containing the point z_0 (which is very different from before) in the proof of (ii). However, we only consider domain outside of the point of the origin in the conclusion (ii). We don't know whether or not Theorem 1.6(ii) and Corollary 1.7 as follows remain valid if $G \setminus \{\mathbf{0}\}$ is replaced by G.

Theorem 1.6. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G of $\mathbb{C}^m$.*

(i) *If $\alpha > 1$ and $\mathcal{F}$ is normal in G, then*

$$\mathcal{F}_\alpha^1 := \left\{ \frac{|Df(z)|}{1 + |f(z)|^\alpha} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G.

(ii) *If*

$$\mathcal{F}_2^1 := \left\{ \frac{|Df(z)|}{1+|f(z)|^2} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G, *then* $\mathcal{F}$ *is normal in* $G \setminus \{\mathbf{0}\}$.

From Theorem 1.6, we get immediately the following corollary which is almost an accurate extension of Theorem 1.2 when omitting the point of origin.

Corollary 1.7. *Let* $\mathcal{F}$ *be a family of holomorphic functions in a domain* G *of* $\mathbb{C}^m$. *Then* $\mathcal{F}$ *is normal in* $G \setminus \{\mathbf{0}\}$ *if and only if*

$$\mathcal{F}_2^1 := \left\{ \frac{|Df(z)|}{1+|f(z)|^2} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in $G \setminus \{\mathbf{0}\}$.

For a general domain containing the origin, we get the following result on the extension of the Marty's theorem, in which an assumption of $f(z) \neq 0$ in a neighborhood of the origin is added.

Theorem 1.8. *Let* $\mathcal{F}$ *be a family of holomorphic functions in a domain* G *containing the origin in* $\mathbb{C}^m$.

(i) *If there exists* $\delta(>0)$ *such that* $f(z) \neq 0$ *for all* $f \in \mathcal{F}$ *and all* $z \in \Delta_\delta(\mathbf{0}) \subset G$, *and if* $\alpha > 0$ *and*

$$\mathcal{F}_\alpha^1 := \left\{ \frac{|Df(z)|}{1+|f(z)|^\alpha} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G, *then* $\mathcal{F}$ *is normal in* G.

(ii) *If* $f(z) \neq 0$ *for any* $f \in \mathcal{F}$ *and all* $z \in G$, *and if* $\alpha > 0$, $k \in \mathbb{N}^+$ *and*

$$\mathcal{F}_\alpha^k := \left\{ \frac{|D^k f(z)|}{1+|f(z)|^\alpha} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G, *then* $\mathcal{F}$ *is normal in* G.

By Theorem 1.6(i) and Theorem 1.8(i), we obtain immediately another corollary which is an extension of Theorem 1.2 when considering the origin.

Corollary 1.9. *Let* $\mathcal{F}$ *be a family of holomorphic functions in a domain* G *containing the origin in* $\mathbb{C}^m$. *Suppose that there exists* $\delta(>0)$ *such that* $f(z) \neq 0$ *for all* $f \in \mathcal{F}$ *and* $z \in \Delta_\delta(\mathbf{0}) \subset G$. *Then for* $\alpha > 1$, *the family* $\mathcal{F}$ *is normal in* G *if and only if*

$$\mathcal{F}_\alpha^1 := \left\{ \frac{|Df(z)|}{1+|f(z)|^\alpha} : f \in \mathcal{F} \right\}$$

is locally uniformly bounded in G.

Similarly as in [5, 9, 10, 19] to consider the lower bound of $\mathcal{F}_\alpha^k$ in a domain, we get the following results.

Theorem 1.10. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G containing the origin in $\mathbb{C}^m$, and $h(x)$ be a strictly monotonically increasing function in $x \in [0, +\infty)$ satisfying $h(0) \geq 0$.*

(i) *Suppose that $\alpha > 1$. If there exists $\delta(> 0)$ such that $f(z) \neq 0$ for all $f \in \mathcal{F}$ and $z \in \Delta_\delta(\mathbf{0}) \subset G$ and*

$$\frac{|Df(z)|}{1 + |f(z)|^\alpha} \geq h(\|z\|), \ z \in G \tag{1}$$

holds for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G;

(ii) *Suppose that $0 < \alpha < 1$, $k \in \mathbb{N}^+$. If $f(z) \neq 0$ for all $f \in \mathcal{F}$ and all $z \in G$, and*

$$\frac{|D^k f(z)|}{1 + |f(z)|^\alpha} \geq h(\|z\|), \ z \in G \tag{2}$$

holds for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.

Corollary 1.11. *Instead of the inequality* (1) *and* (2), *the conclusion* (i) *and* (ii) *of Theorem 1.10 are also true under the inequality*

$$\frac{|Df(z)|}{1 + |f(z)|^\alpha} \geq \varepsilon, \ z \in G$$

and

$$\frac{|D^k f(z)|}{1 + |f(z)|^\alpha} \geq \varepsilon, \ z \in G,$$

where $\varepsilon > 0$, respectively.

Corollary 1.12. *Corresponding to the conclusion* (i) *and* (ii) *of Theorem 1.10, there exists a positive constant C such that*

$$\inf_{z \in G} \frac{|Df(z)|}{1 + |f(z)|^\alpha} \leq C$$

and

$$\inf_{z \in G} \frac{|D^k f(z)|}{1 + |f(z)|^\alpha} \leq C$$

for all holomorphic function $f(z)$ in $G \subset \mathbb{C}^m$, respectively.

At last, we investigate the extensions of results of the Hayman's conjecture concerning the total derivative under the condition that the domain G contains the origin.

Theorem 1.13. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G containing the origin in $\mathbb{C}^m$, $k \in \mathbb{N}^+$, and $h(z) \neq 0$ be continuous in G. Take*

$$E_f := \{z \in G : f^k \cdot Df = h(z), f \in \mathcal{F}\}.$$

If there exist $\delta(> 0)$ and $M(> 0)$ such that for all $f \in \mathcal{F}$, $f(z) \neq 0$ for all $z \in \Delta_\delta(\mathbf{0}) \subset G$ and $|f(z)| \geq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal in G.

Corollary 1.14. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G containing the origin in $\mathbb{C}^m$, $k \in \mathbb{N}^+$, and $h(z) \neq 0$ be continuous in G. If there exists $\delta(>0)$ such that $f(z) \neq 0$ and $z \in \Delta_\delta(\mathbf{0}) \subset G$, and*

$$f^k \cdot Df \neq h(z)$$

for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.

Theorem 1.15. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G containing the origin in $\mathbb{C}^m$. Let $k \in \mathbb{N}^+(\geq 3)$, a be a nonzero finite complex number, and $h(z) \neq 0$ be continuous in G. If there exists $\delta(>0)$ such that $f(z) \neq 0$ and $z \in \Delta_\delta(\mathbf{0}) \subset G$, and*

$$Df + af^k \neq h(z)$$

for all $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in G.

The remainder of this paper is organized as follows. Theorem 1.6 is proved in Section 2. In Section 3, we mainly introduce the version of the Zalcman lemma concerning the total derivative in several complex variables and then prove Theorem 1.8. In Section 4, we give the proof of Theorem 1.10. At the last section, we will give the proofs of Theorems 1.5, 1.13 and 1.15.

2. Proof of Theorem 1.6

Firstly, we prove the claim of (i). Assume that the conclusion is not true, then there exist a point $z_0 \in G$ and a sequence $\{z_n\} \subseteq G$, such that $z_n \to z_0 (n \to \infty)$, $\{f_n\} \subseteq \mathcal{F}$, and

$$\lim_{n\to\infty} \frac{|Df_n(z_n)|}{1 + |f_n(z_n)|^\alpha} = \infty. \tag{3}$$

Due to the condition that $\mathcal{F}$ is normal in G, there exists an appropriate subsequence $\{f_{n_j}(z)\} \subseteq \{f_n(z)\}$ which converges uniformly to a holomorphic function f or ∞ in G. Without loss of generality, we still say $\{f_n(z)\}$ instead of $\{f_{n_j}(z)\}$. If f is holomorphic, then Df is also holomorphic, this contradicts to (3). So $f(z) \equiv \infty$ and thus $\lim_{n\to\infty} f_n(z_0) = \infty$.

Take positive real number r_0 and R_0 such that

$$\overline{\Delta}_{R_0}(z_0) = \{z \in \mathbb{C}^m : \|z - z_0\| \leq R_0\} \subseteq G$$

and

$$0 < r_0 < R_0 \frac{\alpha^{\frac{1}{2m}} - 1}{\alpha^{\frac{1}{2m}} + 1} \quad (\alpha > 1).$$

It yields

$$\frac{1 + \frac{r_0}{R_0}}{(1 - \frac{r_0}{R_0})^{2m-1}} - \frac{1 - \frac{r_0}{R_0}}{(1 + \frac{r_0}{R_0})^{2m-1}} \alpha < 0$$

and $\Delta_{R_0}(z_0)$ is a spherical neighborhood of z_0. For all $z \in \overline{\Delta}_{r_0}(z_0)$ and large enough n, owing to the Harnack's inequality, we obtain

$$\frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}} \ln|f_n(z_0)| \le \ln|f_n(z)| \le \frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}} \ln|f_n(z_0)|,$$

and thus,

$$|f_n(z_0)|^{\frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}}} \le |f_n(z)| \le |f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}}}. \tag{4}$$

Let $D_m(z_0,\nu) \subseteq \Delta_{r_0}(z_0)$ be a multi-disc neighborhood of z_0 as follows

$$D_m(z_0,\nu) = \left\{ z = (z^1, z^2, \ldots, z^m) \in \mathbb{C}^m : \|z^j - z_0^j\| < \nu_j, \quad j = 1,2,\ldots,m \right\},$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_m) \in \mathbb{R}_+^m$, $\|\nu\| \le r_0$. Now for all $z \in D_m(z_0,\nu)$, applying the Cauchy's inequality and combining with (4), we have

$$|(f_n(z))_{z^j}| \le \frac{1}{\nu_j} \sup_{z \in \overline{D_m}(z_0,\nu)} |f_n(z)| \le \frac{1}{\nu_j} |f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}}},$$

where $j = 1,2,\ldots,m$. Hence,

$$\begin{aligned} |Df_n(z)| &\le |z^1(f_n(z))_{z^1}| + |z^2(f_n(z))_{z^2}| + \cdots + |z^m(f_n(z))_{z^m}| \\ &\le (\|z_0\| + r_0) \left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \cdots + \frac{1}{\nu_n} \right) |f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}}}. \end{aligned}$$

The above inequality implies that $\lim_{n\to\infty} f_n(z_0) = \infty$.
Take $M = (\|z_0\| + r_0)\left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \cdots + \frac{1}{\nu_n}\right)$, thus

$$\begin{aligned} \frac{|Df_n(z_n)|}{1+|f_n(z_n)|^\alpha} &\le \frac{M|f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}}}}{1+|f_n(z_0)|^{\frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}}\alpha}} \\ &\le M|f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}} - \frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}}\alpha} \to 0 \ (n \to \infty). \end{aligned}$$

This is a contradiction to (3).

Secondly, we prove the claim of (ii). Take any non-origin $z_0 = (z_0^1, z_0^2, \ldots, z_0^m) \in G \subseteq \mathbb{C}^m$, $\|z_0\| = r_0 > 0$. Owing to the definition of the total derivative, we take a special domain containing the point z_0 as follows (different from before)

$$\Delta(z_0) := \left\{ z \,|\, z = z_0 e^{t+i\theta},\ \ln(1-\frac{\delta}{r_0}) < t < \ln(1+\frac{\delta}{r_0}),\ \theta \in (-\varepsilon,\ \varepsilon) \right\},$$

where $0 < \delta < r_0$ and ε is a real number small enough. Below it is sufficient to prove the $\mathcal{F}$ is normal in $\Delta(z_0)$.

Let $f(z) \in \mathcal{F}$. According to the assumption of the sufficiency, there exists a positive number M such that

$$\frac{|Df(z)|}{1+|f(z)|^2} \le M, z \in \overline{\Delta}(z_0).$$

Take $z = z_0 e^{t^*+i\theta^*} \in \Delta(z_0)$, where $\ln(1-\frac{\delta}{r_0}) < t^* < \ln(1+\frac{\delta}{r_0})$, $-\varepsilon < \theta^* < \varepsilon$. Define a function as follows:

$$h(\tau,\theta) := \arctan |f(z_0 e^{\tau+i\theta})|,$$

where $0 \le \tau \le t^*, 0 \le \theta \le \theta^*$. From

$$\begin{aligned}\frac{dh(\tau,\theta)}{d\tau} &= \frac{1}{1+|f(z_0e^{\tau+i\theta})|^2}\frac{d}{d\tau}|f(z_0e^{\tau+i\theta})| \\ &= \frac{1}{1+|f(z_0e^{\tau+i\theta})|^2}\frac{d}{d\tau}e^{\mathrm{Re}(\ln f(z_0e^{\tau+i\theta}))} \\ &= \frac{|f(z_0e^{\tau+i\theta})|}{1+|f(z_0e^{\tau+i\theta})|^2}\mathrm{Re}\left(\frac{d}{d\tau}\ln(f(z_0e^{\tau+i\theta}))\right) \\ &= \frac{|f(z_0e^{\tau+i\theta})|}{1+|f(z_0e^{\tau+i\theta})|^2}\mathrm{Re}\left(\frac{\frac{d}{d\tau}f(z_0e^{\tau+i\theta})}{f(z_0e^{\tau+i\theta})}\right)\end{aligned}$$

and

$$\left|\frac{d}{d\tau}f(z_0e^{\tau+i\theta})\right| = \left|\sum_{j=1}^{m} z_0^j e^{\tau+i\theta} f_{z_0^j e^{\tau+i\theta}}\right| = |Df(z_0e^{\tau+i\theta})|,$$

we get

$$\left|\frac{dh(\tau,\theta)}{d\tau}\right| \le \frac{|Df(z_0e^{\tau+i\theta})|}{1+|f(z_0e^{\tau+i\theta})|^2} \le M.$$

Similarly, we have

$$\left|\frac{dh(0,\theta)}{d\theta}\right| \le \frac{|Df(z_0e^{i\theta})|}{1+|f(z_0e^{i\theta})|^2} \le M.$$

Hence,

$$\begin{aligned}|\arctan|f(z)| - \arctan|f(z_0)|| &= |h(t^*,\theta^*) - h(0,0)| \\ &\le \left|\int_0^{t^*}\frac{dh(\tau,\theta^*)}{d\tau}d\tau\right| + \left|\int_0^{\theta^*}\frac{dh(0,\theta)}{d\theta}d\theta\right| \\ &\le M\cdot(t^*+\theta^*) \\ &\le M\cdot\left(\ln(1+\frac{\delta}{r_0})+\varepsilon\right).\end{aligned}$$

Choose appropriate numbers δ and ε such that $\ln(1+\frac{\delta}{r_0}) \le \frac{\pi}{12M} - \varepsilon$.

For the case of $|f(z_0)| \le 1$, we have

$$\arctan|f(z)| \le \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3} \quad (z \in \overline{\Delta}(z_0)),$$

therefore

$$|f(z)| \leq \sqrt{3} \quad (z \in \overline{\Delta}(z_0)).$$

For the case of $|f(z_0)| > 1$, we get

$$\arctan |f(z)| > \frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6} \quad (z \in \overline{\Delta}(z_0)),$$

therefore

$$|f(z)| > \frac{1}{\sqrt{3}} \quad (z \in \overline{\Delta}(z_0)).$$

Both of the above cases imply that $\mathcal{F}$ is normal at the point $z_0(\neq 0)$.

3. Proof of Theorem 1.8

We firstly introduce a lemma on normal families concerning the total derivative in a neighborhood of the origin.

Lemma 3.1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain G containing the origin in $\mathbb{C}^m$. If $\mathcal{F}$ is normal in $G \setminus \{\mathbf{0}\}$ and there exists $\delta(> 0)$ such that $f(z) \neq 0$ for all $f \in \mathcal{F}$ and $z \in \Delta_\delta(\mathbf{0})$, then $\mathcal{F}$ is normal in G.*

Proof. Let

$$\overline{\Delta}_r(\mathbf{0}) := \{z \in \mathbb{C}^m : \|z\| \leq r, \delta > r > 0\} \subset \Delta_\delta(\mathbf{0}),$$
$$\partial\Delta_r(\mathbf{0}) := \{z \in \mathbb{C}^m : \|z\| = r, r > 0\}.$$

By the given condition that $\mathcal{F}$ is normal in $G \setminus \{\mathbf{0}\}$, the family $\mathcal{F}$ is normal in $\partial\Delta_r(\mathbf{0})$, so for any sequences $\{f_n\} \subseteq \mathcal{F}$, there exists subsequence $\{f_{n_j}\}$ which converges uniformly to a holomorphic function f or ∞ on $\partial\Delta_r(\mathbf{0})$.

If $f(z) \neq \infty$, then f is analytic on $\partial\Delta_r(\mathbf{0})$. Hence, there exist positive integers N and M such that for any $j \geq N$, we obtain

$$|f_{n_j}(z)| \leq M,$$

where $z \in \partial\overline{\Delta}_r(\mathbf{0})$. Furthermore, there exists a sequence $\{z_j\} \subseteq \partial\Delta_r(\mathbf{0})$ such that

$$\max_{z \in \partial\Delta_r(0)} |f_{n_j}(z)| = f_{n_j}(z_j).$$

By the Maximum modulus principle,

$$|f_{n_j}(z)| \leq |f_{n_j}(z_j)| \leq M$$

holds for all $z \in \overline{\Delta}_r(\mathbf{0})$ and for any positive integer j large enough. So, the subsequence $\{f_{n_j}(z)\}$ is normal at the origin, and there exists a subsequence $\left\{f_{n_{j_k}}\right\}_{k=1}^{\infty}$ of $\{f_{n_j}(z)\}$ converges locally uniformly in a neighborhood of the origin. In view of the fact that $\left\{f_{n_{j_k}}\right\}_{k=1}^{\infty}$ is a subsequence of $\{f_n(z)\}$, we get that $\mathcal{F}$ is normal at the origin.

For the case of $f(z) \equiv \infty$, we get that for arbitrarily positive number M, there are n_j large enough such that $\{f_{n_j}\} \subset \mathcal{F}$ and

$$|f_{n_j}(z)| \geq M$$

for every $z \in \partial\Delta_r(\mathbf{0})$. Considering the condition $f_{n_k}(z) \neq 0$ in $\Delta_\delta(\mathbf{0})$, we know that $\frac{1}{f}$ is holomorphic on $\overline{\Delta}_r(\mathbf{0})$. Based on the minimum modulus principle, for all $z \in \overline{\Delta}_r(\mathbf{0})$, we have

$$\frac{1}{|f_{n_j}(z)|} \leq \frac{1}{M}.$$

So, $\{f_{n_j}(z)\}$ is normal at the origin, and thus there exists $\left\{f_{n_{j_k}}\right\}_{k=1}^{\infty} \subseteq \{f_{n_j}(z)\}$ which converges locally uniformly in a neighborhood of the origin. In view of the fact $\left\{f_{n_{j_k}}\right\}_{k=1}^{\infty} \subseteq \{f_n(z)\}$, we get that $\mathcal{F}$ is normal at the origin.

Therefore, $\mathcal{F}$ is normal in G. □

We next introduce an extension of the classical results due to Pang [26, Lemma 1] concerning the total derivative.

Lemma 3.2. *Let $f(z)$ be a holomorphic function in the unit ball $\mathbb{D}^m = \{z \in \mathbb{C}^m : \|z\| < 1\}$, and let $-1 < k < 1$. If there exists a point z_* satisfying $0 < \|z_*\| < r < 1$ such that*

$$\frac{(\ln \frac{r}{\|z_*\|})^{1+k}|Df(z_*)|}{(\ln \frac{r}{\|z_*\|})^{2k} + |f(z_*)|^2} > 1,$$

then there exist a point z_0 $(0 < \|z_*\| \leq \|z_0\| < r)$ and a number t_0 $(0 < t_0 < 1)$ such that

$$\sup_{\|z\|<r} \frac{(\ln \frac{r}{\|z\|})^{1+k}t_0^{1+k}|Df(z)|}{(\ln \frac{r}{\|z\|})^{2k}t_0^{2k} + |f(z)|^2} = \frac{(\ln \frac{r}{\|z_0\|})^{1+k}t_0^{1+k}|Df(z_0)|}{(\ln \frac{r}{\|z_0\|})^{2k}t_0^{2k} + |f(z_0)|^2} = 1.$$

Proof. Let $E := \{(z,t) : z \in \mathbb{D}^m, 0 < \|z\| < r < 1, 0 < t \leq 1\}$ and define a continuous function

$$F(z,t) := \frac{(\ln \frac{r}{\|z\|})^{1+k}t^{1+k}|Df(z)|}{(\ln \frac{r}{\|z\|})^{2k}t^{2k} + |f(z)|^2}, \ (z,t) \in E.$$

It is sufficient to prove that there exists (z_0, t_0), $0 < \|z_*\| \leq \|z_0\| < r < 1, 0 < t_0 < 1$ such that

$$\sup_{\|z\|<r} F(z,t_0) = F(z_0,t_0) = 1.$$

We claim that

$$\overline{\lim_{(\ln \frac{r}{\|z\|})t \to 0}} F(z,t) = 0. \tag{5}$$

In fact, let $(\ln \frac{r}{\|z_n\|})t_n \to 0$ as $n \to \infty$, where $\|z_n\| < r$, $0 < t_n < 1$, $z_n \to \widetilde{z}_0$ $(n \to \infty)$. Then $\|\widetilde{z}_0\| \leq r$.

(i) If $f(\widetilde{z}_0) \neq 0$, then for $-1 < k$,

$$0 \leq \overline{\lim_{n\to\infty}} F(z_n,t_n) \leq \overline{\lim_{n\to\infty}} \frac{\left(\ln \frac{r}{\|z_n\|}\right)^{1+k} t_n^{1+k}|Df(z_n)|}{|f(z_n)|^2} = 0.$$

(ii) If $f(\widetilde{z}_0) = 0$, then for $k < 1$,

$$0 \leq \overline{\lim_{n\to\infty}} F(z_n, t_n) \leq \overline{\lim_{n\to\infty}} \left(\ln \frac{r}{\|z_n\|}\right)^{1-k} t_n^{1-k} |Df(z_n)| = 0.$$

Let $U = \{(z, t) \in E : F(z, t) > 1\}$. Then $(z_*, 1) \in U$, and thus the set U is nonempty. Taking $t_0 = \inf\{t : (z, t) \in U\}$, then owing to (5), $t_0 \neq 0$. If $t_0 = 1$, then there exists a sequence $\{t_n\}(< 1)$ such that $t_n \to t_0$ $(n \to \infty)$ and $F(z_*, t_n) \leq 1$. Let $n \to \infty$, $F(z_*, t_n) \to F(z_*, 1) \leq 1$, this contradicts that $(z_*, 1) \in U$. Hence, we have $0 < t_0 < 1$. Take z_0, $(z_0, t_0) \in \overline{U}$, satisfying

$$\sup_{\|z\|\leq r} F(z, t_0) = F(z_0, t_0).$$

Let us discuss it in two cases:

- If $F(z_0, t_0) < 1$, then due to the definition of t_0, there exists (z, t_n), $0 < \|z\| < r < 1, 0 < t_n < 1$, $t_n \to t_0$ such that $F(z, t_n) \geq 1$. In addition, $F(z, t_0) \leq F(z_0, t_0) < 1$ and $\lim_{n\to\infty} F(z, t_n) = F(z, t_0)$. Thus for n large enough, $F(z, t_n) < 1$, a contradiction.
- If $F(z_0, t_0) > 1$. Owing to (5), we know $F(z_0, 0) = 0$. Then for the continuity of $F(z, t)$ with respect to t in set E, there exists $0 < t_1 < t_0$ such that $F(z_0, t_1) = 1 + \frac{F(z_0,t_0)-1}{2} > 1$. it is impossible for the definition of t_0.

So there exists (z_0, t_0) such that

$$\sup_{\|z\|<r} F(z, t_0) = F(z_0, t_0) = 1. \tag{6}$$

Considering (5), we know $\|z_0\| < r$. In addition, if $\|z_0\| < \|z_*\|$, then $F(z_*, t_0) < 1$ owing to (6). Combining with $F(z_*, 1) > 1$ and the continuity of $F(z, t)$ with respect to t in set E, it follows that there exists $t_0 < t_* < 1$ such that $F(z_*, t_*) = 1$. That completes this proof. □

Remark 3.3. If $f(z) \neq 0$ in the unit ball $\mathbb{D}^m$ for all $f \in \mathcal{F}$, then Lemma 3.2 holds for all $-1 < k < +\infty$.

It is well-known that the Zalcman's lemma [35] plays an important role in the proofs of many normality criteria in one complex variable. Here, we need propose an extension of the Zalcman's lemma concerning the total derivative as follows.

Lemma 3.4. *Let $\mathcal{F}$ be a family of holomorphic functions in the unit ball $\mathbb{D}^m = \{z \in \mathbb{C}^m : \|z\| < 1\}$ and $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in \Delta_\delta(\mathbf{0})$ $(0 < \delta < 1)$. If $\mathcal{F}$ is not normal in D^m, then for all $-1 < k < 1$, there exist*:

(i) *real number* $0 < r < 1$;
(ii) $\{z_n\} \subseteq \mathbb{D}^m$ *satisfying* $0 < \|z_n\| < r$;
(iii) $\{f_n\} \subseteq \mathcal{F}l$;

(iv) *sequence* $\{\rho_n\} \to 0^+$ *such that*

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^k}, (\zeta \in \mathbb{C})$$

converges locally uniformly to a nonconstant entire function $g(\zeta)$ *in* $\mathbb{C}$, *where* $z_n e^{\rho_n \zeta} \in \mathbb{D}^m$. *Furthermore, if* $f(z) \neq 0$ *for all* $f \in \mathcal{F}$ *and* $z \in \mathbb{D}^m$, *then* k *can be chosen in* $(-1, +\infty)$.

Proof. Suppose $\mathcal{F}$ is not normal in $\mathbb{D}^m$. Then there exists at least one non-origin point $z_0 \in \mathbb{D}^m$ such that $\mathcal{F}$ is not normal at z_0. For otherwise, by Lemma 3.1, $\mathcal{F}$ is normal in $\mathbb{D}^m$. Together with the conclusion (ii) of Theorem 1.6, there exist $0 < r^* < 1, \|z_n^*\| < r^*, \{f_n\} \subseteq \mathcal{F}$ such that

$$\lim_{n\to\infty} \frac{|Df_n(z_n^*)|}{1 + |f_n(z_n^*)|^2} = \infty, \tag{7}$$

where $z_n^* \to z_0$ as $n \to \infty$. Choose r such that $0 < r^* < r < 1$, and define

$$F_n(z,t) = \frac{(\ln \frac{r}{\|z\|})^{1+k} t^{1+k} |Df_n(z)|}{(\ln \frac{r}{\|z\|})^{2k} t^{2k} + |f_n(z)|^2},$$

where $0 < \|z\| < r, 0 < t \leq 1$. For sufficiently large integer n, we can assume $0 < \frac{\|z_0\|}{2} < \|z_n^*\| < r^*$, and then $\ln \frac{r}{r^*} < \ln \frac{r}{\|z_n^*\|} < \ln \frac{2r}{\|z_0\|}$. In addition,

$$F_n(z_n^*, 1) = \frac{(\ln \frac{r}{\|z_n^*\|})^{1+k} |Df_n(z_n^*)|}{(\ln \frac{r}{\|z_n^*\|})^{2k} + |f_n(z_n^*)|^2} = \left(\ln \frac{r}{\|z_n^*\|}\right)^{1-k} \frac{|Df_n(z_n^*)|}{1 + \frac{|f(z_n^*)|^2}{(\ln \frac{r}{\|z_n^*\|})^{2k}}}.$$

Note that $-1 < k < 1$. Then $F_n(z_n^*, 1) \geq (\ln |\frac{r}{r^*}|)^{1-k} \frac{|Df_n(z_n^*)|}{1+\frac{|f(z_n^*)|^2}{\ln |\frac{r}{r^*}|^{2k}}}$. Together with (7), we have

$$\lim_{n\to\infty} F_n(z_n^*, 1) = \infty.$$

Thus, for n large enough,

$$F_n(z_n^*, 1) > 1.$$

Owing to Lemma 3.2, there exist $\{z_n\}$ and $\{a_n\}$ satisfying

$$0 < \frac{\|z_0\|}{2} < \|z_n^*\| \leq \|z_n\| < r, \quad 0 < a_n < 1,$$

related to every $f_n \in \mathcal{F}$ such that

$$\sup_{\|z\|<r} F_n(z, a_n) = F_n(z_n, a_n) = 1.$$

Hence,

$$1 = F_n(z_n, a_n) \geq F_n(z_n^*, a_n) \geq a_n^{1+k} F_n(z_n^*, 1),$$

which yields

$$\lim_{n\to\infty} a_n = 0.$$

Let $\rho_n = \ln \frac{r}{\|z_n\|} a_n \to 0^+$, and it follows that

$$\lim_{n\to\infty} \frac{\rho_n}{\ln \frac{r}{\|z_n\|}} = 0. \tag{8}$$

Thus the function

$$g_n(\zeta) := \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^k}$$

is defined in $|\zeta| < R_n = \frac{\ln \frac{r}{\|z_n\|}}{\rho_n} \to \infty$, and satisfies

$$g_n'(\zeta) = \rho_n^{1-k} Df(z_n e^{\rho_n \zeta}),$$

$$\frac{|g_n'(\zeta)|}{1+|g_n(\zeta)|^2} = \frac{\rho_n^{1+k}|Df(z_n e^{\rho_n\zeta})|}{\rho_n^{2k} + |f(z_n e^{\rho_n \zeta})|^2} \quad (|\zeta| < R < R_n).$$

Owing to (8), then for $|\zeta| < R$,

$$\frac{\ln \frac{r}{\|z_n\|}}{\ln \frac{r}{\|z_n e^{\rho_n \zeta}\|}} \to 1 \ (n \to \infty).$$

Hence, there exists $\varepsilon_n \to 0^+$ such that

$$\rho_n^{1+k} \le \left((1+\varepsilon_n)(\ln \frac{r}{\|z_n e^{\rho_n\zeta}\|})\right)^{1+k} a_n^{1+k},$$

$$\rho_n^{2k} \ge \left((1-\varepsilon_n)(\ln \frac{r}{\|z_n e^{\rho_n\zeta}\|})\right)^{2k} a_n^{2k}.$$

Furthermore, we have

$$\begin{aligned}\frac{|g_n'(\zeta)|}{1+|g_n(\zeta)|^2} &\le \frac{(1+\varepsilon_n)^{1+k}}{(1-\varepsilon_n)^{2k}} \cdot \frac{(\ln \frac{r}{\|z_n e^{\rho_n\zeta}\|})^{1+k} a_n^{1+k} |Df_n(z_n e^{\rho_n\zeta})|}{(\ln \frac{r}{\|z_n e^{\rho_n\zeta}\|})^{2k} a_n^{2k} + |f_n(z_n e^{\rho_n\zeta})|^2} \\ &= \frac{(1+\varepsilon_n)^{1+k}}{(1-\varepsilon_n)^{2k}} F_n(z_n e^{\rho_n\zeta}, a_n) \\ &\le \frac{(1+\varepsilon_n)^{1+k}}{(1-\varepsilon_n)^{2k}}.\end{aligned}$$

By the Marty's theorem (Theorem 1.2), $\{g_n(\zeta)\}$ is normal in $\mathbb{C}$. Without loss of generality, we may assume $\{g_n(\zeta)\}$ converges locally uniformly to a holomorphic function $g(\zeta)$ in $\mathbb{C}$ or ∞. In addition,

$$\frac{|g_n'(0)|}{1+|g_n(0)|^2} = \frac{\rho_n^{1+k}|Df_n(z_n)|}{\rho_n^{2k} + |f_n(z_n)|^2} = F_n(z_n, a_n) = 1,$$

it implies that $g(\zeta) \not\equiv \infty$. Thus $g(\zeta)$ is a holomorphic function and

$$\frac{|g'(0)|}{1+|g(0)|^2} = \lim_{n\to\infty} \frac{|g_n'(0)|}{1+|g_n(0)|^2} = 1.$$

Hence, $g(\zeta)$ is a nonconstant entire function in $\mathbb{C}$. □

Proof of Theorem 1.8. Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}^m$. Note that the origin is in G. Without loss of generality, we may assume that G is the unit ball $\mathbb{D}^m$ and there exists at least non-origin point $z_0 \in \mathbb{D}^m$ such that $\mathcal{F}$ is not normal at z_0.

(i) In view of Lemma 3.4, there exist $0 < r < 1$, $\{z_n\} \subset \Delta_r(\mathbf{0}) \subset G$, $\{f_n\} \subseteq \mathcal{F}$, $\{\rho_n\} \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^{\frac{1}{2}}}$$

converges locally uniformly to nonconstant entire function $g(\zeta)$. Furthermore, we have

$$\rho_n^{\frac{1}{2}} \cdot Df_n(z_n e^{\rho_n \zeta}) = g_n'(\zeta) \to g'(\zeta)(n \to \infty), \zeta \in \mathbb{C}. \tag{9}$$

For any given $\zeta \in \mathbb{C}$, there exists an integer n large enough such that $z_n e^{\rho_n \zeta} \in \Delta_r(\mathbf{0})$. Thus by the condition given, we have

$$\begin{aligned} |g_n'(\zeta)| &= \rho_n^{\frac{1}{2}} \cdot |Df_n(z_n e^{\rho_n \zeta})| \\ &\leq \rho_n^{\frac{1}{2}} \cdot M(1 + |f_n(z_n e^{\rho_n \zeta})|^{\alpha}) \\ &= \rho_n^{\frac{1}{2}} \cdot M(1 + |g_n(\zeta)\rho^{\frac{1}{2}}|^{\alpha}) \\ &\to 0 \ (n \to \infty), \end{aligned}$$

where M is only dependent on the domain $\Delta_r(\mathbf{0})$. Combining with (9), we have $g'(\zeta) \equiv 0$, i.e., $g(\zeta)$ is a constant, a contradiction.

(ii) In view of Lemma 3.4, there exist $0 < r < 1$, $\{z_n\} \subset \Delta_r(\mathbf{0}) \subset G$, $\{f_n\} \subseteq \mathcal{F}$, $\{\rho_n\} \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^{\frac{k}{2}}}$$

converges locally uniformly to a nonconstant entire function $g(\zeta)$ in $\mathbb{C}$. In addition, since $f(z) \neq 0$ for every $f \in \mathcal{F}$ and for all $z \in G$, we have $g_n(\zeta) \neq 0$ and thus $g(\zeta) \neq 0$. Furthermore, in $\mathbb{C}$, we have

$$\rho_n^{\frac{k}{2}} \cdot D^k f_n(z_n e^{\rho_n \zeta}) = g_n^{(k)}(\zeta) \to g^{(k)}(\zeta) \ (n \to \infty). \tag{10}$$

For any given $\zeta \in \mathbb{C}$, there exists a integer n large enough such that $z_n e^{\rho_n \zeta} \in \Delta_r(\mathbf{0})$. Thus by the condition of (ii), we have

$$g_n^{(k)}(\zeta) = \rho_n^{\frac{k}{2}} \cdot D^k f_n(z_n e^{\rho_n \zeta}) \leq \rho_n^{\frac{k}{2}} \cdot M(1 + |f_n(z_n e^{\rho_n \zeta})|^{\alpha}) \to 0$$

as $n \to \infty$, where M depends only on the domain $\Delta_r(0)$. Combining with (10), we have $g^{(k)}(\zeta) \equiv 0$, i.e., $g(\zeta)$ is a polynomial of degree $\leq k$ with respect to ζ. Thus there exists a root ζ^* such that $g(\zeta^*) = 0$ which contradicts to the fact $g(\zeta) \neq 0$. $\square$

4. Proof of Theorem 1.10

Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}^m$. Note that the origin is in G. Without loss of generality, we may assume that G is the unit ball $\mathbb{D}^m$ and there exists at least non-origin point $z_0 \in \mathbb{D}^m$ such that $\mathcal{F}$ is not normal at z_0.

(i) Let $\Delta_r(z_0)$ be a neighborhood of z_0, $r = \frac{\|z_0\|}{2}$. For all $z \in \Delta_r(z_0)$, it is easy to see that

$$|Df(z)| \geq h(\|z\|) > h(\frac{\|z_0\|}{2}) > 0.$$

Hence, we get that the family $\{Df : f \in \mathcal{F}\}$ is normal in $\Delta_r(z_0)$ and thus there exists an appropriate subsequence $\{Df_{n_j}(z)\} \subseteq \{Df_n(z)\}$ converges uniformly to a holomorphic function d in $\Delta_r(z_0))$ or ∞. In addition, for any sequence $\{f_n\} \subseteq \mathcal{F}$, we have

$$|f_n(z)|^\alpha \leq \frac{|Df_n(z)|}{h(\|z\|)} < \frac{|Df_n(z)|}{h(\frac{\|z_0\|}{2})}.$$

Now if d is holomorphic, then $\{f_{n_j}\}$ is locally bounded as well as $\{Df_n(z)\}$ in $\Delta_r(z_0)$. Thus, $\{f_{n_j}\}$ is normal in $\Delta_r(z_0)$ and $\mathcal{F}$ is normal at z_0, a contradiction.

For the case of $d \equiv \infty$, repeating the same reason as in the proof of the part (i) of Theorem 1.6 implies also that

$$\begin{aligned}\frac{|Df_n(z_n)|}{1+|f_n(z_n)|^\alpha} &\leq \frac{M|f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}}}}{1+|f_n(z_0)|^{\frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}}\alpha}} \\ &\leq M|f_n(z_0)|^{\frac{1+\frac{r_0}{R_0}}{(1-\frac{r_0}{R_0})^{2m-1}} - \frac{1-\frac{r_0}{R_0}}{(1+\frac{r_0}{R_0})^{2m-1}}\alpha} \to 0 \ (n \to \infty).\end{aligned}$$

Thus, we have

$$0 < h(\frac{\|z_0\|}{2}) < \frac{|Df_n(z_n)|}{1+|f_n(z_n)|^\alpha} \to 0 \ (n \to \infty).$$

This is a contradiction. Hence, $\mathcal{F}$ is normal in G.

(ii) In view of Lemma 3.4, there exist $0 < r < 1$, $\{z_n\} \subset \Delta_r(\mathbf{0}) \subset G$, $\{f_n\} \subseteq \mathcal{F}$, $\{\rho_n\} \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^\beta}$$

converges locally uniformly to a nonconstant entire function $g(\zeta)$ in $\mathbb{C}$, where β is a positive number large enough such that

$$k - \beta(1-\alpha) < 0 \quad (0 < \alpha < 1).$$

For n large enough, $z_n e^{\rho_n \zeta} \in \Delta_r(0)$ and $h(\|z_n e^{\rho_n \zeta}\|) > h(0) \geq 0$. Since $f(z) \neq 0$ for every $f \in \mathcal{F}$ and all $z \in G$, we get $g(\zeta) \neq 0$ and thus $g(\zeta) \neq 0$ in $\mathbb{C}$. Let

$\zeta_0 \in \mathbb{C}$, $g(\zeta_0) \neq 0$ and we have

$$\rho_n^{k-\beta} \cdot D^k f_n(z_n e^{\rho_n \zeta}) = g_n^{(k)}(\zeta) \to g^{(k)}(\zeta)(n \to \infty). \tag{11}$$

Thus by the given condition, we have

$$\begin{aligned} |g_n^{(k)}(\zeta)| &= |\rho_n^{k-\beta} \cdot D^k f_n(z_n e^{\rho_n \zeta})| \\ &\geq \rho_n^{k-\beta} \cdot h(\|z_n e^{\rho_n \zeta}\|)(1 + |f_n(z_n e^{\rho_n \zeta})|^\alpha) \\ &\geq \rho_n^{k-\beta(1-\alpha)} \cdot h(|z_n e^{\rho_n \zeta}|)(\rho_n^{-\beta} |f_n(z_n e^{\rho_n \zeta})|)^\alpha. \end{aligned}$$

Combining with (11), we get that the right part of the above inequality tends to ∞ as $n \to \infty$, which contradicts to the fact that the left part of the above inequality trends to a finite real number for any given ζ. Thus $\mathcal{F}$ is normal in G.

5. Proofs of Theorems 1.5, 1.13 and 1.15

Proof of Theorem 1.5. Assume that $\mathcal{F}$ is not normal in the domain $G \subset \mathbb{C}^m$. Note that the origin is in G. Without loss of generality, we may assume that G is the unit ball $\mathbb{D}^m$.

Since $f(z) \neq 0$ for any $f \in \mathcal{F}$ and all $z \in G \subset \mathbb{C}^m$, by Lemma 3.4 we get that for positive integer $k \geq 1$, there exist $0 < r < 1$, $0 < \|z_n\| < r$, $\{f_n\} \subseteq \mathcal{F}$, $\{\rho_n\} \to 0^+$ such that

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^k}$$

converges locally uniformly to a nonconstant entire function $g(\zeta)$ in $\mathbb{C}$. In addition, $f(z) \neq 0$ for all $f \in \mathcal{F}$ yields $g_n(\zeta) \neq 0$, and thus $g(\zeta) \neq 0$. Considering the continuous sequence of functions

$$g_n^{(k)}(\zeta) = D^k f_n(z_n e^{\rho_n \zeta}) \neq 1,$$

and applying the generalized Hurwitz theorem, we get that either $g^{(k)}(\zeta) \equiv 1$ or $g^{(k)}(\zeta) \neq 1$ holds in $\mathbb{C}$.

- If $g^{(k)}(\zeta) \equiv 1$ holds in $\mathbb{C}$, then $g(\zeta)$ is a polynomial of degree k in $\mathbb{C}$ with respect to ζ. This contradicts to the conclusion $g(\zeta) \neq 0$.
- If $g^{(k)}(\zeta) \neq 1$ holds in $\mathbb{C}$. Note that $g(\zeta) \neq 0$. Then by the Picard theorem concerning k-th derivatives, we get that $g(\zeta)$ is a constant. This is a contradiction.

Hence, $\mathcal{F}$ is normal in G. □

Proof of Theorem 1.13. By Lemma 3.4 and considering

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^{\frac{1}{k+1}}} \quad (k \in \mathbb{N}^+),$$

one can complete the proof of Theorem 1.13 similarly as the proof of Theorem 1.5. We omit the detail. □

Proof of Theorem 1.15. By Lemma 3.4 and considering

$$g_n(\zeta) = \frac{f_n(z_n e^{\rho_n \zeta})}{\rho_n^{\frac{1}{1-k}}} \quad (\frac{1}{1-k} > -1),$$

one can complete the proof of Theorem 1.15 similarly as the proof of Theorem 1.5. We omit the detail. □

References

[1] D. Bargmann, M. Bonk, A. Hinkkanen, and G. J. Martin, *Families of meromorphic functions avoiding continuous functions*, J. Anal. Math. **79** (1999), 379–387.

[2] W. Bergweiler and W. Eremenko, *On the singularities of the inverse to a mermorphic function of finite order*, Rev. Mat. Iberoam **11** (1995), no. 2, 335–373.

[3] A. Bloch, *Sur les systèmes de fonctions holomorphes à variétés linéaires lacunaires*, Ann. Sci. École Norm. Sup. **43** (1926), no. 3, 309–362.

[4] H. H. Chen ad M. L. Fang, *On value distribution of $f^n f'$*, Sci. China Ser. A **38** (1995), 789–798.

[5] Q. Chen, S. Nevo, and X. C. Pang, *A general differential inequality of the k-th derivative that leads to normality*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 2, 691–695.

[6] D. Drasin, *Normal families and the Nevanlinna theory*, Acta Math. **122** (1969), 231–263.

[7] M. Erwin, *Uberein Problem Von Hayman*, Math. Z. **8** (1979), no. 1, 239–259.

[8] H. Fujimoto, *Extensions of the big Picard's theorem*, Tôhoku Math J. **24** (1972), 415–422.

[9] J. Grahl and S. Nevo, *Spherical derivatives and normal families*, J. Anal. Math. **117** (2012), 119–128.

[10] ———, *An extension of one direction in Marty's normality criterion*, Monatsh. Math. **174** (2014), no. 2, 205–217.

[11] M. Green, *Holomorphic maps into complex projective space omitting hyperplanes*, Trans. Amer. Math. Soc. **169** (1972), 89–103.

[12] Y. X. Gu, *On normal families of meromorphic functions*, Sci. China Ser. A (1978), no. 4, 373–384.

[13] W. K. Hayman, *Research Problems in Function Theory*, London: Athlone Press, 1967.

[14] Z. Hu, *Extended Ces'aro operators on mixed norm spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2171–2179.

[15] L. Jin, *Theorem of Picard type for entire functions of several complex variables*, Kodai Math. J. **26** (2003), no. 2, 221–229.

[16] S. Y. Li, *The normality criterion of a class of meromorphic functions*, J. Fujian Norm. Univ. **2** (1984), 156–158.

[17] S. Y. Li and C. H. Xie, *On normal families of meromorphic functions*, Acta Math. Sin. **4** (1986), 468–476.

[18] B. Li and C. Ouyang, *Higher radial derivative of functions of Q_p spaces and its applications*, J. Math. Anal. Appl. **327** (2007), no. 2, 1257–1272.

[19] X. J. Liu, S. Nevo, and X. C. Pang, *Differential inequalities, normality and quasi-normality*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 277–282.

[20] F. Lü, *Theorems of Picard type for meromorphic function of several complex variables*, Complex Var. Elliptic Equ. **58** (2013), no. 8, 1085–1092.

[21] F. Marty, *Recherches sur la répartition des valeurs dune fonction méromorphe*, Ann. Fac. Sci. Univ. Toulouse Sci. Math. Sci. Phys. (3) **23** (1931), 183–261.

[22] C. Miranda, *Sur un nouveau critere de normalite pour les familles des fonctions holomorphes*, Bull. Sci. Math. France **63** (1935), 185–196.

[23] P. Montel, *Lecons sur les families normales de fonctions analytiques et leur applicaeions*, Coll. Borel, 1927.
[24] E. Nochka, *On the theory of meromorphic functions*, Soviet Math. Dokl. **27** (1983), 377–381.
[25] I. Oshkin, *A normal criterion of families of holomorphic functions*, (Russian) Usp. Mat. Nauk. **37** (1982), no. 2, 221–222.
[26] X. C. Pang, *Bloch's principle and normal criterion*, Sci. China Ser. A **32** (1989), no. 7, 782–791.
[27] ———, *On normal criterion of meromorphic functions*, Sci. China Ser. A **33** (1990), no. 5, 521–527.
[28] H. L. Royden, *A criterion for the normality of a family of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 499–500.
[29] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York-Berlin, 1980.
[30] J. Schiff, *Normal Families*, Springer, New York, 1993.
[31] Z. H. Tu, *Normality criteria for families of holomorphic mappings of several complex variables into* $P^n(\mathbb{C})$, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1039–1049.
[32] Z. H. Tu and S. S. Zhang, *Normal families of holomorphic mappings of several complex variables into* $\mathbb{P}^1(\mathbb{C})$, Acta. Math. Sin. (Chin. Ser.) **53** (2010), no. 6, 1045–1050.
[33] L. Yang and G. H. Zhang, *Recherches sur la normalité des familiesn de fonctions analytiques àdes valeurs multiples. I. Un nouveau critère at quelques applications*, Sci. Sinica **14** (1965), 1258-1271; *II. Géneralisations*, Ibid. **15** (1996), 433–453.
[34] Y. S. Ye, *A new normal criterion and its application*, Chin. Ann. Math. Ser. A(Supplement) **12** (1991), 44–49.
[35] L. Zalcman, *A heuristic principle in complex function theory*, Amer. Math. Monthly **82** (1975), no. 8, 813–817.
[36] ———, *On some quesitions of Hayman*, unpublished manuscript, 5 pp., 1994.
[37] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Text in Mathematics 226, Springer, New York, 2005.

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