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K-G-FRAMES AND STABILITY OF K-G-FRAMES IN HILBERT SPACES

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Abstract. A K-g-frame is a generalization of a g-frame. It can be used to reconstruct elements from the range of a bounded linear operator K in Hilbert spaces. K-g-frames have a certain advantage compared with g-frames in practical applications. In this paper, the interchangeability of two g-Bessel sequences with respect to a K-g-frame, which is different from a g-frame, is discussed. Several construction methods of K-g-frames are also proposed. Finally, by means of the methods and techniques in frame theory, several results of the stability of K-g-frames are obtained.

1. Introduction

Frames as generalized bases in Hilbert spaces were first introduced by Duffin and Schaeffer [8] during their study of nonharmonic Fourier series in 1952. In 1986, Daubechies, Grossmann and Meyer [5] reintroduced the concept of frames. Now frame theory has been widely used in many fields such as filter theory [2], image processing [3], numerical analysis and other areas. We refer to [4, 13, 20, 21] for an introduction to frame theory in Hilbert spaces and its applications.

With the deepening of research on frame theory and its applications in Hilbert spaces, many generalized frames have been appeared. Sun [16] introduced the concepts of g-Riesz bases and g-frames. K-frames were proposed by Găvruta [11] in Hilbert spaces to study atomic systems. In [6, 9, 10, 19], some properties and conclusions of K-frames were given. In [22], the authors put forward the concept of K-g-frames, which are more general than ordinary g-frames in Hilbert spaces. Naturally, K-g-frames have become one of the most active fields in frame theory in recent years. What we discuss in this paper is K-g-frames, which are limited to the range of a bounded linear operator in Hilbert spaces and have gained greater flexibility in practical applications

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relative to g-frames. In [1, 23], several properties and characterizations of K-g-frames were obtained. However, many problems about K-g-frames have not been studied, such as whether the two g-Bessel sequences with respect to a K-g-frame are interchangeable, and how to find efficient approaches for construction of K-g-frames. In this paper, we study the interchangeability of two g-Bessel sequences with respect to a K-g-frame, which is different from the traditional g-frame. Besides, we give several methods to construct K-g-frames. Finally, we present a version of Paley-Wiener Theorem for K-g-frames which is closely related to the results of g-frames.

Throughout this paper, H denotes separable Hilbert space, and I represents the identity operator. $L(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2, where H_1 and H_2 are two Hilbert spaces. In particular, $L(H)$ is a collection of all bounded linear operators from H to H. For any $T \in L(H_1, H_2)$, $N(T)$ and $R(T)$ are the kernel and the range of T, respectively. T^* is the adjoint operator of T. The pseudo-inverse of K is denoted as $K^\dagger$. $\{H_j : j \in J\}$ is a sequence of closed subspaces of H, where J is a subset of integers $\mathbb{Z}$. $L(H, H_j)$ is the collection of all bounded linear operators from H into H_j. $l^2\left(\{H_j\}_{j\in J}\right)$ is defined by

$$l^2\left(\{H_j\}_{j\in J}\right) = \left\{\{a_j\}_{j\in J} : a_j \in H_j, j \in J, \sum_{j\in J} \|a_j\|^2 < +\infty\right\},$$

with the inner product given by

$$\left\langle \{a_j\}_{j\in J}, \{b_j\}_{j\in J} \right\rangle = \sum_{j\in J} \langle a_j, b_j\rangle_{H_j},$$

it is clear that $l^2\left(\{H_j\}_{j\in J}\right)$ is a complex Hilbert space.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Definition 2.1 ([16, Definition 1.1])**.** A sequence $\{\Lambda_j \in L(H, H_j) : j \in J\}$ is called a g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist two positive constants A and B such that, for all $f \in H$,

$$A\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2. \tag{2.1}$$

The constants A and B are called the lower and upper bounds of g-frame, respectively. If the right hand inequality holds, then we say that $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. If $A = B$, we call this g-frame a tight g-frame. If a g-frame ceases to be a g-frame whenever any one of its elements is removed from $\{\Lambda_j\}_{j\in J}$, we say that the g-frame is an exact g-frame.

Definition 2.2 ([1, Theorem 2.5])**.** Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)$ for any $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \forall f \in H. \tag{2.2}$$

The constants A and B are called the lower and upper bounds of K-g-frame, respectively.

Remark 1. Every K-g-frame is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$.

Remark 2. When $K = I$, K-g-frame is the g-frame as defined in Definition 2.1.

Now, we will assume that $R(K)$ is closed. Under this condition, $K^\dagger$ must exist.

From Remark 1, if $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. So we can define the bounded linear operator $T : l^2\left(\{H_j\}_{j\in J}\right) \to H$ as follows

$$T\left(\{g_j\}_{j\in J}\right) = \sum_{j\in J} \Lambda_j^* g_j, \forall \{g_j\}_{j\in J} \in l^2\left(\{H_j\}_{j\in J}\right).$$

The adjoint operator $T^* : H \to l^2\left(\{H_j\}_{j\in J}\right)$ is given by

$$T^* f = \{\Lambda_j f\}_{j\in J}, \forall f \in H.$$

By the above bounded linear operators T and T^*, we can define the bounded linear operator $S : H \to H$ as follows

$$Sf = TT^* f = T\{\Lambda_j f\}_{j\in J} = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \ \forall f \in H.$$

We call T, T^* and S the pre-frame operator, analysis operator and frame operator of K-g-frame, respectively. These operators play an important role in studying K-g-frame theory.

Proposition 2.3 ([15, Definition 2.4])**.** *Let $\{\Lambda_j \in L(H, H_j) : j \in J\}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence with bound B if and only if*

$$T : l^2\left(\{H_j\}_{j\in J}\right) \to H, T\left(\{g_j\}_{j\in J}\right) = \sum_{j\in J} \Lambda_j^* g_j \tag{2.3}$$

is well-defined and $\|T\| \leq \sqrt{B}$.

Definition 2.4 ([16, p. 442])**.** Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B. Suppose that S is g-frame operator of $\{\Lambda_j\}_{j\in J}$. Then, $\{\Lambda_j S^{-1}\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$\frac{1}{A}$, $\frac{1}{B}$. We say that $\left\{\Lambda_j S^{-1}\right\}_{j\in J}$ is the canonical dual g-frame of $\{\Lambda_j\}_{j\in J}$. In other words, $\left\{\Lambda_j S^{-1}\right\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ are dual g-frames with respect to each other.

Definition 2.5 ([4, p. 23])**.** Let U be a complex Hilbert space. Suppose that $S : H \to U$ is a bounded linear operator with closed range $R(S)$. Then there exists a unique bounded linear operator $S^\dagger : U \to H$ satisfying

$$N\left(S^\dagger\right) = R(S)^\perp,\ R\left(S^\dagger\right) = N(S)^\perp,\ SS^\dagger f = f,\ f \in R(S). \tag{2.4}$$

The operator $S^\dagger$ is called the pseudo-inverse operator of S.

Proposition 2.6 ([4])**.** *Let $R(K)$ be closed. If there exists $K^\dagger$ of K such that $KK^\dagger f = f$, $\forall f \in R(K)$, then $KK^\dagger|_{R(K)} = I|_{R(K)}$.*

From Proposition 2.6 and the definition of K-g-frame, it is easy to get that

$$\|f\| = \left\|\left(K^\dagger\,|_{R(K)}\right)^* K^* f\right\| \le \left\|K^\dagger\right\| \|K^* f\|.$$

That is, $\|K^* f\|^2 \ge \left\|K^\dagger\right\|^{-2} \|f\|^2$. By (2.2), we have

$$A\left\|K^\dagger\right\|^{-2}\|f\|^2 \le A\|K^* f\|^2 \le \left\langle \sum_{j\in J} \Lambda_j^* \Lambda_j f, f \right\rangle = \langle Sf, f\rangle \le \|Sf\|\,\|f\|.$$

Therefore,

$$A\left\|K^\dagger\right\|^{-2}\|f\| \le \|Sf\| \le B\|f\|.$$

Proposition 2.7 ([19, Theorem 2.5])**.** *Let $\{f_j\}_{j=1}^\infty$ be a Bessel sequence in H. Then $\{f_j\}_{j=1}^\infty$ is a K-frame for H if and only if there exists constant $A > 0$ such that $S \ge AKK^*$, where S is the frame operator for $\{f_j\}_{j=1}^\infty$.*

Motivated by the characterization of K-frames, we give the characterization of K-g-frames as follows:

Lemma 2.8. *Let $\{\Lambda_j\}_{j\in J}$ be a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if there exists constant $A > 0$ such that $S \ge AKK^*$, where S is the frame operator for $\{\Lambda_j\}_{j\in J}$.*

Proof. $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A, B if and only if

$$A\|K^* f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2, \forall f \in H,$$

that is,

$$\langle AKK^* f, f\rangle \le \langle Sf, f\rangle \le \langle Bf, f\rangle, \forall f \in H,$$

where S is the frame operator of K-g-frame $\{\Lambda_j\}_{j\in J}$. Therefore, the conclusion holds. $\square$

Lemma 2.9 ([12, Lemma 1]). *Let $T : H \to H$ be a linear operator, and suppose that there exist constants $\lambda_1, \lambda_2 \in [0,1)$ such that*

$$\|Tx - x\| \leq \lambda_1 \|x\| + \lambda_2 \|Tx\|, \ \forall x \in H. \tag{2.5}$$

Then $T \in L(H)$, and

$$\begin{aligned} &\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|Tx\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \\ &\frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \left\|T^{-1}x\right\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|, \ \forall x \in H. \end{aligned} \tag{2.6}$$

3. Main results of K-g-frames for Hilbert spaces

In this section, we first discuss whether the two g-Bessel sequences related to a K-g-frame are interchangeable, and then give several methods to construct K-g-frames.

In [1], the author gives the following equivalent characterizations of K-g-frames.

Proposition 3.1 ([1, Theorem 2.5]). *Let $K \in L(H)$. Then the following statements are equivalent*:

(1) *$\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

(2) *$\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ and there exists a g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ such that*

$$Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \ \forall f \in H. \tag{3.1}$$

Because the two g-Bessel sequences related to a g-frame are interchangeable, naturally, we may wonder whether the two g-Bessel sequences $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ with respect to a K-g-frame are interchangeable or not. In fact, the answer is negative. The Example 3.2 bellow illustrates this.

Example 3.2. Suppose that $H = \mathbb{R}^3$, $J = \{1,2,3\}$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of H, and $H_j = \overline{\text{span}}\{e_j\}$. We define $K \in L(H)$, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ as follows

$$K : H \to H, Kf = \langle f, e_1\rangle e_2 + \langle f, e_2\rangle e_2 + \langle f, e_2\rangle e_3;$$

$$\Lambda_j : H \to H_j, \begin{cases} \Lambda_1 f = (\langle f, e_1\rangle + \langle f, e_2\rangle)e_1 \\ \Lambda_2 f = (\langle f, e_3\rangle - \langle f, e_1\rangle)e_2 \\ \Lambda_3 f = \langle f, e_2\rangle e_3; \end{cases}$$

$$\Gamma_j : H \to H_j, \begin{cases} \Gamma_1 f = \langle f, e_2\rangle e_1 \\ \Gamma_2 f = \langle f, e_2\rangle e_2 \\ \Gamma_3 f = \langle f, e_1\rangle e_3. \end{cases}$$

By a simple calculation, we have

$$\begin{cases} \Lambda_1^*: H_1 \to H, \Lambda_1^*(a_1e_1) = a_1e_1 + a_1e_2 \\ \Lambda_2^*: H_2 \to H, \Lambda_2^*(a_2e_2) = a_2e_3 - a_2e_1 \\ \Lambda_3^*: H_3 \to H, \Lambda_3^*(a_3e_3) = a_3e_2 \end{cases}; \begin{cases} \Gamma_1^*: H_1 \to H, \Gamma_1^*(a_1e_1) = a_1e_2 \\ \Gamma_2^*: H_2 \to H, \Gamma_2^*(a_2e_2) = a_2e_2 \\ \Gamma_3^*: H_3 \to H, \Gamma_3^*(a_3e_3) = a_3e_1. \end{cases}$$

For any $f = (c_1, c_2, c_3) \in H$, we have

$$\|K^*f\|^2 = (c_1+c_2)^2 + c_2^2 \le \sum_{j=1}^{3} \|\Lambda_j f\|^2 = (c_1+c_2)^2 + (c_3-c_1)^2 + c_2^2 \le 4\|f\|^2,$$

$$\sum_{j=1}^{3} \|\Gamma_j f\|^2 = c_2^2 + c_2^2 + c_1^2 \le 2\|f\|^2.$$

Obviously, $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. Since

$$\begin{aligned} Kf &= c_1e_2 + c_2e_2 + c_2e_3 = (c_1+c_2)e_2 + c_2e_3, \\ \sum_{j\in J} \Lambda_j^*\Gamma_j f &= \Lambda_1^*(c_2e_1) + \Lambda_2^*(c_2e_2) + \Lambda_3^*(c_1e_3) = (c_1+c_2)e_2 + c_2e_3, \\ \sum_{j\in J} \Gamma_j^*\Lambda_j f &= \Gamma_1^*(c_1+c_2)e_1 + \Gamma_2^*(c_3-c_1)e_2 + \Gamma_3^*(c_2e_3) = (c_2+c_3)e_2 + c_2e_1. \end{aligned}$$

It follows that $Kf = \sum_{j\in J} \Lambda_j^*\Gamma_j f$ for any $f \in H$. Therefore

$$Kf \ne \sum_{j\in J} \Gamma_j^*\Lambda_j f, \ \forall f \in H.$$

Note that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are not interchangeable in general. However, if we strengthen the condition, there exists another type of dual such that $\{\Lambda_j\}_{j\in J}$ and a sequence $\{\Theta_j\}_{j\in J}$ introduced by $\{\Gamma_j\}_{j\in J}$ are interchangeable in the subspace $R(K)$ of H.

Theorem 3.3. *Suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences as in* (3.1). *Then there exists a sequence $\{\Theta_j\}_{j\in J}$ such that*

$$f = \sum_{j\in J} \Lambda_j^*\Theta_j f, \quad \forall f \in R(K), \tag{3.2}$$

where $\Theta_j = \Gamma_j\left(K^\dagger\big|_{R(K)}\right)$. Moreover, $\{\Lambda_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are interchangeable for any $f \in R(K)$.

Proof. Since $R(K)$ is closed, there exists pseudo-inverse $K^\dagger$ of K such that $f = KK^\dagger f, \forall f \in R(K)$. It follows from (3.1) that

$$f = KK^\dagger f = \sum_{j\in J} \Lambda_j^*\Gamma_j K^\dagger f, \forall f \in R(K), \tag{3.3}$$

where $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences for H with respect to $\{H_j\}_{j\in J}$ and satisfy (3.1). Now, let $\Theta_j = \Gamma_j\left(K^\dagger\big|_{R(K)}\right)$. Since $K^\dagger\big|_{R(K)}$:

$R(K) \to H$, we obtain $\Theta_j : R(K) \to H_j$. For any $f \in R(K)$, we have $K^\dagger f \in H$, it follows that

$$\sum_{j\in J} \left\|\Gamma_j K^\dagger f\right\|^2 \le B \left\|K^\dagger f\right\|^2 \le B \left\|K^\dagger\right\|^2 \|f\|^2 . \tag{3.4}$$

Therefore, $\{\Theta_j\}_{j\in J}$ is a g-Bessel sequence for $R(K)$ with respect to $\{H_j\}_{j\in J}$.

Next, we prove that $\{\Lambda_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are interchangeable on $R(K)$. In fact, for any $f, g \in R(K)$, we have

$$\langle f, g\rangle = \left\langle \sum_{j\in J} \Lambda_j^*\Theta_j f, g \right\rangle = \sum_{j\in J} \langle \Theta_j f, \Lambda_j g\rangle = \left\langle f, \sum_{j\in J} \Theta_j^*\Lambda_j g \right\rangle, \tag{3.5}$$

that is, $f = \sum\limits_{j\in J} \Theta_j^*\Lambda_j f,\ \forall f \in R(K)$. □

Example 3.4. Let H, $\{\Lambda_j\}_{j\in J}$, $\{\Gamma_j\}_{j\in J}$ and K be as in Example 3.2. It is easy to get $R(K) = \overline{\text{span}}\{e_2, e_3\}$. Since $R(K)$ is closed, there exists the pseudo-inverse $K^\dagger$ of K such that $f = KK^\dagger f$ for any $f \in R(K)$. Hence, we obtain $KK^\dagger e_2 = e_2$, $KK^\dagger e_3 = e_3$. By the definition of K-g-frame and Definition 2.5, we have

$$K^\dagger e_1 = 0,\ K^\dagger e_2 = e_1,\ K^\dagger e_3 = e_2 - e_1.$$

For any $f = c_2e_2 + c_3e_3 \in R(K)$, we have

$$\begin{aligned}
\Theta_1 f &= \Gamma_1(K^\dagger|_{R(K)})f = \Gamma_1(c_2e_1 + c_3(e_2 - e_1)) = c_3e_1 = \langle f, e_3\rangle e_1,\\
\Theta_2 f &= \Gamma_2(K^\dagger|_{R(K)})f = \Gamma_2(c_2e_1 + c_3(e_2 - e_1)) = c_3e_2 = \langle f, e_3\rangle e_2,\\
\Theta_3 f &= \Gamma_3(K^\dagger|_{R(K)})f = \Gamma_3(c_2e_1 + c_3(e_2 - e_1))\\
&= (c_2 - c_3)e_3 = (\langle f, e_2\rangle - \langle f, e_3\rangle)e_3.
\end{aligned}$$

By direct calculation, we get $\Theta_1^*(a_1e_1) = a_1e_3$, $\Theta_2^*(a_2e_2) = a_2e_3$, $\Theta_3^*(a_3e_3) = a_3e_2 - a_3e_3$.

For any $f = c_2e_2 + c_3e_3 \in R(K)$, we have

$$\begin{aligned}
\sum_{j\in J} \Lambda_j^*\Theta_j f &= \Lambda_1^*(c_3e_1) + \Lambda_2^*(c_3e_2) + \Lambda_3^*(c_2 - c_3)e_3\\
&= c_3e_2 + c_3e_3 + (c_2 - c_3)e_2 = c_2e_2 + c_3e_3,\\
\sum_{j\in J} \Theta_j^*\Lambda_j f &= \Theta_1^*(c_2e_1) + \Theta_2^*(c_3e_2) + \Theta_3^*(c_2e_3)\\
&= c_2e_3 + c_3e_3 + c_2e_2 - c_2e_3 = c_2e_2 + c_3e_3.
\end{aligned}$$

Therefore, we have $f = \sum_{j\in J} \Lambda_j^*\Theta_j f = \sum_{j\in J} \Theta_j^*\Lambda_j f$ for any $f = c_2e_2 + c_3e_3 \in R(K)$.

In order to further enrich the theory of atomic systems of K-g-frames, we give several methods to construct K-g-frames in the following.

Theorem 3.5. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. By Proposition 3.1, we only need to prove that $\{\Lambda_j K^*\}_{j\in J}$ is a g-Bessel sequence and there exists a g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ such that

$$Kf = \sum_{j\in J} (\Lambda_j K^*)^* \Gamma_j f, \quad \forall f \in H. \tag{3.6}$$

Since $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$, we have

$$f = \sum_{j\in J} \Lambda_j^* \Lambda_j S^{-1} f, \quad \forall f \in H, \tag{3.7}$$

where S is the g-frame operator for $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda_j S^{-1}\}_{j\in J}$ is the canonical dual g-frame of $\{\Lambda_j\}_{j\in J}$.

For any $f \in H$, we have $K^* f \in H$. Then, it follows that

$$\sum_{j\in J} \|\Lambda_j K^* f\|^2 \le B \|K^* f\|^2 \le B \|K^*\|^2 \|f\|^2. \tag{3.8}$$

It implies that $\{\Lambda_j K^*\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. By (3.7), we have

$$Kf = \sum_{j\in J} K\Lambda_j^* \Lambda_j S^{-1} f = \sum_{j\in J} (\Lambda_j K^*)^* \Lambda_j S^{-1} f, \quad \forall f \in H. \tag{3.9}$$

By Proposition 3.1, we get that $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Corollary 3.6. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a g-basis for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Theorem 3.7. *If $T, K \in L(H)$ and $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j T^*\}_{j\in J}$ is a TK-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. By the definition of K-g-frame, we get

$$A \|K^* f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B \|f\|^2, \quad \forall f \in H. \tag{3.10}$$

Since $T \in L(H)$, we have $T^* f \in H$ for any $f \in H$. Then,

$$\begin{aligned} &A \left\|(TK)^* f\right\|^2 = A \|K^* T^* f\|^2 \\ \le &\sum_{j\in J} \|\Lambda_j T^* f\|^2 \le B \|T^* f\|^2 \le B \|T\|^2 \|f\|^2, \ \forall f \in H. \end{aligned} \tag{3.11}$$

Therefore, $\{\Lambda_j T^*\}_{j\in J}$ is a TK-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Corollary 3.8. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\left\{\Lambda_j (K^*)^N\right\}_{j\in J}$ is a K^{N+1}-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Remark 3. In Theorems 3.5 and 3.7, $R(K)$ doesn't need to be closed. In Theorem 3.7, even though $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$, it doesn't mean that $\{\Lambda_j T^*\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.

4. Stability of K-g-frames for Hilbert spaces

The Paley-Wiener theorems for frames have been given by Ole Christensen [4]. Perturbations of g-frames and their duals have been discussed in [15] and [17]. Stability of K-g-frames has been investigated in [1] and [23]. In this paper, we give other results about perturbations of K-g-frames.

Theorem 4.1. *Assume that $K \in L(H)$ and K has closed range. Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B and $\{\Gamma_j\}_{j\in J}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. If there exist constants $\lambda_1, \lambda_2, \mu \in [0,1)$ such that $\max\left\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\right\} < 1$ and*

$$\begin{aligned} &\left\|\sum_{j\in I}\left(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j\right) f\right\| \\ &\leq \lambda_1 \left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| + \lambda_2 \left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| + \mu\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}, \ \forall f \in H, \end{aligned} \tag{4.1}$$

where I is any finite subset of J, then $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$$A\frac{1-\left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)}{1+\lambda_2}\ ,\ B\frac{1+\lambda_1+\frac{\mu}{\sqrt{B}}}{1-\lambda_2}. \tag{4.2}$$

Proof. Assume that $I \subset J, |I| < +\infty$. For any $f \in H$, we have

$$\begin{aligned} \left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| &\leq \left\|\sum_{j\in I}\left(\Gamma_j^*\Gamma_j - \Lambda_j^*\Lambda_j\right) f\right\| + \left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| \\ &\leq (1+\lambda_1)\left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| + \lambda_2\left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| + \mu\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| \leq \frac{1+\lambda_1}{1-\lambda_2}\left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| + \frac{\mu}{1-\lambda_2}\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}.$$

Also, since

$$\begin{aligned}\left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| &= \sup_{\|g\|=1}\left|\left\langle\sum_{j\in I}\Lambda_j^*\Lambda_j f, g\right\rangle\right| = \sup_{\|g\|=1}\left|\left\langle\sum_{j\in I}\Lambda_j f, \Lambda_j g\right\rangle\right| \\ &\leq \left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}\sup_{\|g\|=1}\left(\sum_{j\in I}\|\Lambda_j g\|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{B}\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, for all $f\in H$, we have

$$\left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| \leq \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \leq \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\sqrt{B}\|f\|.$$

So $\sum_{j\in I}\Gamma_j^*\Gamma_j f$ is unconditionally convergent. Let

$$G: H\to H,\ Gf=\sum_{j\in I}\Gamma_j^*\Gamma_j f,\ f\in H.$$

Then G is well-defined and bounded and $\|G\| \leq \frac{(1+\lambda_1)B+\mu\sqrt{B}}{1-\lambda_2}$. For each $f\in H$, we have $\sum_{j\in J}\|\Gamma_j f\|^2 = \langle Gf, f\rangle \leq \|G\|\,\|f\|^2$. It implies that $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$.

Let S be the K-g-frame operator of $\{\Lambda_j\}_{j\in J}$. From (4.1), we obtain

$$\|(S-G)f\| \leq \lambda_1\|Sf\| + \lambda_2\|Gf\| + \mu\left(\sum_{j\in I}\|\Lambda_j f\|^2\right)^{\frac{1}{2}},\ \forall f\in H.$$

Then,

$$\begin{aligned}\left\|f-GS^{-1}f\right\| &\leq \lambda_1\|f\| + \lambda_2\left\|GS^{-1}f\right\| + \mu\left(\sum_{j\in I}\left\|\Lambda_j S^{-1}f\right\|^2\right)^{\frac{1}{2}} \\ &\leq \left(\lambda_1+\frac{\mu}{\sqrt{A}}\right)\|f\| + \lambda_2\left\|GS^{-1}f\right\|.\end{aligned}$$

Since $0 \le \max\left\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\right\} < 1$, by Lemma 2.9, we get

$$\frac{1-\lambda_2}{1+\left(\lambda_1+\frac{\mu}{\sqrt{A}}\right)} \le \left\|SG^{-1}\right\| \le \frac{1+\lambda_2}{1-\left(\lambda_1+\frac{\mu}{\sqrt{A}}\right)}.$$

Therefore,

$$\|G\| \ge \frac{A}{\|SG^{-1}\|}\|KK^*\| \ge \frac{A-\left(A\lambda_1+\mu\sqrt{A}\right)}{(1+\lambda_2)}\|KK^*\|.$$

By Lemma 2.8, $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Corollary 4.2. *Let $K \in L(H)$ and $\{\Lambda_j : \Lambda_j \in L(H, H_j)\}_{j\in J}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A, B and $\{\Gamma_j\}_{j\in J}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. If there exists constant $0 < R < A$ such that*

$$\left\|\sum_{j\in J}\left(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j\right)f\right\| \le R\|K^*f\|,\ \forall f \in H, \tag{4.3}$$

then $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $A-R$ and $B+R\sqrt{\frac{B}{A}}$.

Proof. It is clear that $\sum_{j\in J}\Gamma_j^*\Gamma_j f$ is convergent for any $f \in H$. Then,

$$\left\|\sum_{j\in J}\left(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j\right)f\right\| \le R\|K^*f\| \le \frac{R}{\sqrt{A}}\left(\sum_{j\in J}\|\Lambda_j f\|^2\right)^{\frac{1}{2}},\ \forall f \in H.$$

By letting $\lambda_1,\ \lambda_2 = 0$, $\mu = \frac{R}{\sqrt{A}}$ in Theorem 4.1, $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds $A-R$ and $B+R\sqrt{\frac{B}{A}}$. □

Theorem 4.3. *Let $K \in L(H)$ and $\{\Lambda_j : \Lambda_j \in L(H, H_j)\}_{j\in J}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A , B and $\{\Gamma_j\}_{j\in J}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. If there exist constants $\lambda_1, \lambda_2, \mu \in [0, 1)$ such that* $\max\left\{\lambda_1 + \frac{\mu}{A}, \lambda_2\right\} < 1$ *and*

$$\left\|\sum_{j\in I}\left(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j\right)f\right\| \le \lambda_1\left\|\sum_{j\in I}\Lambda_j^*\Lambda_j f\right\| + \lambda_2\left\|\sum_{j\in I}\Gamma_j^*\Gamma_j f\right\| + \mu\|K^*f\|,\ \forall f \in H, \tag{4.4}$$

where I is any finite subset of J, then $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$$A\frac{1-\left(\lambda_1+\frac{\mu}{A}\right)}{1+\lambda_2},\ B\frac{1+\lambda_1+\frac{\mu}{\sqrt{AB}}}{1-\lambda_2}. \tag{4.5}$$

Proof. The proof is similar to that of Theorem 4.1. $\square$

Theorem 4.4. *Let $K \in L(H)$ and $\{f_j\}_{j\in J}$ be a K-frame of H with bounds A, B and let $\{g_j\}_{j\in J}$ be a sequence of H. If there exist constants $\lambda_1, \lambda_2, \mu \in [0,1)$ such that* $\max\left\{\lambda_1 + \frac{\mu}{A}, \lambda_2\right\} < 1$ *and*

$$\left\|\sum_{j\in I} \langle f, f_j\rangle f_j - \langle f, g_j\rangle g_j\right\| \leq \lambda_1 \left\|\sum_{j\in I} \langle f, f_j\rangle f_j\right\| + \lambda_2 \left\|\sum_{j\in I} \langle f, g_j\rangle g_j\right\| + \mu \|K^* f\|,\ \forall f \in H, \tag{4.6}$$

where I is any finite subset of J, then $\{g_j\}_{j\in J}$ is a K-frame for H with bounds

$$A\frac{1-\left(\lambda_1+\frac{\mu}{A}\right)}{1+\lambda_2}\ ,\ B\frac{1+\lambda_1+\frac{\mu}{\sqrt{AB}}}{1-\lambda_2}. \tag{4.7}$$

Proof. For any $j \in J$, let

$$\Lambda_j : H \to C, \Lambda_j(f) = \langle f, f_j\rangle,$$

$$\Gamma_j : H \to C, \Gamma_j(f) = \langle f, g_j\rangle.$$

Then, $\{\Lambda_j : \Lambda_j \in L(H, \mathbb{C})\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$. Thus, condition (4.6) means (4.4). From Theorem 4.3, the conclusion is established. $\square$

Theorem 4.5. *Let $K \in L(H)$ and $\{\Lambda_j : \Lambda_j \in L(H, H_j)\}_{j\in J}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A , B and $\{\Gamma_j\}_{j\in J}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. If there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \in [0,1)$ such that* $\max\left\{\lambda_1 + \frac{\mu_1}{\sqrt{A}} + \frac{\mu_2}{A}, \lambda_2\right\} < 1$ *and for any $f \in H$,*

$$\left\|\sum_{j\in I}\left(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j\right) f\right\| \leq \lambda_1 \left\|\sum_{j\in I} \Lambda_j^*\Lambda_j f\right\| + \lambda_2 \left\|\sum_{j\in I} \Gamma_j^*\Gamma_j f\right\| + \mu_1 \left(\sum_{j\in I} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \mu_2 \|K^* f\|, \tag{4.8}$$

where I is any finite subset of J, then $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$$A\frac{1-\left(\lambda_1+\frac{\mu_1}{\sqrt{A}}+\frac{\mu_2}{A}\right)}{1+\lambda_2}\ ,\ B\frac{1+\lambda_1+\frac{\mu_1}{\sqrt{B}}+\frac{\mu_2}{\sqrt{AB}}}{1-\lambda_2}. \tag{4.9}$$

Proof. The proof is analogous to that of Theorem 4.1. $\square$

Based on the stability of frames in [17] and the knowledge of operator theory in [7, 12, 18], the corollary immediately follows from Theorem 3.7 of [1].

Corollary 4.6. *Let $K \in L(H)$, $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A , B and $\{\Gamma_j\}_{j\in J}$ be a sequence for H with respect to $\{H_j\}_{j\in J}$. If $\{\Lambda_j - \Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ with bound M and*

$$\sum_{j\in J} \|(\Lambda_j - \Gamma_j)\, f\|^2 \le R\, \|K^* f\|^2\, , \forall f \in H, \tag{4.10}$$

where $R < A$, then $\{\Gamma_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$$A\left(1 - \sqrt{\frac{R}{A}}\right)^2,\ \min\left\{ B\left(1 + \sqrt{\frac{R}{A}}\right)^2, B\left(1 + \sqrt{\frac{M}{B}}\right)^2 \right\}. \tag{4.11}$$

Proof. Since $\{\Lambda_j - \Gamma_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ with bound M, it follows that

$$\sum_{j\in J} \|\Lambda_j - \Gamma_j\|^2 \le M\|f\|^2, \forall f \in H.$$

By the triangle inequality, for any $f \in H$, we have

$$\begin{aligned}
\left(\sum_{j\in J} \|\Gamma_j f\|^2\right)^{\frac{1}{2}} &\le \left(\sum_{j\in J} \|(\Lambda_j - \Gamma_j) f\|^2\right)^{\frac{1}{2}} + \left(\sum_{j\in J} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\
&\le \left(\sqrt{M} + \sqrt{B}\right) \|f\| \\
&= \sqrt{B}\left(1 + \sqrt{\frac{M}{B}}\right) \|f\|.
\end{aligned}$$

Also, for any $f \in H$, we have

$$\begin{aligned}
\left(\sum_{j\in J} \|\Gamma_j f\|^2\right)^{\frac{1}{2}} &\ge \left(\sum_{j\in J} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} - \left(\sum_{j\in J} \|(\Lambda_j - \Gamma_j)\, f\|^2\right)^{\frac{1}{2}} \\
&\ge \left(\sum_{j\in J} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} - \sqrt{R}\, \|K^* f\| \\
&\ge \sqrt{A}\left(1 - \sqrt{\frac{R}{A}}\right) \|K^* f\|\, .
\end{aligned}$$

Similarly, we obtain $\left(\sum\limits_{j\in J}\|\Gamma_j f\|^2\right)^{\frac{1}{2}} \leq \sqrt{B}\left(1+\sqrt{\frac{R}{A}}\right)\|f\|$. Hence, the upper bound for K-g-frame $\{\Gamma_j\}_{j\in J}$ is

$$\min\left\{B\left(1+\sqrt{\frac{R}{A}}\right)^2, B\left(1+\sqrt{\frac{M}{B}}\right)^2\right\}$$

and the lower bound is $A\left(1-\sqrt{\frac{R}{A}}\right)^2$. $\square$

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