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CONVERGENCE RATES FOR SEQUENCES OF CONDITIONALLY INDEPENDENT AND CONDITIONALLY IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Abstract. The Marcinkiewicz-Zygmund strong law of large numbers for conditionally independent and conditionally identically distributed random variables is an existing, but merely qualitative result. In this paper, for the more general cases where the conditional order of moment belongs to $(0, \infty)$ instead of $(0, 2)$, we derive results on convergence rates which are quantitative ones in the sense that they tell us how fast convergence is obtained. Furthermore, some conditional probability inequalities are of independent interest.

1. Introduction

We will be working on a fixed probability space $(\Omega,\ \mathcal{A},\ P)$ and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. We should interpret $\mathcal{F}$ as information available, for instance, $\mathcal{F}$ may be the collection $\{\Omega,\ A,\ A^c,\ \emptyset\}$ where A represents an event of particular importance such as a massive disaster resulting from an earthquake or a hurricane.

A finite sequence $\{X_k, 1 \leq k \leq n\}$ of random variables is said to be conditionally independent given $\mathcal{F}$ ($\mathcal{F}$-independent, in short) if

$$P\left\{\bigcap_{k=1}^{n} (X_k \in B_k) \,|\mathcal{F}\right\} = \prod_{k=1}^{n} P\left(X_k \in B_k \,|\mathcal{F}\right) \text{ a.s. for all } B_k \in \mathcal{B},$$

where $\mathcal{B}$ is the Borel σ-algebra in $\mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ of random variables is said to be $\mathcal{F}$-independent if every finite subsequence is $\mathcal{F}$-independent.

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Of course, $\mathcal{F}$-independence reduces to unconditional or ordinary independence when $\mathcal{F} = \{\Omega, \emptyset\}$, but independence of random variables neither implies nor is implied by $\mathcal{F}$-independence.

We next recall the concept of conditionally identical distributiveness. A pair of random variables X and Y are said to be conditionally identically distributed given $\mathcal{F}$ ($\mathcal{F}$-identically distributed, in short) if

$$P(X \in B \,|\mathcal{F}) = P(Y \in B \,|\mathcal{F}) \text{ a.s. for all } B \in \mathcal{B}.$$

A collection of random variables is said to be $\mathcal{F}$-identically distributed if every pair of random variables in the collection is $\mathcal{F}$-identically distributed.

Just as in the case of conditional independence, $\mathcal{F}$-identical distributiveness reduces to ordinary identical distributiveness when $\mathcal{F} = \{\Omega, \emptyset\}$. It should be pointed out that conditionally identical distributiveness implies identical distributiveness, but the converse implication need not always be true, such an example can be found in [21].

An important example of conditionally independent and conditionally identically distributed sequences of random variables is the so-called exchangeable random variables. Let $\{X_n, n \geq 1\}$ be such a sequence, that is, the joint distribution of $(X_1, X_2, \ldots, X_n)$ is the same as that of $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$ for each $n \geq 1$ and any permutation π of $(1, 2, \ldots, n)$. By de Finetti's theorem, $\{X_n\}$ is conditionally independent and conditionally identically distributed given either its tail σ-algebra or its σ-algebra of permutable events, c.f. Theorem 7.3.2 of Chow and Teicher [3].

The statistical perspective of conditional independence and conditionally identical distribution is that of a Bayesian. A problem begins with a parameter θ with its prior probability distribution that exists only in mind of the investigator. The statistical model that is most commonly in use is that of a sequence $\{X_n, n \geq 1\}$ of observable random variables that is independent and identically distributed for each given value of θ. As such, $\{X_n, n \geq 1\}$ is $\mathcal{F}$-independent and $\mathcal{F}$-identically distributed but neither necessarily independent nor necessarily identically distributed, where $\mathcal{F} = \sigma(\theta)$.

Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}$-independent and $\mathcal{F}$-identically distributed random variables. As usual, let $S_n = \sum_{k=1}^n X_k$, $n \geq 1$, denote their partial sums. Majerek et al. [11] proved that

$$\lim_{n\to\infty} n^{-1} S_n = Y \text{ a.s.}$$

if and only if $E^{\mathcal{F}} X = Y$ a.s., which is a conditional version of the Kolmogorov strong law of large numbers.

The further derivations are conditional versions of generalized Kolmogorov's inequality and Hájek-Rényi's inequality due to Prakasa Rao [13], and conditional versions of Hoeffding's identity and Fubini's theorem due to Roussas [14].

Hence we are just wondering which results on independent and identically distributed random variables have analogous ones in a conditional setting? As pointed out by Prakasa Rao [13], one does have to derive results under conditioning if there is a need even though the results and proofs of such results may be analogous to those under the non-conditioning setup. This motivates our original interest in investigating conditionally independent and conditionally identically distributed random variables.

Starting from the conditional independence given a sub-σ-algebra $\mathcal{F}$, the past a decade has witnessed the active development of a rich probability theory of conditional dependence and many important theoretical results have been obtained. See, for instance, Christofides and Hadjikyriakou [4] for conditional demimartingale, Liu and Prakasa Rao [9] for conditional Borel-Cantelli lemma, Ordóñez Cabrera et al. [12] for conditionally negatively quadrant dependence, Wang and Wang [16] for conditional demimartingale and conditional N-demimartingale, Yuan and Xie [22] for conditionally linearly negatively quadrant dependence, Yuan and Lei [17] for conditional strong mixing. These achievements also stimulate us to study conditionally independent and conditionally identically distributed random variables.

In Yuan and Li [20], they established the Marcinkiewicz-Zygmund strong law of large numbers for conditionally independent and conditionally identically distributed random variables, but it is merely a qualitative result. In this paper, for the more general cases where the order of conditional moment belongs to $(0, \infty)$ instead of $(0, 2)$, we derive quantitative results in the sense that they tell us how fast convergence is obtained, which link integrability of the summands to convergence rates in strong laws of large numbers. The main result is given in Section 2, a number of conditional inequalities of independent interest are prepared in Section 3, and the proof of main result is put in Section 4.

Following Prakasa Rao [13] for the sake of convenience we will use the notation $P^{\mathcal{F}}(A)$ to denote $P(A\,|\mathcal{F})$ and $E^{\mathcal{F}}X$ to denote $E(X\,|\mathcal{F})$. Furthermore, $a \vee b = \max\{a, b\}$.

2. Main results

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with mean zero and put $S_n = \sum_{k=1}^{n} X_k$. Hsu and Robbins [8] prove that if $EX_1^2 < \infty$, then

$$\sum_{n=1}^{\infty} P(|S_n| \geq n\varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

Somewhat latter, Erdös [5, 6] proved the converse. In addition to being a result on a kind convergence, Hsu-Robbins-Erdös result can be viewed as a result on the rate of convergence in the law of large numbers. Namely, not only does the term $P(|S_n| \geq n\varepsilon)$ have to tend to zero, the sum of them has to converge, what is a little more.

The Hsu-Robbins-Erdös result was later extended in a series of papers which culminated in the paper by Baum and Katz [1], bridging the integrability of summands and the rate of convergence in the Marcinkiewicz-Zygmund strong law of large numbers, and showing that if $0 < p < 2$, $r \geq p$, then

$$E\left|X_1\right|^r < \infty \text{ where } EX_1 = 0 \text{ whenever } r \geq 1$$

is equivalent to

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\left|S_n\right| \geq n^{1/p}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0,$$

and also equivalent to

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\max_{1\leq k\leq n}\left|S_k\right| \geq n^{1/p}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0,$$

the last equivalence is due to Chow [2].

Our goal is to extend the equivalences above to the case where $\{X_n, n \geq 1\}$ is a sequence of conditionally independent and conditionally identically distributed random variables.

Theorem 2.1. *Let $0 < p < 2$ and $r \geq p$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of $\mathcal{F}$-independent and $\mathcal{F}$-identically distributed random variables. If there exists some real number s greater than $2 \vee \frac{2(r-p)}{r(2-p)}$ such that*

$$E\left(E^{\mathcal{F}}\left|X_1\right|^r\right)^s < \infty,\ E^{\mathcal{F}} X_1 = 0 \text{ a.s. whenever } r \geq 1, \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\left|S_n\right| \geq \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0 \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\max_{1\leq k\leq n}\left|S_k\right| \geq \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.3}$$

Conversely, if one of (2.2) *and* (2.3) *is finite for some $\varepsilon > 0$, then*

$$E\left|X_1\right|^r < \infty,\ E^{\mathcal{F}} X_1 = 0 \text{ a.s. whenever } r \geq 1. \tag{2.4}$$

Remark 2.2. If $\mathcal{F} = \{\emptyset, \Omega\}$, then (2.1) is equivalent to

$$E\left|X_1\right|^r < \infty,\ EX_1 = 0 \text{ whenever } r \geq 1,$$

so Theorem 2.1 is an extension to the classical non-conditional setting. Moreover, if $\mathcal{F}$ is not the trivial σ-algebra, then $E\left(E^{\mathcal{F}}\left|X_1\right|^r\right)^s < \infty$ is stronger than $E\left|X_1\right|^r < \infty$ and $E^{\mathcal{F}} X_1 = 0$ a.s. is stronger than $EX_1 = 0$.

Remark 2.3. Theorem 2.1 remains valid provided that $\{X_n, n \geq 1\}$ is a sequence of exchangeable random variables and $\mathcal{F}$ is taken as its tail σ-algebra or σ-algebra of permutable events. This result has, to the best of our knowledge, never been established in the literature before.

3. Several conditional probability inequalities

In order to prove Theorem 2.1, we establish several conditional inequalities in this section. Since these inequalities are of independent interest, we formulate them as propositions.

A random variable X is said to be $\mathcal{F}$-symmetric if X and $-X$ are $\mathcal{F}$-identically distributed. In this terminology we obtain $E^{\mathcal{F}}[XI(|X| \leq c)] = 0$ for any $c > 0$ and further establish a truncated inequality for $\mathcal{F}$-symmetric random variables as follows.

Proposition 3.1. *Let $X_1, X_2, \ldots, X_n$ be $\mathcal{F}$-independent random variables, let $b_1, b_2, \ldots, b_n$ be positive reals, and define*

$$Y_k = X_k I\left(|X_k| < b_k\right), k = 1, 2, \ldots, n.$$

If each X_k is $\mathcal{F}$-symmetry, then, for any $\varepsilon > 0$,

$$P\left(\left|\sum_{k=1}^n X_k\right| \geq \varepsilon\right) \leq \varepsilon^{-2} \sum_{k=1}^n EY_k^2 + \sum_{k=1}^n P\left(|X_k| \geq b_k\right). \tag{3.1}$$

Proof. Assertion (3.1) follows upon observing that

$$\begin{aligned} P\left(\left|\sum_{k=1}^n X_k\right| \geq \varepsilon\right) &= P\left\{\left(\left|\sum_{k=1}^n X_k\right| \geq \varepsilon\right) \cap \left[\bigcap_{k=1}^n \left(|X_k| < b_k\right)\right]\right\} \\ &\quad + P\left\{\left(\left|\sum_{k=1}^n X_k\right| \geq \varepsilon\right) \cap \left[\bigcup_{k=1}^n \left(|X_k| \geq b_k\right)\right]\right\} \\ &\leq P\left(\left|\sum_{k=1}^n Y_k\right| \geq \varepsilon\right) + P\left(\bigcup_{k=1}^n \left(|X_k| > b_k\right)\right) \\ &\leq \varepsilon^{-2} E\left(\sum_{k=1}^n Y_k\right)^2 + \sum_{k=1}^n P\left(|X_k| \geq b_k\right) \\ &= \varepsilon^{-2} \sum_{k=1}^n EY_k^2 + \sum_{k=1}^n P\left(|X_k| \geq b_k\right), \end{aligned}$$

the last equality holding since $E\left(Y_iY_j\right) = E\left[E^{\mathcal{F}}\left(Y_iY_j\right)\right] = E\left[E^{\mathcal{F}}Y_i \cdot E^{\mathcal{F}}Y_j\right] = 0$ for $i \neq j$. $\square$

Before continuing with our conditional probability inequalities we pause for recalling two concepts. The first concept is conditional median. Let ξ be a random variable, its conditional median with respect to $\mathcal{F}$, is defined as an $\mathcal{F}$-measurable random variable, say $\mathrm{Med}_{\mathcal{F}}\xi$, such that

$$P^{\mathcal{F}}\left(\xi \geq \mathrm{Med}_{\mathcal{F}}\xi\right) \geq \frac{1}{2} \leq P^{\mathcal{F}}\left(\xi \leq \mathrm{Med}_{\mathcal{F}}\xi\right) \text{ a.s.}$$

This concept was introduced by Tomkins [15], which refined the original definition proposed by Loève [10] and became slightly tougher. Note that $\mathrm{Med}_{\mathcal{F}}\xi$

is just a usual median of ξ if $\mathcal{F} = \{\Omega,\ \emptyset\}$, say Med$\xi$, which is, of course, not necessarily unique; it is easy to concoct examples to show that this is also the case when $\mathcal{F}$ is not trivial.

The second concept is conditional symmetrization. Let X and X' be $\mathcal{F}$-independent and $\mathcal{F}$-identically distributed random variables, then $X^s_{\mathcal{F}} = X - X'$ is $\mathcal{F}$-symmetric in view of Corollary 3.9 of Yuan and Lei [18] and we call $X^s_{\mathcal{F}}$ the $\mathcal{F}$-symmetrization of X.

With the help of the two concepts mentioned above we now present conditional versions of the weak symmetrization inequalities that relate tail probabilities of a random variable to tail probabilities of its $\mathcal{F}$-symmetrization.

Proposition 3.2. *Let X be a random variable. Then, for any real x and any $\mathcal{F}$-measurable random variable η,*

$$\frac{1}{2} P\left(X - Med_{\mathcal{F}} X \geq x\right) \leq P\left(X^s_{\mathcal{F}} \geq x\right)$$

and

$$\frac{1}{2} P\left(|X - Med_{\mathcal{F}} X| \geq x\right) \leq P\left(|X^s_{\mathcal{F}}| \geq x\right) \leq 2P\left(|X - \eta| \geq \frac{x}{2}\right).$$

In particular,

$$\frac{1}{2} P\left(|X - Med_{\mathcal{F}} X| \geq x\right) \leq P\left(|X^s_{\mathcal{F}}| \geq x\right) \leq 2P\left(|X - Med_{\mathcal{F}} X| \geq \frac{x}{2}\right).$$

Proof. Since X and X' are $\mathcal{F}$-identically distributed, $Med_{\mathcal{F}} X$ is also a conditional median of X' with respect to $\mathcal{F}$. In addition, Lemma 2.4 of Yuan et al. [21] shows that $X - Med_{\mathcal{F}} X$ and $X' - Med_{\mathcal{F}} X$ are $\mathcal{F}$-independent. Thus

$$\begin{aligned}
&P\left(X^s_{\mathcal{F}} \geq x\right) = P\left\{(X - Med_{\mathcal{F}} X) - (X' - Med_{\mathcal{F}} X) \geq x\right\} \\
&\geq P\left(X - Med_{\mathcal{F}} X \geq x, X' - Med_{\mathcal{F}} X \leq 0\right) \\
&= E\left[P^{\mathcal{F}}\left(X - Med_{\mathcal{F}} X \geq x\right) P^{\mathcal{F}}\left(X' - Med_{\mathcal{F}} X \leq 0\right)\right] \\
&\geq \frac{1}{2} E\left[P^{\mathcal{F}}\left(X - Med_{\mathcal{F}} X \geq x\right)\right] = \frac{1}{2} P\left(X - Med_{\mathcal{F}} X \geq x\right).
\end{aligned}$$

This justifies the first inequality, which, together with the inequality obtained by changing X into $-X$, proves the left-most inequality in the second one. The right-most inequality follows from the observation

$$\begin{aligned}
&P\left(|X^s_{\mathcal{F}}| \geq x\right) = P\left\{|(X - \eta) - (X' - \eta)| \geq x\right\} \\
&\leq P\left(|X - \eta| \geq \frac{x}{2}\right) + P\left(|X' - \eta| \geq \frac{x}{2}\right) = 2P\left(|X - \eta| \geq \frac{x}{2}\right),
\end{aligned}$$

where $P\left(|X - \eta| \geq x/2\right) = P\left(|X' - \eta| \geq x/2\right)$ since $X - \eta$ and $X' - \eta$ are $\mathcal{F}$-identically distributed in view of Theorem 3.7 of [18]. $\square$

Similarly, we can obtain the conditional versions of the strong symmetrization inequalities.

Proposition 3.3. *Let $X_1, X_2, \ldots, X_n$ be random variables. Then, for any real x and any $\mathcal{F}$-measurable random variables $\eta_1, \eta_2, \ldots, \eta_n$,*

$$\frac{1}{2}P\left\{\max_{1\le k\le n}(X_k - Med_{\mathcal{F}}X_k) \ge x\right\} \le P\left(\max_{1\le k\le n} X^s_{k,\mathcal{F}} \ge x\right) \tag{3.2}$$

and

$$\begin{aligned}\frac{1}{2}P\left(\max_{1\le k\le n}|X_k - Med_{\mathcal{F}}X_k| \ge x\right) &\le 2P\left(\max_{1\le k\le n}\left|X^s_{k,\mathcal{F}}\right| \ge x\right)\\ &\le 4P\left(\max_{1\le k\le n}|X_k-\eta_k| \ge \frac{x}{2}\right).\end{aligned} \tag{3.3}$$

In particular,

$$\begin{aligned}\frac{1}{2}P\left(\max_{1\le k\le n}|X_k - Med_{\mathcal{F}}X_k| \ge x\right) &\le 2P\left(\max_{1\le k\le n}\left|X^s_{k,\mathcal{F}}\right| \ge x\right)\\ &\le 4P\left(\max_{1\le k\le n}|X_k - Med_{\mathcal{F}}X_k| \ge \frac{x}{2}\right).\end{aligned}$$

Proof. Let $(X'_1, X'_2, \ldots, X'_n)$ be an $\mathcal{F}$-independent copy of $(X_1, X_2, \ldots, X_n)$, namely, let $(X'_1, X'_2, \ldots, X'_n)$ and $(X_1, X_2, \ldots, X_n)$ be $\mathcal{F}$-independent and $\mathcal{F}$-identically distributed. Put

$$A_1 = \{X_1 - Med_{\mathcal{F}}X_1 \ge x\},$$

$$A_k = \left\{\max_{1\le j\le k-1}(X_j - Med_{\mathcal{F}}X_j) < x,\ X_k - Med_{\mathcal{F}}X_k \ge x\right\},\ k = 2, \ldots, n,$$

$$B_k = \{X'_k - Med_{\mathcal{F}}X_k \le 0\}, k = 1, 2, \ldots, n$$

and

$$C_k = \{X^s_k \ge x\},\ k = 1, 2, \ldots, n.$$

Since the A_k are pairwise disjoint with $A_k \cap B_k \subset C_k$,

$$\begin{aligned}P\left(\bigcup_{k=1}^{n} C_k\right) &\ge P\left\{\bigcup_{k=1}^{n}(A_k\cap B_k)\right\} = \sum_{k=1}^{n} P(A_k\cap B_k)\\ &= \sum_{k=1}^{n} E\left[P^{\mathcal{F}}(A_k)\,P^{\mathcal{F}}(B_k)\right] \ge \frac{1}{2}\sum_{k=1}^{n} E\left[P^{\mathcal{F}}(A_k)\right]\\ &= \frac{1}{2}\sum_{k=1}^{n} P(A_k) = \frac{1}{2}P\left(\bigcup_{k=1}^{n} A_k\right),\end{aligned}$$

this completes the proof of (3.2). By changing X_k into $-X_k$, we obtain

$$\frac{1}{2}P\left\{\max_{1\le k\le n}\left[-(X_k - Med_{\mathcal{F}}X_k)\right] \ge x\right\} \le P\left\{\max_{1\le k\le n}\left(-X^s_{k,\mathcal{F}}\right) \ge x\right\},$$

which, together with (3.2), yields

$$\frac{1}{2}P\left\{\max_{1\le k\le n}|X_k - Med_{\mathcal{F}}X_k| \ge x\right\}$$

$$\leq \frac{1}{2}P\left\{\max_{1\leq k\leq n}\left(X_k - Med_{\mathcal{F}}X_k\right)\geq x\right\}$$
$$+\frac{1}{2}P\left\{\max_{1\leq k\leq n}\left[-\left(X_k - Med_{\mathcal{F}}X_k\right)\right]\geq x\right\}$$
$$\leq P\left\{\max_{1\leq k\leq n}X^s_{k,\mathcal{F}}\geq x\right\}+P\left\{\max_{1\leq k\leq n}\left(-X^s_{k,\mathcal{F}}\right)\geq x\right\}$$
$$\leq 2P\left\{\max_{1\leq k\leq n}\left|X^s_{k,\mathcal{F}}\right|\geq x\right\},$$

which is precisely the left-most inequality of (3.3). The proof of the right-most inequality is similar to that in the last part of Proposition 3.2. □

The forthcoming inequalities extend the Lévy inequalities (see Theorem 3.7.1 of Gut [7]) to conditional case.

Proposition 3.4. *Let $X_1, X_2, \ldots, X_n$ be $\mathcal{F}$-independent random variables. Then, for any real x,*

$$P\left\{\max_{1\leq k\leq n}\left[S_k - Med_{\mathcal{F}}\left(S_k - S_n\right)\right]\geq x\right\}\leq 2P\left(S_n\geq x\right) \tag{3.4}$$

and

$$P\left\{\max_{1\leq k\leq n}\left|S_k - Med_{\mathcal{F}}\left(S_k - S_n\right)\right|\geq x\right\}\leq 2P\left(\left|S_n\right|\geq x\right). \tag{3.5}$$

If, in addition, each X_k is $\mathcal{F}$-symmetric, then

$$P\left(\max_{1\leq k\leq n}S_k\geq x\right)\leq 2P\left(S_n\geq x\right)$$

and

$$P\left(\max_{1\leq k\leq n}\left|S_k\right|\geq x\right)\leq 2P\left(\left|S_n\right|\geq x\right). \tag{3.6}$$

Proof. Set $M_k = \max_{1\leq j\leq k}\left[S_j - Med_{\mathcal{F}}\left(S_j - S_n\right)\right]$ for $k = 1, 2, \ldots, n$. Write

$$A_1 = \left\{S_1 - Med_{\mathcal{F}}\left(S_1 - S_n\right)\geq x\right\},$$
$$A_k = \left\{M_{k-1} < x, S_k - Med_{\mathcal{F}}\left(S_k - S_n\right)\geq x\right\},\ k = 2, \ldots, n$$

and

$$B_k = \left\{S_n - S_k - Med_{\mathcal{F}}\left(S_n - S_k\right)\geq 0\right\},\ k = 1, 2, \ldots, n.$$

Clearly the A_k are pairwise disjoint with $\cup_{k=1}^n A_k = \{M_n \geq x\}$, $P^{\mathcal{F}}\left(B_k\right)\geq 1/2$ and

$$\left\{S_n\geq x\right\}\supset\bigcup_{k=1}^{n}\left(A_k\cap B_k\right)$$

upon noticing that $Med_{\mathcal{F}}\left(S_k - S_n\right) = -Med_{\mathcal{F}}\left(S_n - S_k\right)$. Also, Lemma 2.4 of Yuan et al. [21] shows that A_k and B_k are $\mathcal{F}$-independent. It follows that

$$P\left(S_n\geq x\right)\geq\sum_{k=1}^{n}P\left(A_k\cap B_k\right) = \sum_{k=1}^{n}E\left[P^{\mathcal{F}}\left(A_k\right)P^{\mathcal{F}}\left(B_k\right)\right]$$

$$\geq \frac{1}{2}\sum_{k=1}^{n} P\left(A_k\right) = \frac{1}{2}P\left(M_n \geq x\right),$$

which is precisely (3.4). To prove (3.5), we replace X_k by $-X_k$, $1 \leq k \leq n$, in (3.4) to get

$$P\left\{-S_n \geq x\right\} \geq \frac{1}{2}P\left\{\max_{1\leq k\leq n}\left[-\big(S_k - Med_{\mathcal{F}}\left(S_k - S_n\right)\big)\right] \geq x\right\}.$$

Combining the last inequality with (3.4) leads to

$$\begin{aligned} P\left\{|S_n| \geq x\right\} &= P\left\{S_n \geq x\right\} + P\left\{-S_n \geq x\right\} \\ &\geq \frac{1}{2}P\left\{\max_{1\leq k\leq n}\left|S_k - Med_{\mathcal{F}}\left(S_k - S_n\right)\right| \geq x\right\}, \end{aligned}$$

which is just the desired conclusion. □

Remark 3.5. A nice by-product of the above proof is that all results in Theorem 3.4 also remain true in the sense of almost sure by replacing P whenever it occurs by $P^{\mathcal{F}}$. For example, (3.6) can turn into

$$P^{\mathcal{F}}\left(\max_{1\leq k\leq n}|S_k| \geq x\right) \leq 2P^{\mathcal{F}}\left(|S_n| \geq x\right) \text{ a.s.,} \tag{3.7}$$

which will be used in the proof of Proposition 3.6.

Finally, we extend the Kahane-Hoffmann-Jørgensen inequalities (see Theorem 3.7.5 of Gut [7]) to conditional case.

Proposition 3.6. *Let $X_1, X_2, \ldots, X_n$ be $\mathcal{F}$-independent and $\mathcal{F}$-symmetric random variables.*

(i) *For any $x, y > 0$,*

$$\begin{aligned} P\left(|S_n| \geq 2x + y\right) &\leq P\left(\max_{1\leq k\leq n}|X_k| \geq y\right) + 4E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^2 \\ &\leq \sum_{k=1}^{n} P\left(|X_k| \geq y\right) + 4E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^2. \end{aligned} \tag{3.8}$$

(ii) *For any integer $j \geq 1$ and any $x > 0$,*

$$P\left(|S_n| \geq 3^j x\right) \leq C_j P\left(\max_{1\leq k\leq n}|X_k| \geq y\right) + D_j E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^{2^j}, \tag{3.9}$$

where C_j and D_j are numerical constants depending only on j. If, in addition, $X_1, X_2, \ldots, X_n$ are $\mathcal{F}$-identically distributed, then

$$P\left(|S_n| \geq 3^j x\right) \leq C_j n P\left(|X_1| \geq y\right) + D_j E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^{2^j}.$$

Proof. Set $Y_n = \max_{1\leq k\leq n}|X_k|$,

$$A_1 = \left\{|S_1| \geq x\right\}, A_k = \left\{\max_{1\leq j\leq k-1}|S_j| < x,\ |S_k| \geq x\right\}, k = 2, \ldots, n.$$

Then the A_k are pairwise disjoint with

$$\{|S_n| \geq 2x + y\} \subset \bigcup_{k=1}^{n} A_k = \left\{ \max_{1 \leq k \leq n} |S_n| \geq x \right\}.$$

Consequently,

$$P\left(|S_n| \geq 2x + y\right) = P\left\{ \left(|S_n| \geq 2x + y\right) \cap \left(\bigcup_{k=1}^{n} A_k \right) \right\} = \sum_{k=1}^{n} P\left\{ \left(|S_n| \geq 2x + y\right) \cap A_k \right\}. \tag{3.10}$$

Since by the triangular inequality,

$$|S_n| \leq |S_{k-1}| + |X_k| + |S_n - S_k| \text{ for } 1 \leq k \leq n,$$

it follows that, on the set $\{|S_n| \geq 2x + y\} \cap A_k$,

$$|S_n - S_k| \geq |S_n| - |S_{k-1}| - |X_k| \geq 2x + y - x - Y_n = x + y - Y_n,$$

so that, noticing that $S_n - S_k$ and A_k are $\mathcal{F}$-independent by Lemma 2.4 of Yuan et al. [21],

$$\begin{aligned}
&P\left\{ \left(|S_n| \geq 2x + y\right) \cap A_k \right\} \leq P\left\{ \left(|S_n - S_k| \geq x + y - Y_n\right) \cap A_k \right\} \\
&= P\left\{ \left(|S_n - S_k| \geq x + y - Y_n\right) \cap A_k \cap \left(Y_n \geq y\right) \right\} \\
&\quad + P\left\{ \left(|S_n - S_k| \geq x + y - Y_n\right) \cap A_k \cap \left(Y_n < y\right) \right\} \\
&\leq P\left\{ A_k \cap \left(Y_n \geq y\right) \right\} + P\left\{ \left(|S_n - S_k| \geq x\right) \cap A_k \right\} \\
&= P\left\{ A_k \cap \left(Y_n \geq y\right) \right\} + E\left[P^{\mathcal{F}}\left(|S_n - S_k| \geq x\right) P^{\mathcal{F}}\left(A_k\right) \right] \\
&\leq P\left\{ A_k \cap \left(Y_n \geq y\right) \right\} + 2E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right) P^{\mathcal{F}}\left(A_k\right) \right],
\end{aligned}$$

the last inequality being a consequence of (3.7). Joining this with (3.10) finally yields

$$\begin{aligned}
&P\left(|S_n| \geq 2x + y\right) \\
&\leq \sum_{k=1}^{n} P\left\{ A_k \cap \left(Y_n \geq y\right) \right\} + 2\sum_{k=1}^{n} E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right) P^{\mathcal{F}}\left(A_k\right) \right] \\
&= P\left\{ \left(\bigcup_{k=1}^{n} A_k \right) \cap \left(Y_n \geq y\right) \right\} + 2E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right) P^{\mathcal{F}}\left(\bigcup_{k=1}^{n} A_k \right) \right] \\
&\leq P\left(Y_n \geq y\right) + 4E\left[P^{\mathcal{F}}\left(|S_n| \geq x\right) \right]^2,
\end{aligned}$$

where we exploited (3.7) in the final step.

(ii) We need only to prove the following almost sure version of (3.9):

$$P^{\mathcal{F}}\left(|S_n| \geq 3^j x\right) \leq C_j P^{\mathcal{F}}\left(\max_{1 \leq k \leq n} |X_k| \geq y \right) + D_j \left[P^{\mathcal{F}}\left(|S_n| \geq x\right) \right]^{2^j} \text{ a.s.,} \tag{3.11}$$

because once this relation has been verified one can obtain (3.9) by taking expectation. To prove (3.11), we notice that (3.8) is also true in the sense of almost sure by replacing P by $P^{\mathcal{F}}$, namely

$$(3.12)\quad P^{\mathcal{F}}\left(|S_n| \geq 2x+y\right) \leq P^{\mathcal{F}}\left(\max_{1\leq k\leq n}|X_k| \geq y\right) + 4\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^2 \text{ a.s.}$$

Now we prove (3.11) by induction. For $j = 1$, note that (3.11) is covered by (3.12) with $y = x$. Next assume that (3.11) holds for $j-1\,(j \geq 2)$, then, exploiting the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ for positive numbers a, b, it follows from the induction hypothesis that

$$P^{\mathcal{F}}\left(|S_n| \geq 3^j x\right) \leq P^{\mathcal{F}}\left(Y_n \geq 3^{j-1}x\right) + 4\left[P^{\mathcal{F}}\left(|S_n| \geq 3^{j-1}x\right)\right]^2$$

$$\leq P^{\mathcal{F}}\left(Y_n \geq 3^{j-1}x\right) + 4\left\{C_{j-1}P^{\mathcal{F}}\left(Y_n \geq x\right) + D_{j-1}\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^{2^{j-1}}\right\}^2$$

$$\leq P^{\mathcal{F}}\left(Y_n \geq 3^{j-1}x\right) + 8C_{j-1}^2\left[P^{\mathcal{F}}\left(Y_n \geq x\right)\right]^2 + 8D_{j-1}^2\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^{2^j}$$

$$\leq \left(1 + 8C_{j-1}^2\right)P^{\mathcal{F}}\left(Y_n \geq x\right) + 8D_{j-1}^2\left[P^{\mathcal{F}}\left(|S_n| \geq x\right)\right]^{2^j} \text{ a.s.}$$

This proves (3.11) with $C_j = 1 + 8C_{j-1}^2$ and $D_j = 8D_{j-1}^2$. □

4. Proof of Theorem 2.1

The approach of this proof is quite different from that of Theorem 6.12.1 in Gut [7], but we are obliged to repeat some of arguments used there for completeness.

We first consider the proof of the first part.

(i) $\{X_n\}$ is an $\mathcal{F}$-symmetric sequence.

We distinguish four cases to verify (2.2).

(a) For $r = p$, set, for $n \geq 1$,

$$Y_{n,k} = X_k I\left(|X_k| < n^{1/r}\right),\ 1 \leq k \leq n$$

and

$$S_n' = \sum_{k=1}^{n} Y_{n,k}.$$

Proposition 3.1, and the fact that conditionally identical distribution implies identical distribution, then yield

$$P\left(|S_n| \geq \varepsilon n^{1/r}\right) \leq \varepsilon^{-2}n^{1-2/r}EY_{n,1}^2 + nP\left(|X_1| \geq n^{1/r}\right),$$

so that

$$\sum_{n=1}^{\infty} n^{r/p-2}P^{\mathcal{F}}\left(|S_n| \geq \varepsilon n^{1/p}\right) = \sum_{n=1}^{\infty} n^{-1}P^{\mathcal{F}}\left(|S_n| \geq \varepsilon n^{1/p}\right)$$

$$\leq \varepsilon^{-2}\sum_{n=1}^{\infty} n^{-2/r}EY_{n,1}^2 + \sum_{n=1}^{\infty} P\left(|X_1| \geq n^{1/r}\right) =: \varepsilon^{-2}I_1 + I_2.$$

It is well known that $E|X_1|^r < \infty$ guarantees $I_2 < \infty$. As for I_1, by changing the order of expectation and summation and the fact that $0 < r < 2$,

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{-2/r} E\left[X_1^2 I\left(|X_1| \le n^{1/r}\right)\right] \\
&= E\left[X_1^2 \sum_{n=1}^{\infty} n^{-2/r} I\left(|X_1| \le n^{1/r}\right)\right] = E\left(X_1^2 \sum_{n:\ n\ge |X_1|^r \vee 1} n^{-2/r}\right) \\
&= E\left[X_1^2 I\left(0 \le |X_1| < 2^{1/r}\right) \sum_{n=1}^{\infty} n^{-2/r}\right] \\
&\quad + E\left[X_1^2 I\left(|X_1| \ge 2^{1/r}\right) \sum_{n:\ n\ge |X_1|^r} n^{-2/r}\right] \\
&\le 2^{2/r} \sum_{n=1}^{\infty} n^{-2/r} + \frac{2^{2/r-1}}{2/r-1} E\left[X_1^2 \left(|X_1|^r\right)^{-2/r+1}\right] \\
&= 2^{2/r} \sum_{n=1}^{\infty} n^{-2/r} + \frac{2^{2/r-1} r}{2-r} E|X_1|^r < \infty.
\end{aligned}$$

(b) For $r > p$, $r \le 1$, applying, successively, Proposition 3.6 (i), the conditional Markov inequality, and the c_r-inequality yields

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2} P\left(|S_n| \ge \varepsilon n^{1/p}\right) \\
\le\ & \sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \ge n^{1/p}\varepsilon \big/ 3\right) + 4 \sum_{n=1}^{\infty} n^{r/p-2} E\left[\frac{E^{\mathcal{F}} |S_n|^r}{\left(n^{1/p}\varepsilon/3\right)^r}\right]^2 \\
\le\ & \sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \ge n^{1/p}\varepsilon \big/ 3\right) + 4 \sum_{n=1}^{\infty} n^{r/p-2} E\left[\frac{n E^{\mathcal{F}} |X_1|^r}{\left(n^{1/p}\varepsilon/3\right)^r}\right]^2 \\
=\ & \sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \ge n^{1/p}\varepsilon \big/ 3\right) + 4 \times 9^r \varepsilon^{-2r} E\left(E^{\mathcal{F}} |X_1|^r\right)^2 \sum_{n=1}^{\infty} n^{-r/p}.
\end{aligned}$$

The second term is finite because $E\left[E^{\mathcal{F}} |X_1|^r\right]^2 < \infty$ and $\sum_{n=1}^{\infty} n^{-r/p} < \infty$ implied by $r > p$. With regard to the first term, since $E\left(|X_1|^p\right)^{r/p} = E|X_1|^r < \infty$, it follows that from Theorem 2.12.1(iv) in Gut [7]

$$\sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \ge n^{1/p}\varepsilon \big/ 3\right) = \sum_{n=1}^{\infty} n^{r/p-1} P\left(\left|3\varepsilon^{-1} X_1\right|^p \ge n\right) < \infty. \tag{4.1}$$

(c) For $r > p$, $1 < r < 2$, the same procedure as in (b) with the c_r-inequality replaced by Theorem 3.3 in Yuan and Li [20] yields the desired result.

(d) For $r \geq 2$, let $j > 1$ to be specified later. We use Proposition 3.6(ii) and Theorem 3.3 in [20] to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2} P\left(|S_n| \geq \varepsilon n^{1/p}\right) \\
&= \sum_{n=1}^{\infty} n^{r/p-2} P\left(|S_n| \geq 3^j 3^{-j} n^{1/p} \varepsilon\right) \\
&\leq C_j \sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \geq 3^{-j} n^{1/p} \varepsilon\right) + D_j \sum_{n=1}^{\infty} n^{r/p-2} E\left[\frac{E^{\mathcal{F}} |S_n|^r}{\left(3^{-j} n^{1/p} \varepsilon\right)^r}\right]^{2^j} \\
&\leq C_j \sum_{n=1}^{\infty} n^{r/p-1} P^{\mathcal{F}}\left(|X_1| \geq 3^{-j} n^{1/p} \varepsilon\right) \\
&\quad + D_j \sum_{n=1}^{\infty} n^{r/p-2} E\left[\frac{B_r^* n^{r/2} E^{\mathcal{F}} |X_1|^r}{\left(3^{-j} n^{1/p} \varepsilon\right)^r}\right]^{2^j} \\
&= C_j \sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \geq 3^{-j} n^{1/p} \varepsilon\right) \\
&\quad + D_j \left(B_r^* 3^{rj} \varepsilon^{-r}\right)^{2^j} E\left(E^{\mathcal{F}} |X_1|^r\right)^{2^j} \sum_{n=1}^{\infty} n^{\beta},
\end{aligned}$$

where B_r^* is a constant depending only on r and

$$\beta = rp^{-1} - 2 + \left(2^{-1} r\right) 2^j - 2^j rp^{-1} = rp^{-1} - 2 + 2^{j-1} rp^{-1} (p-2).$$

Analogously to the proof of (4.1), one can show that the first sum is finite. The second sum is also finite because $\beta < -1$ if taking $j > \log_2 \frac{2(r-p)}{r(2-p)}$.

Now we prove that (2.3) is true. In fact, (2.2) implies (2.3) according to the fact that

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq n^{1/p} \varepsilon\right) \leq 2P\left(|S_n| \geq n^{1/p} \varepsilon\right),$$

which is a straight-forward consequence of Proposition 3.4.

(ii) $\{X_n\}$ is the original sequence.

Since $E\left|X_1^s\right|^r \leq (2 \vee 2^r) E\left|X_1\right|^r < \infty$, relation (2.2) holds by what have just now been proved for the $\mathcal{F}$-symmetrized case, and therefore we have

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(|S_n - Med_{\mathcal{F}} S_n| \geq n^{1/p} \varepsilon\right) < \infty \text{ for all } \varepsilon > 0$$

according to Proposition 3.2, which implies (2.2) if

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(|Med_{\mathcal{F}} S_n| \geq n^{1/p} \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

To prove this assertion, we first note that

$$P^{\mathcal{F}}\left\{|S_n| \geq 2^{1/r}\left(E^{\mathcal{F}}|S_n|^r\right)^{1/r}\right\} \leq \frac{E^{\mathcal{F}}|S_n|^r}{\left[2^{1/r}\left(E^{\mathcal{F}}|S_n|^r\right)^{1/r}\right]^r} = \frac{1}{2},$$

so that, by the definition of a conditional median,

$$|Med_{\mathcal{F}}S_k| \leq 2^{1/r}\left(E^{\mathcal{F}}|S_k|^r\right)^{1/r} \text{ a.s.},$$

and thus, we need only to prove that

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left\{2^{1/r}\left(E^{\mathcal{F}}|S_n|^r\right)^{1/r} \geq n^{1/p}\varepsilon\right\} < \infty \text{ for all } \varepsilon > 0$$

or, equivalently,

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

(a) For $0 < r < 1$, we have

$$P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \frac{E\left(E^{\mathcal{F}}|S_n|^r\right)^2}{n^{2r/p}\varepsilon^2} \leq \frac{n^{2r}E\left(E^{\mathcal{F}}|X_1|^r\right)^2}{n^{2r/p}\varepsilon^2},$$

which indicates that

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \varepsilon^{-2}E\left(E^{\mathcal{F}}|X_1|^r\right)^2 \sum_{n=1}^{\infty} n^{-r/p-2+2r},$$

the last series converges because $-r/p - 2 + 2r \leq -3 + 2r < -1$.

(b) For $1 \leq r < 2$, we have

$$P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \frac{E|S_n|^r}{n^{r/p}\varepsilon} \leq \frac{n^{r/2}E|X_1|^r}{n^{r/p}\varepsilon}$$

by Theorem 3.3 in [20], which indicates that

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \varepsilon^{-1}E|X_1|^r \sum_{n=1}^{\infty} n^{-2+r/2},$$

the last series converges because $-2 + r/2 < -1$.

(c) For $r \geq 2$, we have

$$P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \frac{E\left(E^{\mathcal{F}}|S_n|^r\right)^s}{n^{sr/p}\varepsilon^s} \leq \frac{n^{sr/2}E\left(E^{\mathcal{F}}|X_1|^r\right)^s}{n^{sr/p}\varepsilon^s}$$

again by Theorem 3.3 in [20], which indicates that

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(E^{\mathcal{F}}|S_n|^r \geq n^{r/p}\varepsilon\right) \leq \varepsilon^{-s}E\left(E^{\mathcal{F}}|X_1|^r\right)^s \sum_{n=1}^{\infty} n^{r/p-2+sr/2-sr/p},$$

the last series converges because $r/p - 2 + sr/2 - sr/p < -1$ by recalling $s > 2(r-p)/r(2-p)$.

Finally, we point out that the procedure for (2.3) is the same, however, with Proposition 3.3 instead of Proposition 3.2.

For the second part of the theorem, since (2.3) imply (2.2), we only need to prove (2.2) $\Rightarrow$(2.4). Once again we begin with the $\mathcal{F}$-symmetric case. In view of Proposition 3.4, we have

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\max_{1\le k\le n} |S_k| \ge n^{1/p}\varepsilon\right) < \infty \text{ for some } \varepsilon > 0. \tag{4.2}$$

Since $|X_k| \le |S_k| + |S_{k-1}|$, it follows that $\max_{1\le k\le n} |X_k| \le 2\max_{1\le k\le n} |S_k|$, and hence (4.2) implies

$$\sum_{n=1}^{\infty} n^{r/p-2} P\left(\max_{1\le k\le n} |X_k| \ge n^{1/p}\varepsilon/2\right) < \infty \text{ for some } \varepsilon > 0, \tag{4.3}$$

from which, we assert

$$P\left(\max_{1\le k\le n} |X_k| \ge n^{1/p}2^{1/p}\varepsilon/2\right) \to 0 \text{ for some } \varepsilon > 0 \text{ as } n \to \infty. \tag{4.4}$$

In fact, the result is obvious if $r/p \ge 2$, so we assume that $1 \le r/p < 2$. In view of (4.3),

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{r/p-2} P\left(\max_{1\le k\le n} |X_k| \ge n^{1/p}\varepsilon/2\right) \\
&\ge \sum_{j=0}^{\infty} P\left(\max_{1\le k\le 2^j} |X_k| \ge 2^{(j+1)/p}\varepsilon/2\right) \sum_{n=2^j}^{2^{j+1}-1} n^{r/p-2} \\
&\ge \sum_{j=0}^{\infty} P\left(\max_{1\le k\le 2^j} |X_k| \ge 2^{(j+1)/p}\varepsilon/2\right) 2^{(j+1)(r/p-2)}2^j \\
&= 2^{r/p-2} \sum_{j=0}^{\infty} 2^{(r/p-1)j} P\left(\max_{1\le k\le 2^j} |X_k| \ge 2^{(j+1)/p}\varepsilon/2\right),
\end{aligned}$$

which entails since $r/p - 1 \ge 0$ that

$$P\left(\max_{1\le k\le n} |X_k| \ge 2^{(j+1)/p}\varepsilon/2\right) \to 0 \text{ for some } \varepsilon > 0 \text{ as } n \to \infty. \tag{4.5}$$

For each $n \ge 1$, let $j_n \ge 0$ be such that $2^{j_n} \le n < 2^{j_n+1}$, that is, $j_n = [\log_2 n]$, where $\log_2$ denotes the logarithm to the base 2 and $[x]$ denotes the largest integer not exceeding x. By (4.5),

$$\begin{aligned}
P\left(\max_{1\le k\le n} |X_k| \ge n^{1/p}2^{1/p}\varepsilon/2\right) &\le P\left(\max_{1\le k\le 2^{j_n+1}} |X_k| \ge 2^{(j_n+1)/p}\varepsilon/2\right) \\
&\to 0 \text{ as } n \to 0,
\end{aligned}$$

proving (4.4).

From (4.4), we further assert that

$$6P\left(\max_{1\le k\le n}|X_k|\ge n^{1/p}2^{1/p}\varepsilon/2\right)\ge nP\left(|X_1|\ge n^{1/p}2^{1/p}\varepsilon/2\right) \tag{4.6}$$

for sufficient large n. To prove this, set $\tilde{\varepsilon}:=2^{1/p}\varepsilon/2$. Since

$$P\left(\max_{1\le k\le n}|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)=\sum_{k=1}^{n}P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon},\max_{1\le j\le k-1}|X_j|< n^{1/p}\tilde{\varepsilon}\right),$$

we deduce that

$$\begin{aligned}(4.7)\quad nP\left(|X_1|\ge n^{1/p}\tilde{\varepsilon}\right)&=\sum_{k=1}^{n}P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)\\ &=P\left(\max_{1\le k\le n}|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)+\sum_{k=1}^{n}P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon},\max_{1\le j\le k-1}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right).\end{aligned}$$

But

$$\begin{aligned}(4.8)\quad &\sum_{k=1}^{n}P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon},\max_{1\le j\le k-1}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)\\ &\le\sum_{k=1}^{n}EI\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)I\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)\\ &=E\sum_{k=1}^{n}\left[I\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)-P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)\right]I\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)\\ &\quad+nP\left(|X_1|\ge n^{1/p}\tilde{\varepsilon}\right)P\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)\\ &=:I_3+nP\left(|X_1|\ge n^{1/p}\tilde{\varepsilon}\right)P\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right).\end{aligned}$$

By applying the Cauchy-Schwarz inequality and Theorem 2.1 of Yuan and Li [19], we get

$$\begin{aligned}(4.9)\quad &|I_3|\\ &\le\sqrt{E\left\{\sum_{k=1}^{n}\left[I\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)-P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)\right]\right\}^2P\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)}\\ &\le\sqrt{E\sum_{k=1}^{n}E^{\mathcal{F}}\left[I\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)-P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)\right]^2P\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)}\\ &\le\sqrt{\sum_{k=1}^{n}P\left(|X_k|\ge n^{1/p}\tilde{\varepsilon}\right)P\left(\max_{1\le j\le n}|X_j|\ge n^{1/p}\tilde{\varepsilon}\right)}\end{aligned}$$

$$\leq \frac{1}{2} nP\left(|X_1| \geq n^{1/p}\tilde{\varepsilon}\right) + \frac{1}{2} P\left(\max_{1\leq j\leq n} |X_j| \geq n^{1/p}\tilde{\varepsilon}\right).$$

From (4.7)-(4.9), we obtain

$$nP\left(|X_1| \geq n^{1/p}\tilde{\varepsilon}\right) \leq 3P\left(\max_{1\leq j\leq n} |X_j| \geq n^{1/p}\tilde{\varepsilon}\right) + 2nP\left(|X_1| \geq n^{1/p}\tilde{\varepsilon}\right) P\left(\max_{1\leq j\leq n} |X_j| \geq n^{1/p}\tilde{\varepsilon}\right),$$

which in conjunction with (4.4) yields (4.6).

Inserting (4.6) into (4.3), we obtain

$$\sum_{n=1}^{\infty} n^{r/p-1} P\left(|X_1| \geq n^{1/p} 2^{1/p}\varepsilon/2\right) < \infty \text{ for some } \varepsilon > 0,$$

which is equivalent to $E\left(\left|2^{1-1/p}X_1/\varepsilon\right|^p\right)^{r/p} < \infty$, and therefore to $E\left|X_1\right|^r < \infty$.

Now, suppose that (2.2) holds for some $\varepsilon > 0$ in the general case. Then it also does so for the $\mathcal{F}$-symmetricized variables (with $\varepsilon/2$). Hence $E\left|X^s_{1,\mathcal{F}}\right|^r < \infty$, from which we conclude that $E\left|X_1\right|^r < \infty$. Furthermore, if $r \geq 1$, then the conditional mean must be almost surely finite, so that the conditional strong law of large numbers (Theorem 4.2 of Majerek et al. [11]) holds, which, in turn, forces the conditional mean to equal 0 almost surely.

References

[1] L. E. Baum and M. Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. **120** (1965), no. 1, 108–123.

[2] Y. S. Chow, *Delayed sums and Borel summability of independent, identically distributed random variables*, Bull. Inst. Math. Acad. Sinica **1** (1973), no. 2, 207–220.

[3] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd Edition, New York, Springer-Verlag, 1997.

[4] T. C. Christofides and M. Hadjikyriakou, *Conditional demimartingales and related results*, J. Math. Anal. Appl. **398** (2013), no. 1, 380–391.

[5] P. Erdös, *On a theorem of Hsu and Robbins*, Ann. Math. Statist. **20** (1949), no. 2, 286–291.

[6] ———, *Remark on my paper "On a theorem of Hsu and Robbins"*, Ann. Math. Statist. **21** (1950), no. 1, 138.

[7] A. Gut, *Probability: A Graduate Course*, 2nd Edition, New York, Springer-Verlag, 2013.

[8] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), no. 2, 25–31.

[9] J. C. Liu and B. L. S. Prakasa Rao, *On conditional Borel-Cantelli lemmas for sequences of random variables*, J. Math. Anal. Appl. **399** (2013), no. 1, 156–165.

[10] M. Loève, *Probability Theory*, 3rd Edition, Princeton, Van Nostrand, 1963.

[11] D. Majerek, W. Nowak, and W. Zieba, *Conditional strong law of large number*, Int. J. Pure Appl. Math. **20** (2005), no. 2, 143–157.

[12] M. Ordóñez Cabrera, A. Rosalsky, and A. Volodin, *Some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables*, TEST **21** (2012), no. 2, 369–385.

[13] B. L. S. Prakasa Rao, *Conditional independence, conditional mixing and conditional association*, Ann. Inst. Statist. Math. **61** (2009), no. 2, 441–460.
[14] G. G. Roussas, *On conditional independence, mixing, and association*, Stoch. Anal. Appl. **26** (2008), no. 6, 1274–1309.
[15] R. J. Tomkins, *On conditional medians*, Ann. Probab. **3** (1975), no. 2, 375–379.
[16] X. H. Wang and X. J. Wang, *Some inequalities for conditional demimartingales and conditional N-demimartingales*, Statist. Probab. Lett. **83** (2013), no. 3, 700–709.
[17] D. M. Yuan and L. Lei, *Some conditional results for conditionally strong mixing sequences of random variables*, Sci. China Math. **56** (2013), no. 4, 845–859.
[18] ———, *Some results following from conditional characteristic functions*, Comm. Statist. Theory Methods Meth. doi: 10.1080/03610926.2014.906614.
[19] D. M. Yuan and S. J. Li, *From conditional independence to conditionally negative association: some preliminary results*, Comm. Statist. Theory Methods **44** (2015), no. 18, 3942–3966.
[20] ———, *Extensions of several classical results for independent and identically distributed random variables to conditional cases*, J. Korean Math. Soc. **52** (2015), no. 2, 431–445.
[21] D. M. Yuan, L. R. Wei, and L. Lei, *Conditional central limit theorems for a sequence of conditional independent random variables*, J. Korean Math. Soc. **51** (2014), no. 1, 1–15.
[22] D. M. Yuan and Y. Xie, *Conditional limit theorems for conditionally linearly negative quadrant dependent random variables*, Monatsh. Math. **166** (2012), no. 2, 281–299.

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