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GAUSS MAPS OF RULED SUBMANIFOLDS AND APPLICATIONS *I*

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Abstract. As a generalizing certain geometric property occurred on the helicoid of 3-dimensional Euclidean space regarding the Gauss map, we study ruled submanifolds in a Euclidean space with pointwise 1-type Gauss map of the first kind. In this paper, as new examples of cylindrical ruled submanifolds in Euclidean space, we construct generalized circular cylinders and characterize such ruled submanifolds and minimal ruled submanifolds of Euclidean space with pointwise 1-type Gauss map of the first kind.

1. Introduction

An immersion x of a manifold M into a Euclidean space $\mathbb{E}^m$ is said to be *of finite type* if it can be expressed as

$$x = x_0 + x_1 + \cdots + x_k$$

for some positive integer k, where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here Δ denotes the Laplace operator defined on M. If $\lambda_1, \ldots, \lambda_k$ are mutually different, M is said to be of k-type. In particular, the minimal submanifolds are very typical finite type submanifolds, namely 1-type submanifolds.

A ruled surface or a ruled submanifold is one of the typical geometric objects that many mathematicians have studied with great interest in the classical differential geometry. Due to Catalan's Theorem, the only minimal ruled surfaces

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in Euclidean 3-space $\mathbb{E}^3$ are the planes and the helicoids. J. M. Barbosa, M. Dajczer and L. P. Jorge investigated the minimal ruled submanifolds and showed that those of Euclidean space are the generalized helicoids [1]. By using the notion of finite type immersion, B.-Y. Chen et al. showed that a ruled surface of finite type in an m-dimensional Euclidean space is part of either a cylinder over a curve of finite type or a helicoid in $\mathbb{E}^3$ [4]. In particular, making use of the character of plane curves of finite type, we see that a ruled surface of finite type in $\mathbb{E}^3$ is part of a plane, a circular cylinder or a helicoid. And in [9], F. Dillen extended these results to ruled submanifolds of finite type in Euclidean space.

Since the notion of finite type immersion of Riemannian manifolds into Euclidean space was introduced by B.-Y. Chen in the late 1970's, such a notion has been extended to submanifolds in pseudo-Euclidean space and to smooth functions defined on submanifolds of Euclidean space or pseudo-Euclidean space [2]. Especially, two of the present authors completed the classification of the minimal ruled submanifolds in Minkowski space by considering two aspects whether rulings of the ruled submanifolds are non-degenerate or degenerate [14]. Also, in [13, 19], the ruled surfaces and ruled submanifolds of finite type were examined.

On the other hand, some studies were focused on submanifolds of Euclidean or pseudo-Euclidean space with the Gauss map of finite type. In [5], B.-Y. Chen and P. Piccini initiated the submanifolds in Euclidean space with finite type Gauss map so that they classified compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + C)$, where C is a constant vector and $\lambda \in \mathbb{R}$. After that, quite a few of studies on ruled surfaces and ruled submanifolds with finite type Gauss map in Euclidean space or pseudo-Euclidean space have been studied and classified ([10, 11, 12, 15, 16, 17, 18, 20]).

However, some surfaces including a helicoid have an interesting property concerning the Gauss map which looks like satisfying an eigenvalue problem. As a matter of fact, it is not: The helicoid in $\mathbb{E}^3$ parameterized by

$$x(u, v) = (u \cos v, u \sin v, av), \quad a \neq 0$$

has the Gauss map

$$G = \frac{1}{\sqrt{a^2 + u^2}}(a \sin v, -a \cos v, u).$$

Its Laplacian ΔG is given by

$$\Delta G = \frac{2a^2}{(a^2 + u^2)^2} G.$$

On the other hand, the right (or circular) cone C_a with parametrization

$$x(u, v) = (u \cos v, u \sin v, au), \quad a \geq 0$$

has the Gauss map

$$G = \frac{1}{\sqrt{1+a^2}}(a\cos v, a\sin v, -1)$$

which satisfies

$$\Delta G = \frac{1}{u^2}(G + (0, 0, \frac{1}{\sqrt{1+a^2}})$$

(cf. [6, 7]). The Gauss maps of examples above are similar to 1-type, but obviously different from the usual sense of 1-type Gauss map. Based on these, we define:

Definition 1.1. An oriented n-dimensional submanifold M of the Euclidean space $\mathbb{E}^m$ is said to have *pointwise 1-type* Gauss map if it satisfies the condition

$$\Delta G = f(G + C), \tag{1}$$

where f is a non-zero smooth function on M and C some constant vector. In particular, if C is zero, the Gauss map G is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* ([3, 6, 7, 8, 21]).

In [6, 7], M. Choi et al. proved that a ruled surface in 3-dimensional Euclidean space with pointwise 1-type Gauss map is part of a plane, a circular cylinder, a helicoid, a cylinder over a plane curve of infinite type or a circular cone. And, in [8, 22], ruled surfaces in pseudo-Euclidean space with pointwise 1-type Gauss map were studied.

We now raise a question: *Can we completely classify ruled submanifolds in Euclidean space with pointwise 1-type Gauss map of the first kind?*

In this paper, we study the ruled submanifolds in Euclidean space with pointwise 1-type Gauss map of the first kind and construct the new examples of ruled submanifolds called generalized circular cylinders. As a result, we completely classify ruled submanifolds of Euclidean space with pointwise 1-type Gauss map of the first kind.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let $x : M \to \mathbb{E}^m$ be an isometric immersion of an n-dimensional Riemannian manifold M into $\mathbb{E}^m$. Let $(x_1, x_2, \dots, x_n)$ be a local coordinate system of M. For the components g_{ij} of the Riemannian metric $\langle\cdot,\cdot\rangle$ on M induced from that of $\mathbb{E}^m$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}). Then the Laplace operator Δ on M is defined by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}}\sum_{i,j}\frac{\partial}{\partial x_i}(\sqrt{\mathcal{G}}g^{ij}\frac{\partial}{\partial x_j}).$$

We now choose an adapted local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ in $\mathbb{E}^m$ such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ normal to M.

The Gauss map $G : M \to G(n,m) \subset \mathbb{E}^N$ $(N = {}_mC_n)$, $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ of M is a smooth map which carries a point p in M to an oriented n-plane in $\mathbb{E}^m$ by the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$.

An inner product $\ll \cdot, \cdot \gg$ on $G(n,m) \subset \mathbb{E}^N$ is defined by

$$\ll e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = \det(\langle e_{i_l}, e_{j_k} \rangle),$$

where l, k run over the range $\{1, 2, \ldots, n\}$. Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \,|\, 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N$.

Now, we introduce the definition of a ruled submanifold M in $\mathbb{E}^m$. An $(r+1)$-dimensional submanifold M in $\mathbb{E}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{E}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, a parametrization of a ruled submanifold M in $\mathbb{E}^m$ can be given by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I, \; t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \ldots, r$. For each s, $E(s,r)$ is open in Span$\{e_1(s), e_2(s), \ldots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s), e_2(s), \ldots, e_r(s)$ along the curve α. We call $E(s,r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s,r)$ is parallel along α, or *non-cylindrical* otherwise.

Definition 2.1. An $(r+1)$-dimensional cylindrical ruled submanifold M is called a generalized circular cylinder $\Sigma_a \times \mathbb{E}^{r-1}$ if the base curve α is a circle and the generators of rulings are orthogonal to the plane containing the circle α, where $\Sigma_a = S^1(a) \times \mathbb{R}$ is a circular cylinder over a circle $S^1(a)$ of radius a in $\mathbb{E}^3$.

For later use, we need:

Lemma 2.1 (Lemma 2.2 in [1])**.** *Given a curve α and orthonormal vector fields $e_1, e_2, \ldots, e_n$ along α in a Riemannian manifold $\bar{M}$ with the Riemannian connection $\bar{D}$, we can always choose orthonormal vector fields $f_1, \ldots, f_n$ along α such that*:

(a) *The sets of vectors $\{f_j(s) : 1 \leq j \leq n\}$ and $\{e_j(s) : 1 \leq j \leq n\}$ generate the same subspace of $T_{\alpha(s)}\bar{M}$.*

(b) *The vector fields $(\bar{D}/ds)f_i(s)$ are normal to the subspace of $T_{\alpha(s)}\bar{M}$ spanned by $\{f_j(s) : 1 \leq j \leq n\}$ for all $i = 1, 2, \ldots, n$.*

3. Ruled submanifolds and Gauss map

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{E}^m$. Then, the base curve α can be chosen to be orthogonal to the rulings by taking an integral curve of the field of normal directions to the rulings of M. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.1, we may choose orthonormal vector fields $e_1(s), \ldots, e_r(s)$ along α satisfying

$$\langle \alpha'(s), e_i(s) \rangle = 0, \ \langle e_i'(s), e_j(s) \rangle = 0 \ \text{ for } s \in I \text{ and } i, j = 1, 2, \ldots, r. \tag{2}$$

A parametrization of M is then obtained as

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \ s \in I. \tag{3}$$

In this paper, we always assume that the parametrization (3) satisfies the condition (2). Then, M has the Gauss map

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{q^{1/2}} (\Phi + \Psi), \quad \text{with} \quad \Psi = \sum_{i=1}^{r} t_i \Psi_i, \tag{4}$$

where q is the function of $s, t_1, t_2, \ldots, t_r$ defined by $q = \langle x_s, x_s \rangle$, Φ and Ψ_i $(i = 1, 2, \ldots, r)$ are vector fields along α given by

$$\Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

Now, we separate the cases into two typical types of ruled submanifolds which are cylindrical or non-cylindrical. First of all, we consider the following lemma.

Lemma 3.1. *Suppose that a unit speed curve $\alpha(s)$ in an m-dimensional Euclidean space $\mathbb{E}^m$ defined on an interval I satisfies*

$$\alpha'''(s) = f(s)(\alpha'(s) + C), \tag{5}$$

where f is a function and C a constant vector in $\mathbb{E}^m$. Then, the curve α lies in a 3-dimensional Euclidean space $\mathbb{E}^3$. In particular, if the constant vector C is zero, we see that α is a plane curve.

Proof. We fix a point $s_0 \in I$. Let us denote by V the linear span of $\{\alpha'(s_0), \alpha''(s_0), C\}$. Then V is a at most 3-dimensional space in $\mathbb{E}^m$.

For any vector a in the orthogonal complement $V^\perp$ of V, we consider the function $h_a(s)$ defined by $h_a(s) = \langle a, \alpha'(s) \rangle$. Then, it follows from (5) that

$$h_a''(s) = f(s) h_a(s). \tag{6}$$

Hence, the function $h_a(s)$ is a solution of a second order linear differential equation with initial condition $h_a(s_0) = h_a'(s_0) = 0$. This shows that the function $h_a(s)$ vanishes identically on the interval I. Thus, we have $\alpha'(s) \in V$ for all $s \in I$, which shows that the curve α lies in a parallel displacement $\alpha(s_0) + V$ of the space V. This completes the proof. $\square$

Theorem 3.2. *A cylindrical ruled submanifold M in $\mathbb{E}^m$ has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a generalized circular cylinder.*

Proof. Let M be an $(r+1)$-dimensional cylindrical ruled submanifold in $\mathbb{E}^m$, which is parameterized by (3). We may assume that $e_1, e_2, \ldots, e_r$ generating the rulings are constant vectors.

Then, $q \equiv 1$ and the Laplace operator Δ of M is expressed by

$$\Delta = -\frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \frac{\partial^2}{\partial t_i^2}$$

and the Gauss map G of M is given by

$$G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

If we denote by Δ' the Laplace operator of α, that is $\Delta' = -\frac{\partial^2}{\partial s^2}$, we have the Laplacian ΔG of the Gauss map

$$\Delta G = \Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

We now suppose that the Gauss map G is of pointwise 1-type of the first kind, that is $\Delta G = fG$ for some function f. Then the condition $\Delta G = fG$ is rewritten as

$$\Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r = f\alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

Therefore we have

$$\Delta' \alpha' = f\alpha',$$

which shows that the function f depends only on s. It follows that

$$-\alpha'''(s) = f(s)\alpha'(s). \tag{7}$$

Then, Lemma 3.1 implies that α is a plane curve and the function f is given by $f = \langle \alpha'', \alpha'' \rangle$, which is the squared curvature function of α. By considering the Frenet formula for α satisfying (7), we easily see that the curvature of the base curve is non-zero constant. Thus, the plane curve α is part of a circle. Therefore, M is an open part of a generalized circular cylinder. The converse is straightforward. □

Next, we deal with the case that M is non-cylindrical. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (3) in $\mathbb{E}^m$. Then, we have

$$x_s = \alpha'(s) + \sum_{j=1}^{r} t_j e_j'(s), \quad x_{t_i} = e_i(s)$$

for $s \in I$ and $i = 1, 2, \ldots, r$. The aforementioned function q is given by

$$q = \langle x_s, x_s \rangle = 1 + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j, \tag{8}$$

where $u_i = \langle \alpha', e_i' \rangle$, $w_{ij} = \langle e_i', e_j' \rangle$, $i, j = 1, \dots, r$. Note that q is a polynomial in $t = (t_1, \dots, t_r)$ with functions in s as coefficients and the degree of q is 2. Then, the Laplace operator Δ of M is obtained by

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\frac{\partial^2}{\partial t_i^2}. \tag{9}$$

Proposition 3.3. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold of $\mathbb{E}^m$ parameterized by* (3) *satisfying* (2)*. Suppose some generators $e_{j_1}, e_{j_2}, \dots, e_{j_k}$ $(1 \le k < r)$ of the rulings are constant vectors along α. Then, M has pointwise 1-type Gauss map if and only if the ruled submanifold M_1 has pointwise 1-type Gauss map, where M_1 is the non-cylindrical ruled submanifold over the base curve α with the rulings generated by e_j for $j \ne j_1, j_2, \dots, j_k$.*

Proof. Suppose that M is an $(r+1)$-dimensional non-cylindrical ruled submanifold of $\mathbb{E}^m$ parameterized by (3) with $e_{j_i}' = \mathbf{0}$ for all $i = 1, 2, \dots, k$. By rearranging the indices, we may assume that $j_1, \dots, j_k$ are $r-k+1, \dots, r$. Also, M can be expressed as $M = M_1 \times \mathbb{E}^k$, where M_1 is parameterized by

$$x = x(s, t_1, t_2, \cdots, t_{r-k}) = \alpha(s) + \sum_{i=1}^{r-k} t_i e_i(s). \tag{10}$$

It is easy to show that the Gauss map G on M satisfies

$$G = G_1 \wedge C_0 \quad \text{and} \quad \Delta G = (\Delta_1 G_1) \wedge C_0, \tag{11}$$

where Δ_1 is the Laplace operator on M_1, G_1 the Gauss map on M_1 and C_0 the constant vector field defined by $C_0 = e_{r-k+1} \wedge \cdots \wedge e_r$.

Choose orthonormal vector fields $e_{r+1}, \dots, e_m$ of the normal space of M along α. If we put $e_0(s) = \alpha'(s)$, then $\{e_{i_1} \wedge \cdots \wedge e_{i_{r+1}} \,|\, 0 \le i_1 < \cdots < i_{r+1} \le m\}$ is an orthonormal basis of E^N which contains the Grassmannian manifold $G(r+1, m)$.

Suppose that M has pointwise 1-type Gauss map satisfying (1). Let us denote by $V = E^k \subset E^m$ the space spanned by the constant vectors $e_{r-k+1}, \dots,$ e_r. Then, using the basis elements of $G(r+1, m)$ the constant vector C is uniquely decomposed as follows:

$$C = C_1 \wedge C_0 + D,$$

where C_1 and D are constant vectors such that each component of C_1 is orthogonal to V and each term of D does not contain all of $e_{r-k+1}, \dots, e_r$.

If we compare (1) and (11) and take into account of the linearly independency of the basis elements of $G(r+1, m)$, we see that

$$D = \mathbf{0}$$

and

$$\Delta G = (\Delta_1 G_1) \wedge C_0 = f(G_1 + C_1) \wedge C_0,$$

from which, we see that the function f depends on $s, t_1, t_2, \dots, t_{r-k}$. This shows that the Gauss map G_1 of M_1 is of pointwise 1-type satisfying $\Delta_1 G_1 = f(G_1 + C_1)$.

The converse is straightforward. □

Based on Proposition 3.3, without loss of generality, we may assume that $e_j'(s) \neq \mathbf{0}$ for all $j = 1, 2, \dots, r$ on the domain I of α. From now on, for a polynomial $F(t)$ in $t = (t_1, t_2, \dots, t_r)$, deg $F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \dots, t_r)$ unless otherwise stated.

Now, we suppose that M has pointwise 1-type Gauss map of the first kind, i.e., $\Delta G = fG$. Using (4) and (9), this condition is written as

$$\begin{aligned}&(\frac{\partial q}{\partial s})^2(\Phi + \sum_{j=1}^r \Psi_j t_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\Phi' + \sum_{j=1}^r \Psi_j' t_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(\Phi + \sum_{j=1}^r \Psi_j t_j)\\ &+ q^2(\Phi'' + \sum_{j=1}^r \Psi_j'' t_j) + \frac{1}{2}q\sum_{i=1}^r(\frac{\partial q}{\partial t_i})^2(\Phi + \sum_{j=1}^r \Psi_j t_j) - \frac{1}{2}q^2\sum_{i=1}^r \frac{\partial q}{\partial t_i}\Psi_i\\ &- \frac{1}{2}q^2\sum_{i=1}^r \frac{\partial^2 q}{\partial t_i^2}(\Phi + \sum_{j=1}^r \Psi_j t_j) + fq^3(\Phi + \sum_{j=1}^r \Psi_j t_j) = \mathbf{0}\end{aligned} \tag{12}$$

for some non-zero function f, where $\mathbf{0}$ denotes zero vector. For the vector fields $\Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r$ and $\Psi_j = e_j' \wedge e_1 \wedge \cdots \wedge e_r$ $(j = 1, 2, \dots, r)$, we put

$$\begin{aligned}\ll \Phi, \Phi'' \gg &= -\mu + 2\sum_{k=1}^r u_k^2 - \sum_{k=1}^r w_{kk},\\ \ll \Phi, \Psi_i'' \gg &= \tilde{y}_i + 2\sum_{k=1}^r u_k w_{ik} - \sum_{k=1}^r u_i w_{kk},\\ \ll \Psi_j, \Phi'' \gg &= p_j + 2\sum_{k=1}^r u_k w_{jk} - \sum_{k=1}^r u_j w_{kk},\\ \ll \Psi_j, \Psi_i'' \gg &= \sigma_{ji} + 2\sum_{k=1}^r w_{jk} w_{ik} - \sum_{k=1}^r w_{ji} w_{kk},\\ \ll \Phi, \Psi_i \gg &= u_i, \quad \ll \Phi, \Psi_i' \gg = \tilde{x}_i,\\ \ll \Psi_j, \Phi' \gg &= \tilde{z}_j, \quad \ll \Psi_j, \Psi_i \gg = w_{ji}, \quad \ll \Psi_j, \Psi_i' \gg = \xi_{ji},\end{aligned}$$

where $\mu = \langle \alpha'', \alpha'' \rangle$, $\tilde{x}_i = \langle \alpha', e_i'' \rangle$, $\tilde{y}_i = \langle \alpha', e_i''' \rangle$, $\tilde{z}_i = \langle \alpha'', e_i' \rangle$, $p_i = \langle \alpha''', e_i' \rangle$, $\xi_{ij} = \langle e_i', e_j'' \rangle$ and $\sigma_{ij} = \langle e_i', e_j''' \rangle$ for $i, j = 1, 2, \dots, r$. We easily see that

$$u_i'(s) = \tilde{x}_i(s) + \tilde{z}_i(s) \quad \text{and} \quad w_{ij}'(s) = \xi_{ij}(s) + \xi_{ji}(s). \tag{13}$$

If we take the inner product with the vector Φ to equation (12), then we obtain

$$\begin{aligned}&(\frac{\partial q}{\partial s})^2(1+\sum_{j=1}^r t_ju_j)-\frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^r t_j\tilde{x}_j)-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1+\sum_{j=1}^r t_ju_j)\\&+q^2(\phi+\sum_{j=1}^r t_j\varphi_j)+\frac{1}{2}q\sum_{i=1}^r(\frac{\partial q}{\partial t_i})^2(1+\sum_{j=1}^r t_ju_j)-\frac{1}{2}q^2\sum_{i=1}^r\frac{\partial q}{\partial t_i}u_i\\&-\frac{1}{2}q^2\sum_{i=1}^r\frac{\partial^2 q}{\partial t_i^2}(1+\sum_{j=1}^r t_ju_j)+fq^3(1+\sum_{j=1}^r t_ju_j)=0,\end{aligned}$$

where we put

$$\phi=\ll\Phi,\Phi''\gg\quad\text{and}\quad\varphi_i=\ll\Phi,\Psi_i''\gg.$$

By putting

$$\begin{aligned}P(t)=&(\frac{\partial q}{\partial s})^2(1+\sum_{j=1}^r u_jt_j)-\frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^r \tilde{x}_jt_j)-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1+\sum_{j=1}^r u_jt_j)\\&+q^2(\phi+\sum_{j=1}^r \varphi_jt_j)+\frac{1}{2}q\sum_{i=1}^r(\frac{\partial q}{\partial t_i})^2(1+\sum_{j=1}^r u_jt_j)\\&-\frac{1}{2}q^2\sum_{i=1}^r\frac{\partial q}{\partial t_i}u_i-\frac{1}{2}q^2\sum_{i=1}^r\frac{\partial^2 q}{\partial t_i^2}(1+\sum_{j=1}^r u_jt_j),\end{aligned}\tag{14}$$

we may assume that the function f is the rational function in t with functions in s as coefficients of the form

$$f=-\frac{P(t)}{q^3(1+\sum_{j=1}^r u_jt_j)}.\tag{15}$$

Substituting (15) into (12) and multiplying $(1+\sum_{j=1}^r u_jt_j)$ with the equation obtained in such a way, we get

(16)

$$\begin{aligned}&-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^r t_j\Psi_j')(1+\sum_{k=1}^r u_kt_k)+\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi+\sum_{j=1}^r t_j\Psi_j)(\sum_{k=1}^r\tilde{x}_kt_k)\\&+q^2(\Phi''+\sum_{j=1}^r t_j\Psi_j'')(1+\sum_{k=1}^r u_kt_k)-q^2(\Phi+\sum_{j=1}^r t_j\Psi_j)(\phi+\sum_{k=1}^r\varphi_kt_k)\\&-\frac{1}{2}q^2\sum_{i=1}^r(\frac{\partial q}{\partial t_i})\Psi_i(1+\sum_{k=1}^r u_kt_k)+\frac{1}{2}q^2\sum_{i=1}^r(\frac{\partial q}{\partial t_i})u_i(\Phi+\sum_{j=1}^r t_j\Psi_j)=\mathbf{0}.\end{aligned}$$

We rewrite (16) in the following form

$$-\frac{3}{2}(\frac{\partial q}{\partial s})R(t)=qQ(t),\tag{17}$$

where

$$R(t) = (\Phi' + \sum_{j=1}^{r} t_j\Psi_j')(1 + \sum_{k=1}^{r} u_k t_k) - (\Phi + \sum_{j=1}^{r} t_j\Psi_j)(\sum_{k=1}^{r} \tilde{x}_k t_k) \tag{18}$$

and

$$\begin{aligned} Q(t) = & -(\Phi'' + \sum_{j=1}^{r} t_j\Psi_j'')(1 + \sum_{k=1}^{r} u_k t_k) + (\Phi + \sum_{j=1}^{r} t_j\Psi_j)(\phi + \sum_{k=1}^{r} \varphi_k t_k) \\ & + \frac{1}{2}\sum_{i=1}^{r}(\frac{\partial q}{\partial t_i})\Psi_i(1 + \sum_{k=1}^{r} u_k t_k) - \frac{1}{2}\sum_{i=1}^{r}(\frac{\partial q}{\partial t_i})u_i(\Phi + \sum_{j=1}^{r} t_j\Psi_j). \end{aligned}$$

Suppose that $\frac{\partial q}{\partial s} \neq 0$ on some open interval I_1. We show that it is a contradiction no matter what the function q is of the form either $q = (1+\sum_{i=1}^{r} u_i t_i)^2$ or $q \neq (1 + \sum_{i=1}^{r} u_i t_i)^2$. In order to do that, we need the following lemma.

Lemma 3.4. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3) *in $\mathbb{E}^m$ with pointwise 1-type Gauss map of the first kind. Suppose that $\frac{\partial q}{\partial s} \neq 0$ on some open interval I_1. If $q(t) \neq (1+\sum_{i=1}^{r} u_i t_i)^2$, then $R(t)$ in* (18) *has to be expressed as*

$$R(t) = q(t)B(s) \tag{19}$$

for some vector field $B(s)$ along α.

Proof. We consider the following two cases according to q and $\frac{\partial q}{\partial s}$ whether they are relatively prime or not. First, suppose that q and $\frac{\partial q}{\partial s}$ are relatively prime. It is obvious that (19) holds.

Next, suppose that q and $\frac{\partial q}{\partial s}$ are not relatively prime. Without loss of generality, we may assume that $q = (1 + \sum_{i=1}^{r} a_i(s)t_i)(1 + \sum_{i=1}^{r} b_i(s)t_i)$ and $\frac{\partial q}{\partial s} = (1 + \sum_{i=1}^{r} a_i(s)t_i)(\sum_{j=1}^{r} c_j t_j)$ for some functions $a_i(s), b_i(s), c_i(s)$ of s and $i = 1, 2, \ldots, r$. Since $q = 1 + \sum_i 2u_i t_i + \sum_{i,j} w_{ij} t_i t_j$ and $\frac{\partial q}{\partial s} = \sum_i 2u_i' t_i + \sum_{i,j} w_{ij}' t_i t_j$, we can see that

$$a_i + b_i = 2u_i \quad c_i = 2u_i' \quad \text{and} \quad w_{ij} = a_i(2u_j - a_j). \tag{20}$$

Since $w_{ij} = w_{ji}$ for all i, j, we have

$$a_i u_j = a_j u_i \tag{21}$$

for all $i, j = 1, 2, \ldots, r$. Since a_j, b_j, c_j are not all vanishing functions for all $j = 1, 2, \ldots, r$, there exists some j_0 such that $a_{j_0} \neq 0$. Together with (20) and (21), we see that $a_i = 0$ if and only if $u_i = 0$ and $b_i = 0$ for all $i = 1, 2 \ldots, r$.

On the other hand, from (17), $R(t)$ and $Q(t)$ must be of the form

$$\begin{aligned} R(t) =& (1+\sum_{i=1}^{r}(2u_i-a_i)t_i)(\mathbf{a}(s)+\sum_{j=1}^{r}\mathbf{b}_j(s)t_j), \\ Q(t) =& -\frac{3}{2}(\sum_{j=1}^{r}2u_j't_j)(\mathbf{a}(s)+\sum_{j=1}^{r}\mathbf{b}_j(s)t_j) \end{aligned} \tag{22}$$

for some vector fields $\mathbf{a}(s)$ and $\mathbf{b}_j(s)$ along α for $j=1,2,\dots,r$. By comparing the constant terms with respect to t in (18) and (22), we can see that

$$\mathbf{a}(s) = \Phi'(s). \tag{23}$$

Putting (23) into (22) and then, considering the coefficients of terms containing t_{j_0} in (18), we get

$$\mathbf{b}_{j_0}(s) = (a_{j_0}-u_{j_0})\Phi' + \Psi_{j_0}' - \tilde{x}_{j_0}\Phi. \tag{24}$$

Now, we have two equations to express $Q(t)$. With the aid of (23) and (24), comparing the coefficients of terms containing $t_{j_0}^0$, $t_{j_0}^1$, $t_{j_0}^2$ of these equations, we have the following equations:

$$-\Phi'' + \phi\Phi + \sum_{i=1}^{r}u_i\Psi_i - \sum_{i=1}^{r}u_i^2\Phi = 0, \tag{25}$$

$$\begin{aligned} &-u_{j_0}\Phi'' - \Psi_{j_0}'' + \varphi_{j_0}\Phi + \phi\Psi_{j_0} + (\sum_{i=1}^{r}u_i\Psi_i)u_{j_0} + \sum_{i=1}^{r}w_{ij_0}\Psi_i \\ &-(\sum_{i=1}^{r}u_i^2)\Psi_{j_0} - (\sum_{i=1}^{r}u_iw_{ij_0})\Phi = -3u_{j_0}'\Phi', \end{aligned} \tag{26}$$

$$\begin{aligned} &-u_{j_0}\Psi_{j_0}'' + \varphi_{j_0}\Psi_{j_0} + (\sum_{i=1}^{r}w_{ij_0}\Psi_i)u_{j_0} - (\sum_{i=1}^{r}u_iw_{ij_0})\Psi_{j_0} \\ =& -3u_{j_0}'((a_{j_0}-u_{j_0})\Phi' + \Psi_{j_0}' - \tilde{x}_{j_0}\Phi). \end{aligned} \tag{27}$$

Substituting (25) into (26), we obtain

$$\begin{aligned} &-u_{j_0}\phi\Phi + (\sum_{i=1}^{r}u_i^2)u_{j_0}\Phi - \Psi_{j_0}'' + \varphi_{j_0}\Phi + \phi\Psi_{j_0} + \sum_{i=1}^{r}w_{ij_0}\Psi_i \\ &-(\sum_{i=1}^{r}u_i^2)\Psi_{j_0} - (\sum_{i=1}^{r}u_iw_{ij_0})\Phi = -3u_{j_0}'\Phi'. \end{aligned} \tag{28}$$

Multiplying (28) with u_{j_0} and putting the equation obtained in such a way into (27), we have

$$(u_{j_0}\phi - (\sum_{i=1}^{r}u_i^2)u_{j_0} - \varphi_{j_0} + \sum_{i=1}^{r}u_iw_{ij_0})(\Psi_{j_0} - u_{j_0}\Phi) \tag{29}$$

$$= -3u'_{j_0}(a_{j_0}\Phi' + \Psi'_{j_0} - \tilde{x}_{j_0}\Phi).$$

If we apply Lemma 2.1 to the normal space $T^{\perp}_{\alpha(s)}M$ of M, then there exists an orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $T^{\perp}_{\alpha(s)}M$ satisfying

$$\langle e'_a(s), e_b(s)\rangle = 0$$

for all $a, b = r+1, \ldots, m-1$. Then, we can put

$$\begin{aligned} \alpha'' &= -\sum_{i=1}^{r} u_i e_i - \sum_{a=r+1}^{m-1} u_a e_a, \\ e'_i &= u_i \alpha' + \sum_{a=r+1}^{m-1} \langle e'_i, e_a\rangle e_a, \end{aligned} \tag{30}$$

where $u_a = \langle \alpha', e'_a\rangle$ for $a = r+1, \ldots, m-1$. Together with the definitions of Φ, Ψ_j and (30), by straightforward computations, equation (29) can be rewritten as

$$\begin{aligned} (31)\quad & (u_{j_0}\phi - (\sum_{i=1}^{r} u_i^2)u_{j_0} - \varphi_{j_0} + \sum_{i=1}^{r} u_i w_{ij_0})(\Psi_{j_0} - u_{j_0}\Phi) \\ = & -3u'_{j_0}\{ \sum_{a=r+1}^{m-1} (-a_{j_0}u_a + \langle e''_{j_0}, e_a\rangle)\xi_a \\ & + \sum_{i=1}^{r}\sum_{a=r+1}^{m-1} \langle (a_{j_0} + u_{j_0})e'_i - u_i e'_{j_0}, e_a\rangle \eta_{ia} \\ & + \sum_{i=1}^{r}\sum_{a,b=r+1}^{m-1} \langle e'_{j_0}, e_a\rangle\langle e'_i, e_b\rangle e_a \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_b \wedge e_{i+1} \wedge \cdots \wedge e_r\}, \end{aligned}$$

where $\xi_a = e_a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r$ and $\eta_{ia} = \alpha' \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_a \wedge e_{i+1} \wedge \cdots \wedge e_r$ for all $a = r+1, \ldots, m-1$. Note that the vectors $\eta_{ia} = \alpha' \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_a \wedge e_{i+1} \wedge \cdots \wedge e_r$ are orthogonal to all of other vectors in (31) for all $a = r+1, \ldots, m-1$. This implies that for all $i = 1, 2, \ldots, r$ and $a = r+1, \ldots, m-1$,

$$u'_{j_0}\langle (a_{j_0} + u_{j_0})e'_i - u_i e'_{j_0}, e_a\rangle = 0.$$

Suppose that $u'_i = 0$ for all $i = 1, 2, \ldots, r$ on $I_2 \subset I_1$. Then, $c_i = 0$ on I_2 and hence $\frac{\partial q}{\partial s} = 0$ on I_2, which is a contradiction. Thus, we may assume that $u'_{j_0} \neq 0$. Therefore, we get

$$(a_{j_0} + u_{j_0})e'_i - u_i e'_{j_0} = a_{j_0} u_i \alpha' \tag{32}$$

for all $i = 1, 2, \ldots, r$ and $a = r+1, \ldots, m-1$. Taking the inner product e'_k to the both sides of (32) for some $k \in \{1, 2, \ldots, r\}$, we get

$$(a_{j_0} + u_{j_0})w_{ik} - u_i w_{j_0 k} = a_{j_0} u_i u_k \tag{33}$$

for all $i, k = 1, 2, \ldots, r$.

Since for any i, j, $w_{ij} = a_i(2u_j - a_j)$, (33) implies

$$2a_i a_{j_0} u_k - a_i a_{j_0} a_k + 2a_i u_{j_0} u_k - a_i a_k u_{j_0} - 3a_{j_0} u_i u_k + a_{j_0} a_k u_i = 0 \tag{34}$$

for all $i, k = 1, 2, \dots, r$. Making use of (21), we see that (34) can be simplified as

$$a_i a_{j_0} u_k - a_i a_{j_0} a_k - a_{j_0} u_i u_k + a_{j_0} a_k u_i = 0$$

for all $i, k = 1, 2, \dots, r$. Since $a_{j_0} \neq 0$, we have

$$a_i u_k - a_i a_k - u_i u_k + a_k u_i = 0,$$

which implies

$$(a_i - u_i)(a_k - u_k) = 0$$

for all $i, k = 1, 2, \dots, r$. Thus, we can see that $a_j = u_j$ for all $j = 1, 2, \dots, r$. Hence, $w_{ij} = u_i u_j$ for all $i, j = 1, 2, \dots, r$ by virtue of (21), which leads to a contradiction to our assumption: $q(t) \neq (1 + \sum_{i=1}^r u_i t_i)^2$. Therefore, we conclude that q and $\frac{\partial q}{\partial s}$ are relatively prime. This completes the proof. $\square$

We now prove $\frac{\partial q}{\partial s} = 0$ by considering the following two cases depending on the function q which can be expressed as $q(t) \neq (1 + \sum_{i=1}^r u_i t_i)^2$ or $q(t) = (1 + \sum_{i=1}^r u_i t_i)^2$.

Case 1. Suppose that $\frac{\partial q}{\partial s} \neq 0$ on an open interval I_1. Let $q(t) \neq (1 + \sum_{i=1}^r u_i t_i)^2$. By Lemma 3.4, we may put $R(t)$ by

$$R(t) = q(t)B(s)$$

for some vector field $B(s)$ along α.

$$\begin{aligned}
&(\Phi' + \sum_{j=1}^r t_j \Psi_j')(1 + \sum_{k=1}^r u_k t_k) - (\Phi + \sum_{j=1}^r t_j \Psi_j)(\sum_{k=1}^r \tilde{x}_k t_k) \\
&= B(s)(1 + \sum_{i=1}^r 2u_i t_i + \sum_{j,i=1}^r w_{ij} t_i t_j).
\end{aligned} \tag{35}$$

Considering the constant terms in (35) with respect to t, we see that

$$B(s) = \Phi'(s).$$

Next, comparing the coefficients of the terms containing t_i and $t_i t_j$ for any i and j in (35) $(i, j = 1, 2, \dots, r)$, we have the following:

$$\Psi_i' = u_i \Phi' + \tilde{x}_i \Phi, \tag{36}$$

$$u_i \Psi_j' + u_j \Psi_i' - \tilde{x}_i \Psi_j - \tilde{x}_j \Psi_i = 2w_{ij} \Phi'. \tag{37}$$

Taking the inner product with Ψ_j to the both sides of (36), we obtain

$$\xi_{ji} = u_i \tilde{z}_j + \tilde{x}_i u_j.$$

So we get

$$\begin{aligned}\xi_{ji} + \xi_{ij} &= (u_i\tilde{z}_j + \tilde{x}_i u_j) + (u_j\tilde{z}_i + \tilde{x}_j u_i) \\ &= u_i(\tilde{x}_j + \tilde{z}_j) + u_j(\tilde{x}_i + \tilde{z}_i).\end{aligned} \tag{38}$$

Due to (13), (38) yields

$$w'_{ij} = u_i u'_j + u_j u'_i$$

for $i, j = 1, 2, \ldots, r$. Therefore, we have

$$w_{ij} = u_i u_j + c_{ij} \tag{39}$$

for some constants c_{ij} and $i, j = 1, 2, \ldots, r$.

Let $e_{r+1}, e_{r+2}, \ldots, e_{m-1}$ be the orthogonal normal vector fields to M along α. If we put

$$e'_i = u_i\alpha' + \sum_{a=r+1}^{m-1} \langle e'_i, e_a\rangle e_a,$$

then the constants c_{ij} are given by

$$c_{ij} = \sum_{a=r+1}^{m-1} \langle e'_i, e_a\rangle\langle e'_j, e_a\rangle$$

for $i, j = 1, 2, \ldots, r$.

Putting (39) and (36) into (37), we obtain

$$2c_{ij}\Phi' = u_i\tilde{x}_j\Phi + u_j\tilde{x}_i\Phi - \tilde{x}_i\Psi_j - \tilde{x}_j\Psi_i. \tag{40}$$

Again, taking the inner product with Ψ_k to (40) for $k = 1, 2, \ldots, r$, we have

$$2c_{ij}\tilde{z}_k = u_i\tilde{x}_j u_k + u_j\tilde{x}_i u_k - \tilde{x}_i w_{jk} - \tilde{x}_j w_{ik} \tag{41}$$

for $i, j, k = 1, 2, \ldots, r$. By (39) and (41), we get

$$2c_{ij}\tilde{z}_k = -c_{jk}\tilde{x}_i - c_{ik}\tilde{x}_j. \tag{42}$$

Because the function $q \neq (1+\sum_{i=1}^{r} u_i t_i)^2$, there must be a non-zero constant c_{ik} defined in (39) for some i and k . If $c_{ik} \neq 0$, we can see easily that $c_{ii} \neq 0$ and $c_{kk} \neq 0$. Then, by replacing j, k with i in (42), for the case $c_{ii} \neq 0$, we obtain

$$u'_i = \tilde{x}_i + \tilde{z}_i = 0.$$

Now, we consider the case that $c_{ik} = 0$ for some i, k. If $c_{ii} \neq 0$ and $c_{kk} \neq 0$, we see easily that u_i and u_k are constant. Note that if $c_{i_0 i_0} = 0$ for some i_0, then $c_{i_0 k} = 0$ for all $k = 1, 2, \ldots, r$. Indeed, since $c_{i_0 i_0} = 0$, $w_{i_0 i_0} = u_{i_0}^2$ which implies $e'_{i_0} = u_{i_0}\alpha'$. Then, by definition of the functions, we have $w_{i_0 k} = u_{i_0} u_k$ for all $k = 1, 2, \ldots, r$.

So we consider the set $\Lambda = \{i \,|\, c_{ii} = 0\} \subset \{1, 2, \ldots, r\}$. For $i \in \Lambda$, $w_{ik} = u_i u_k$ for all $k = 1, 2, \ldots, r$. Then, the function $q = 1 + \sum 2u_i t_i + \sum w_{ij} t_i t_j$ can be rewritten as

$$q = (1 + \sum_{i\in\Lambda} u_i t_i)^2 + 2\sum_{i\notin\Lambda} u_i t_i + 2\sum_{i\in\Lambda}(\sum_{k\notin\Lambda} w_{ik} t_i t_k) + \sum_{k,h\notin\Lambda} w_{kh} t_k t_h.$$

Since u_k and w_{kh} are constant for $k, h \notin \Lambda$,

$$\frac{\partial q}{\partial s} = 2(1 + \sum_{i\in\Lambda} u_i t_i)(\sum_{i\in\Lambda} u_i' t_i) + 2\sum_{i\in\Lambda}(\sum_{k\notin\Lambda} u_k u_i' t_i t_k)$$
$$= 2(1 + \sum_{j=1}^{r} u_j t_j)(\sum_{i\in\Lambda} u_i' t_i).$$

Then, (17) implies

$$\begin{aligned} & -3(1+\sum_{j=1}^{r} u_j t_j)(\sum_{i\in\Lambda} u_i' t_i)\Phi' \\ = & -(\Phi'' + \sum_{j=1}^{r} t_j \Psi_j'' - \sum_{j=1}^{r} u_j \Psi_j - \sum_{l=1}^{r}(\sum_{j=1}^{r} w_{jl}\Psi_j) t_l)(1+\sum_{j=1}^{r} u_j t_j) \\ & + (\Phi + \sum_{j=1}^{r} t_j \Psi_j)(\phi + \sum_{j=1}^{r} \varphi_j t_j - \sum_{j=1}^{r} u_j^2 - \sum_{l=1}^{r}(\sum_{j=1}^{r} u_j w_{jl}) t_l). \end{aligned} \tag{43}$$

In (43), considering the constant terms with respect to t and the coefficients of terms containing t_i for $i \in \Lambda$, we have the following equations

$$-\Phi'' + \sum_{j=1}^{r} u_j \Psi_j + \phi\Phi - (\sum_{j=1}^{r} u_j^2)\Phi = \mathbf{0}, \tag{44}$$

$$\begin{aligned} -3u_i'\Phi' = & -u_i\Phi'' + (\sum_{j=1}^{r} u_j\Psi_j)u_i - \Psi_i'' + \sum_{j=1}^{r} w_{ij}\Psi_j \\ & + \varphi_i\Phi - (\sum_{j=1}^{r} u_j w_{ij})\Phi + \phi\Psi_i - (\sum_{j=1}^{r} u_j^2)\Psi_i. \end{aligned} \tag{45}$$

Putting (44) into (45) and using the fact that $\Psi_i = u_i\Phi$ for $i \in \Lambda$, we get

$$-3u_i'\Phi' = -\Psi_i'' + \sum_{j=1}^{r} w_{ij}\Psi_j + \varphi_i\Phi - (\sum_{j=1}^{r} u_j w_{ij})\Phi. \tag{46}$$

From $\Psi_i = u_i\Phi$, we have

$$\Psi_i'' = u_i''\Phi + 2u_i'\Phi' + u_i\Phi'' \quad \text{and} \quad \varphi_i = u_i'' + u_i\phi. \tag{47}$$

By (44) and (47), equation (46) implies

$$u_i'\Phi' = \mathbf{0} \tag{48}$$

for $i \in \Lambda$.

We now suppose that $\Phi' \equiv \mathbf{0}$. By definition,

$$\Phi' = \alpha'' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{k\notin\Lambda} \alpha' \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_r.$$

It implies that

$$\alpha' \wedge e_1 \wedge \cdots \wedge e'_k \wedge \cdots \wedge e_r \wedge e_k = \mathbf{0}$$

for $k \notin \Lambda$. Therefore, the vector fields $\alpha', e_1, \ldots, e_r, e'_k$ are linearly dependent for all s which means that $e'_k = u_k \alpha'$ for $k \notin \Lambda$. But it contradicts $q \neq (1 + \sum_{i=1}^r u_i t_i)^2$. Therefore, by (48) we have

$$u'_i = 0$$

for $i \in \Lambda$.

Summing up the above results, we can see that u_j are constant functions for $j = 1, 2, \ldots, r$ and hence the functions w_{ij} are constant for all $i, j = 1, 2, \ldots, r$ because of (39). Therefore, we can conclude that

$$\frac{\partial q}{\partial s} = 0 \tag{49}$$

for all s, which contradicts $\frac{\partial q}{\partial s} \neq 0$ on the open interval I_1.

Case 2. Suppose that the function q is of the form $q(t) = (1 + \sum_{i=1}^r u_i t_i)^2$. Then, we can see that $w_{ij} = u_i u_j$ for all $i, j = 1, 2, \ldots, r$ and hence $G = \Phi$. Therefore, $\Delta G = fG$ becomes

$$\frac{1}{2q^2} \frac{\partial q}{\partial s} \Phi' - \frac{1}{q} \Phi'' = f\Phi. \tag{50}$$

Taking the inner product with Φ to the both sides of (50), we find the function f given as

$$f = -\frac{\phi(s)}{q(t)}. \tag{51}$$

Substituting f into (50) implies

$$(\sum_{i=1}^r u'_i t_i)\Phi' - (1 + \sum_{i=1}^r u_i t_i)\Phi'' = -\phi(1 + \sum_{i=1}^r u_i t_i)\Phi. \tag{52}$$

It follows that

$$\Phi'' = \phi\Phi. \tag{53}$$

By (52) and (53), we get

$$\sum_{i=1}^r u'_i \Phi' t_i = \mathbf{0}$$

and hence $u'_i \Phi' = \mathbf{0}$ for all i.

If $\Phi' \equiv \mathbf{0}$, it follows from (53) that the function ϕ is identically zero because Φ is non-zero vector field for all $s \in I$. Then, the function f is also identically zero by virtue of (51) that is a contradiction. Therefore, we have $u'_i = 0$ for all $i = 1, 2, \ldots, r$ and we can conclude that

$$\frac{\partial q}{\partial s} = 0$$

for all $s \in I$. This is a contradiction.

According to Cases 1 and 2, we conclude from equation (12) that

$$\frac{\partial q}{\partial s} = 0$$

for all $s \in I$. Therefore, we have:

Proposition 3.5. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3) *in $\mathbb{E}^m$ with pointwise 1-type Gauss map of the first kind. Then the functions*

$$u_i(s) = \langle \alpha'(s), e_i'(s) \rangle \quad \text{and} \quad w_{ij}(s) = \langle e_i'(s), e_j'(s) \rangle$$

are constant for all $i, j = 1, 2, \ldots, r$.

Now, we need the following lemma to examine the mean curvature of the ruled submanifold of $\mathbb{E}^m$ with pointwise 1-type Gauss map of the first kind:

Lemma 3.6. *Let M be an n-dimensional submanifold of a Euclidean space $\mathbb{E}^m$ with pointwise 1-type Gauss map G of the first kind. Then, the mean curvature vector field H is parallel in the normal bundle.*

Proof. See Lemma 5.1 of [22]. □

We prove that a minimality of a non-cylindrical ruled submanifold M is equivalent for M to have pointwise 1-type Gauss map of the first kind.

Theorem 3.7. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold in $\mathbb{E}^m$. Then, M has pointwise 1-type Gauss map G of the first kind if and only if M is minimal.*

Proof. Suppose that a ruled submanifold M parameterized by (3) has pointwise 1-type Gauss map of the first kind. The mean curvature vector field H is given by

$$\begin{aligned} H &= \frac{1}{r+1}\{h(\frac{x_s}{||x_s||}, \frac{x_s}{||x_s||}) + \sum_{i=1}^{r} h(x_{t_i}, x_{t_i})\} \\ &= \frac{1}{r+1}\{\frac{1}{q}h(x_s, x_s) + \sum_{i=1}^{r} h(e_i, e_i)\}, \end{aligned} \tag{54}$$

where h is the second fundamental form on M. Since $x_{t_i t_i} = 0$, (54) is reduced to

$$H = \frac{1}{(r+1)q}\{x_{ss} - \langle x_{ss}, x_s\rangle x_s - \sum_{i=1}^{r}\langle x_{ss}, e_i\rangle e_i\}.$$

By straightforward computation, we get

$$\langle x_{ss}, x_s\rangle = \sum_{i,j=1}^{r} \xi_{ij} t_i t_j \quad \text{and} \quad \langle x_{ss}, e_i\rangle = -u_i - \sum_{j=1}^{r} w_{ij} t_j.$$

According to Proposition 3.5, w_{ij} are constant for all i, $j = 1, 2, \ldots, r$ and thus

$$\sum_{i,j=1}^{r} \xi_{ij} t_i t_j = \sum_{i \le j} (\xi_{ij} + \xi_{ji}) t_i t_j = 0.$$

So, the mean curvature vector field H is expressed as

$$H = \frac{1}{(r+1)q} \{\alpha'' + \sum_{i=1}^{r} t_i e_i'' + \sum_{i=1}^{r} u_i e_i + \sum_{j=1}^{r} (\sum_{i=1}^{r} w_{ij} e_i) t_j\}, \tag{55}$$

which yields

(56)

$$\begin{aligned}\langle H, H\rangle = \frac{1}{(r+1)^2 q^2} \{\langle \alpha'', \alpha''\rangle - \sum_{k=1}^{r} u_k^2 + 2\sum_{i=1}^{r} \langle \alpha'', e_i''\rangle t_i - 2\sum_{k,i=1}^{r} u_k w_{ki} t_i \\ + \sum_{i,j=1}^{r} \langle e_i'', e_j''\rangle t_i t_j - \sum_{i,j=1}^{r} (\sum_{k=1}^{r} w_{ik} w_{jk}) t_i t_j\}.\end{aligned}$$

Differentiating (56) with respect to t_{i_0} for some i_0 and using Lemma 3.6, we have

$$\begin{aligned}0 = \frac{-2}{(r+1)^2 q^3} (\frac{\partial q}{\partial t_{i_0}}) \{\langle \alpha'', \alpha''\rangle - \sum_{k=1}^{r} u_k^2 + 2\sum_{i=1}^{r} \langle \alpha'', e_i''\rangle t_i - 2\sum_{k,i=1}^{r} u_k w_{ki} t_i \\ + \sum_{i,j=1}^{r} \langle e_i'', e_j''\rangle t_i t_j - \sum_{i,j=1}^{r} (\sum_{k=1}^{r} w_{ik} w_{jk}) t_i t_j\} \\ + \frac{2}{(r+1)^2 q^2} \{\langle \alpha'', e_{i_0}''\rangle - \sum_{k=1}^{r} u_k w_{ki_0} + \sum_{j=1}^{r} \langle e_{i_0}'', e_j''\rangle t_j \\ - \sum_{j=1}^{r} (\sum_{k=1}^{r} w_{i_0 k} w_{jk}) t_j\},\end{aligned}$$

or, equivalently,

(57)

$$\begin{aligned}0 = & -2(u_{i_0} + \sum_{j=1}^{r} w_{i_0 j} t_j) \{\langle \alpha'', \alpha''\rangle - \sum_{k=1}^{r} u_k^2 + 2\sum_{i=1}^{r} \langle \alpha'', e_i''\rangle t_i - 2\sum_{k,i=1}^{r} u_k w_{ki} t_i \\ & + \sum_{i,j=1}^{r} \langle e_i'', e_j''\rangle t_i t_j - \sum_{i,j=1}^{r} (\sum_{k=1}^{r} w_{ik} w_{jk}) t_i t_j\} \\ & + (1 + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j) \{\langle \alpha'', e_{i_0}''\rangle - \sum_{k=1}^{r} u_k w_{ki_0} \\ & + \sum_{j=1}^{r} \langle e_{i_0}'', e_j''\rangle t_j - \sum_{j=1}^{r} (\sum_{k=1}^{r} w_{i_0 k} w_{jk}) t_j\}.\end{aligned}$$

Considering the coefficients of terms containing t_j, t_j^2 and t_j^3 for some $j = 1, 2, \ldots, r$ in (57), we have

$$\begin{aligned} &- 4u_{i_0}\langle\alpha'', e''_j\rangle + 4u_{i_0}(\sum_{k=1}^{r} u_k w_{kj}) - 2w_{i_0 j}\langle\alpha'', \alpha''\rangle + 2w_{i_0 j}(\sum_{k=1}^{r} u_k^2) \\ &+ \langle e''_{i_0}, e''_j\rangle - \sum_{k=1}^{r} w_{i_0 k} w_{jk} + 2u_j\langle\alpha'', e''_{i_0}\rangle - 2u_j(\sum_{k=1}^{r} u_k w_{ki_0}) = 0, \end{aligned} \tag{58}$$

$$\begin{aligned} &- 2u_{i_0}\langle e''_j, e''_j\rangle + 2u_{i_0}(\sum_{k=1}^{r} w_{jk}^2) - 4w_{i_0 j}\langle\alpha'', e''_j\rangle + 4w_{i_0 j}(\sum_{k=1}^{r} u_k w_{kj}) \\ &+ 2u_j\langle e''_{i_0}, e''_j\rangle - 2u_j(\sum_{k=1}^{r} w_{i_0 k} w_{jk}) + w_{jj}\langle\alpha'', e''_{i_0}\rangle - w_{jj}(\sum_{k=1}^{r} u_k w_{ki_0}) = 0, \end{aligned} \tag{59}$$

$$-2w_{i_0 j}\langle e''_j, e''_j\rangle + 2w_{i_0 j}(\sum_{k=1}^{r} w_{jk}^2) + w_{jj}\langle e''_{i_0}, e''_j\rangle - w_{jj}(\sum_{k=1}^{r} w_{i_0 k} w_{jk}) = 0. \tag{60}$$

Since $w_{i_0 i_0} \neq 0$, by replacing j with i_0 in (58), (59) and (60), we can obtain easily

$$\langle\alpha'', \alpha''\rangle = \sum_{k=1}^{r} u_k^2, \quad \langle\alpha'', e''_{i_0}\rangle = \sum_{k=1}^{r} u_k w_{ki_0} \quad \text{and} \quad \langle e''_{i_0}, e''_{i_0}\rangle = \sum_{k=1}^{r} w_{i_0 k}^2. \tag{61}$$

Equation (58) with the help of (61) yields

$$\langle e''_i, e''_j\rangle = \sum_{k=1}^{r} w_{ik} w_{jk}. \tag{62}$$

Together with equations (56), (61) and (62), we conclude that the mean curvature vector field H vanishes on M.

Conversely, suppose that a non-cylindrical ruled submanifold M is minimal. The mean curvature vector field H is given by

$$\begin{aligned} H &= \frac{1}{(r+1)q}\{x_{ss} - \langle x_{ss}, x_s\rangle x_s - \sum_{i=1}^{r}\langle x_{ss}, e_i\rangle e_i\} \\ &= \frac{1}{(r+1)q}\{\alpha'' + \sum_{i=1}^{r} t_i e''_i - (\sum_{k,j=1}^{r} \xi_{kj} t_k t_j)(\alpha' + \sum_{i=1}^{r} t_i e'_i) \\ &\qquad + \sum_{i=1}^{r}(u_i + \sum_{j=1}^{r} w_{ij} t_j) e_i\}, \end{aligned}$$

from which, $H = \mathbf{0}$ implies

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i, \quad e''_i = -\sum_{j=1}^{r} w_{ji} e_j \quad \text{and} \quad \xi_{kj} = 0$$

for all $i, j, k = 1, \ldots, r$. It follows that

$$u_i' = \langle \alpha'', e_i' \rangle + \langle \alpha', e_i'' \rangle = 0.$$

Therefore, we see that u_i and w_{ij} are constant functions for all $i, j = 1, \ldots, r$ which means that $\frac{\partial q}{\partial s} = 0$ on M.

By straightforward computation, we get

$$\Phi'' + \sum_{i=1}^{r} \Psi_i'' t_i = \frac{1}{2} \sum_{k=1}^{r} \frac{\partial q}{\partial t_k} \Psi_k - \sum_{k=1}^{r} w_{kk} (\Phi + \sum_{i=1}^{r} \Psi_i t_i).$$

Then, by using terms in (12), we have

$$\Delta G = \frac{1}{q^{5/2}} \{ q \sum_{k=1}^{r} w_{kk} - \frac{1}{2} \sum_{k=1}^{r} (\frac{\partial q}{\partial t_k})^2 + \frac{1}{2} q \sum_{k=1}^{r} \frac{\partial^2 q}{\partial t_k^2} \} (\Phi + \sum_{i=1}^{r} \Psi_i t_i),$$

which is reduced to

$$\Delta G = fG$$

for some function

$$f = \frac{1}{q^2} \{ q \sum_{k=1}^{r} w_{kk} - \frac{1}{2} \sum_{k=1}^{r} (\frac{\partial q}{\partial t_k})^2 + \frac{1}{2} q \sum_{k=1}^{r} \frac{\partial^2 q}{\partial t_k^2} \}.$$

Therefore, a minimal non-cylindrical ruled submanifold has pointwise 1-type Gauss map of the first kind. It completes the proof. □

Thus, combining Theorem 3.2, Theorem 3.7 and the result on generalized helicoid in [1], we have:

Theorem 3.8 (Classification). *The only ruled submanifold M of Euclidean space $\mathbb{E}^m$ with pointwise 1-type Gauss map of the first kind is an open part of a generalized circular cylinder $\Sigma_a \times \mathbb{E}^{r-1}$ or a generalized helicoid.*

Combining the result of [9] with Theorem 3.7, we have:

Theorem 3.9. *Let M be a non-cylindrical ruled submanifold of $\mathbb{E}^m$. Then, the following are equivalent:*
(1) *M is minimal.*
(2) *M is a generalized helicoid.*
(3) *M is a finite type submanifold.*
(4) *M has pointwise 1-type Gauss map of the first kind.*

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