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SOME EXPRESSIONS FOR THE INVERSE INTEGRAL TRANSFORM VIA THE TRANSLATION THEOREM ON FUNCTION SPACE

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Abstract. In this paper, we analyze the necessary and sufficient condition introduced in [5]: that a functional F in $L^2(C_{a,b}[0,T])$ has an integral transform $\mathcal{F}_{\gamma,\beta}F$, also belonging to $L^2(C_{a,b}[0,T])$. We then establish the inverse integral transforms of the functionals in $L^2(C_{a,b}[0,T])$ and then examine various properties with respect to the inverse integral transforms via the translation theorem. Several possible outcomes are presented as remarks. Our approach is a new method to solve some difficulties with respect to the inverse integral transform.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of real-valued continuous functions x on $[0,T]$ with $x(0) = 0$. The concept of the integral transform $\mathcal{F}_{\gamma,\beta}$ was introduced by Lee in his unifying paper [15]. In [4, 12, 13, 14, 16], the authors studied the integral transform and the convolution product of functionals in various classes. Recently paper [6, 9], the authors established basic formulas for integral transforms, convolution products and inverse integral transforms, and they also established a Fubini theorem for integral transforms and convolution products of functionals in $L_2(C_0[0,T])$.

The function space $C_{a,b}[0,T]$ induced by a generalized Brownian motion was introduced by J. Yeh in [20] and was used extensively in [5, 7, 8, 9, 10]. In [5], the authors gave a necessary and sufficient condition that a functional F in $L^2(C_{a,b}[0,T])$ has an integral transform $\mathcal{F}_{\gamma,\beta}F$ also belonging to $L^2(C_{a,b}[0,T])$.

Previous researches have established the inverse transform T^{-1} of a transform T on function space (e.g., analytic Fourier-Feynman transform, (modified) Fourier-Wiener transform, or integral transform) namely;

$$T^{-1}T(F) = TT^{-1}(F)$$

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as well as basic formulas

$$T(F*G)=T(F)T(G) \text{ and } T(F)*T(G)=T(FG),$$

where $*$ is the convolution product with respect to T. While it is well known that establishing the inverse transform and two basic formulas on function space can be difficult. The functionals in $L^2(C_{a,b}[0,T])$ are particularly problematic. Recently [7], the authors have established a version of the inverse integral transform and various basic formulas involving the inverse integral transform of functionals. In particular, they have established the inverse integral transform of functionals in an appropriate class as follows:

$$\mathcal{F}_{\gamma,\beta}^{-1}=\mathcal{F}_{-i\gamma,1}\circ\mathcal{F}_{i\gamma,1}\circ\mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}}, \tag{1.1}$$

where $\circ$ denotes the composition of operators. However, for the existence of the generalized integral transform $\mathcal{F}_{\gamma,\beta}$ of the functionals in $L^2(C_{a,b}[0,T])$, we assumed that $\gamma^2+\beta^2=1$. In this case, γ must be zero. Therefore, this inverse integral transform should not be used for functionals in $L^2(C_{a,b}[0,T])$, as in equation (1.1). We need to establish a new version with respect to the inverse integral transform.

In this paper, we introduce a dense set in $L^2(C_{a,b}[0,T])$, and then determine the inverse integral transform for the functionals in $L^2(C_{a,b}[0,T])$. Several properties of the inverse integral transform are examined. Finally, we present several applications via the translation theorem on function space. The most important feature of this paper is that we demonstrated the ability to represent the generalized integral transform, and the inverse integral transform of functionals in $L^2(C_{a,b}[0,T])$ via the translation theorem.

However, when $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$ and so most of the results in [6] follow immediately from the results in this paper. The Wiener process used in [4, 6, 11, 13, 14, 15, 16, 19] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [5, 7, 8, 18], is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [17].

2. Preliminaries

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t)=\min\{b(s),b(t)\}$. By Theorem 14.2 in [21], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

A subset E of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho E \in \mathcal{W}(C_{a,b}[0,T])$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere.

Let

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T |v^2(s)| db(s) < \infty \text{ and } \int_0^T |v^2(s)| d|a|(s) < \infty \right\},$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$. Then $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space with the norm $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ and $(u,v)_{a,b} = \int_0^T u(t)\overline{v(t)} d[b(t) + |a|(t)]$.

For each $v \in L^2_{a,b}[0,T]$, let $\langle v, x \rangle$ denote the Paley-Wiener-Zygmund(PWZ) stochastic integral. Note that the properties of the PWZ integral studied several times in many papers. For more details see, [5, 7, 8, 10].

Throughout this paper we will assume that each functional $F : C_{a,b}[0,T] \to \mathbb{C}$ we consider is scale-invariant measurable and that $\int_{C_{a,b}[0,T]} |F(\rho x)| d\mu(x) < \infty$ for each $\rho > 0$.

We are now ready to state the definition of the generalized integral transform used in [5, 7, 10].

Definition 2.1. Let $K_{a,b}[0,T]$ be the complexification of $C_{a,b}[0,T]$ and let F be a functional defined on $K_{a,b}[0,T]$. For each pair of nonzero complex numbers γ and β, the generalized integral transform $\mathcal{F}_{\gamma,\beta}F$ of F is defined by

$$\mathcal{F}_{\gamma,\beta}F(y) \equiv \mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) d\mu(x), \tag{2.1}$$

$y \in K_{a,b}[0,T]$ if it exists.

Throughout this paper, in order to ensure that various integrals exist, we will assume that $\beta = c + id$ is a nonzero complex number satisfying the inequality

$$\mathrm{Re}(1 - \beta^2) = 1 + d^2 - c^2 > 0. \tag{2.2}$$

Note that $\beta = c + id$ satisfies equality (2.2) if and only if $(c,d) \in \mathbb{R}^2$ lies in the open region determined by the hyperbola $c^2 - d^2 = 1$ containing the d-axis. Next, let

$$\gamma = \sqrt{1 - \beta^2}, \quad -\pi/4 < \arg(\gamma) < \pi/4 \tag{2.3}$$

and note that $\gamma^2 + \beta^2 = 1$ and $\mathrm{Re}(\gamma^2) = \mathrm{Re}(1 - \beta^2) > 0$.

3. Inverse integral transforms of functionals in $L^2(C_{a,b}[0,T])$

In this section, we introduce a class $\mathcal{A}$ of functionals in $L^2(C_{a,b}[0,T])$. We then establish the inverse integral transforms of functionals in $\mathcal{A}$.

We shall analyze the condition introduced in [5, 14] to obtain the existence of inverse integral transform. In one-parameter Wiener space $C_0[0,T]$ (i.e., where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ for this study), the existence of the integral transform depends on the size of β (see [14]). If $|\beta| \le 1$, then the integral transform of F in $L_2(C_0[0,T])$ always exists, and is an element of $L_2(C_0[0,T])$. On function space, the existence of the integral transform depends on the pairs of (γ,β) for more detailed see [5, Theorem 7].

Now we introduce a class $\mathcal{A}$ of functionals which is used in this paper. Let

$$\mathcal{G} = \{(\gamma,\beta) \in \mathbb{C}\times\mathbb{C} : \mathcal{F}_{\gamma,\beta}(F) \in L^2(C_{a,b}[0,T]), F \in L^2(C_{a,b}[0,T])\}.$$

Also, let $\mathbb{E}_0^{(m)}$ (m is fixed) be the class of functionals of the form

$$F(x) = f(\langle\alpha_1,x\rangle,\dots,\langle\alpha_m,x\rangle) = f(\langle\vec{\alpha},x\rangle),$$

where $\{\alpha_1,\dots,\alpha_m\}$ is a orthonormal set in $L^2_{a,b}[0,T]$ and f is an entire function on $\mathbb{C}^m$ and

$$|f(\vec{u})| \le L_F \exp\{K_F \sum_{j=1}^m |u_j|\}$$

for some positive real numbers L_F and K_F. Then for all nonzero complex numbers γ and β (and hence $(\gamma,\beta) \in \mathcal{G}$), the integral transform $\mathcal{F}_{\gamma,\beta}(F)$ exists and is an element of $\mathbb{E}_0^{(m)}$, for more details see [7]. Also, we can easily check that $\mathbb{E}_0^{(m)} \subset L^2(C_{a,b}[0,T])$ for all $m = 1,2,\dots$. Next, the generalized Fourier-Hermite functional $\Phi_{(m_1,\dots,m_N)}(x)$ satisfies all definitions of the class $\mathbb{E}_0^{(N)}$ for each $N = 1,2,\dots$ and likewise that the functional $F_N(x) = \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)}\Phi_{(m_1,\dots,m_N)}(x)$ also belong to $\cup_{N=1}^\infty \mathbb{E}_0^{(N)}$. Let

$$\mathcal{A} = \cup_{N=1}^\infty \mathbb{E}_0^{(N)}.$$

Then the class $\mathcal{A}$ is dense in $L^2(C_{a,b}[0,T])$ since the fact that

$$\mathcal{M} \equiv \{\Phi_{(m_1,\dots,m_N)}\}_{N=1}^\infty$$

is an orthonormal set in $L^2(C_{a,b}[0,T])$. Hence by using general theories in vector space, we could extend all results and formulas of the space $\mathcal{A}$ to the $L^2(C_{a,b}[0,T])$.

Remark 3.1. To apply the translation theorem, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < \infty;$$

from which it follows that $\int_0^T \left[\frac{a'(t)}{b'(t)}\right]^2 d[b(t)+|a|(t)] < \infty$. These tell us that the mean function a can be written like as $a(t) = \int_0^t z(s)db(s)$ where $z(s) = \frac{a'(s)}{b'(s)} \in L^2_{a,b}[0,T]$.

The following result was established in [10].

Theorem 3.2. *Let the function $a = a(t)$ be as in Remark 3.1 above. Let F be a μ-integrable functional on $K_{a,b}[0,T]$. Then for nonzero complex number c, $F(x+ca)$ is μ-integrable as a function of $x \in C_{a,b}[0,T]$ and*

$$\begin{aligned}&\int_{C_{a,b}[0,T]} F(x+ca)d\mu(x)\\ &= \exp\Big\{-\frac{c^2+2c}{2}(\frac{a'}{b'},a')\Big\}\int_{C_{a,b}[0,T]} F(x)\exp\Big\{c\langle\frac{a'}{b'},x\rangle\Big\}d\mu(x),\end{aligned}$$

where $(z,a') = \int_0^T z(t)da(t)$ for some $z \in L^2_{a,b}[0,T]$.

As mentioned in Section 1, establishing the inverse integral transform has proven to be difficult for functionals in $L^2(C_{a,b}[0,T])$. Hereafter, the operator T_c is defined as specified below, to solve these difficulties. We then establish the inverse integral transform in Theorem 3.4. Define an operator T_c from $\mathcal{A}$ into $\mathcal{A}$ by

$$T_c(F)(x) = F(x+ca) \tag{3.1}$$

for $x \in C_{a,b}[0,T]$ and complex number c. The operator T_c is well-defined for all complex number c from Theorem 3.2 above. When $a(t) \equiv 0$ on $[0,T]$, the operator T_c is the identity operator for all complex number c and it has an inverse operator T_{-c} for all complex number c.

The following lemma was established in [8] and used in [10]. The formula (3.2) is called the Fubini theorem with respect to the function space integrals.

Lemma 3.3. *Let F be μ-integrable defined on $K_{a,b}[0,T]$. Then for all complex numbers γ and β,*

$$\int_{C^2_{a,b}[0,T]} F(\gamma x+\beta y)d(\mu\times\mu)(x,y) = \int_{C_{a,b}[0,T]} F(\sqrt{\gamma^2+\beta^2}w+ca)d\mu(w), \tag{3.2}$$

where $c = \gamma+\beta-\sqrt{\gamma^2+\beta^2}$.

The following theorem is the first main result in this paper. Throughout this theorem, we establish the inverse integral transform of functionals in $\mathcal{A}$. By the denseness of $\mathcal{A}$, we can easily extend the result in Theorem 3.4 to the functionals in $L^2(C_{a,b}[0,T])$.

Theorem 3.4. *Let F be an element of $\mathcal{A}$ and let $c = -\frac{\gamma}{\beta}(1+i)$ for nonzero complex numbers γ and β. Then for all $(\gamma,\beta) \in \mathcal{G}$ with $(\frac{i\gamma}{\beta},\frac{1}{\beta}) \in \mathcal{G}$,*

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F))(y) = F(y) = \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cF))(y) \tag{3.3}$$

for $y \in C_{a,b}[0,T]$. This tells us that the inverse integral transform $\mathcal{F}^{-1}_{\gamma,\beta}$ of integral transform $\mathcal{F}_{\gamma,\beta}$ is given by

$$\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c.$$

Proof. First for all $(\gamma,\beta)\in\mathcal{G}$, $T_c(F)$ is an element of $\mathcal{A}$ and so for all $(\frac{i\gamma}{\beta},\frac{1}{\beta})\in\mathcal{G}$, $\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F)$ is an element of $\mathcal{A}$. Next, using equations (3.1) and (2.1) it follows that for $y\in C_{a,b}[0,T]$,

$$\begin{aligned}\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F))(y) &= \int_{C_{a,b}[0,T]} T_c(\mathcal{F}_{\gamma,\beta}F)\left(i\frac{\gamma}{\beta}x+\frac{1}{\beta}y\right)d\mu(x)\\ &= \int_{C_{a,b}[0,T]} \mathcal{F}_{\gamma,\beta}(F)\left(i\frac{\gamma}{\beta}x+\frac{1}{\beta}y-\frac{\gamma}{\beta}(1+i)a\right)d\mu(x)\\ &= \int_{C^2_{a,b}[0,T]} F(\gamma z+i\gamma x+y-\gamma(1+i)a)d\mu(z)d\mu(x).\end{aligned}$$

Now let $F_y(x)=F(x+y)$. Then we have

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F))(y) = \int_{C^2_{a,b}[0,T]} F_{y-\gamma(1+i)a}(\gamma z+i\gamma x)d\mu(z)d\mu(x).$$

Now using equation (3.2), the last expression above equals to the expression

$$\begin{aligned}&\int_{C_{a,b}[0,T]} F_{y-\gamma(1+i)a}(\sqrt{\gamma^2+(i\gamma)^2}w+(\gamma+i\gamma-\sqrt{\gamma^2+(i\gamma)^2})a)d\mu(z)\\ &= \int_{C_{a,b}[0,T]} F_{y-\gamma(1+i)a}((\gamma+i\gamma)a)d\mu(z)\\ &= F_{y-\gamma(1+i)a}((\gamma+i\gamma)a)=F(y).\end{aligned}$$

On the other hands, using the similar method in statement above, we can conclude that

$$\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(F)))(y)=F(y).$$

Hence we complete the proof of Theorem 3.4 as desired. □

Next, we give two examples illustrating the use of equation (3.3) in Theorem 3.4.

Example 3.5. For $x\in C_{a,b}[0,T]$, let $F(x)=x(T)$. Then clearly F is an element of $\mathcal{A}$. Now we will calculate all expressions in equation (3.3) and verify equality.

(i) Using equation (2.1), it follows that for all $y\in C_{a,b}[0,T]$,

$$\mathcal{F}_{\gamma,\beta}(F)(y)=\beta y(T)+\gamma a(T)$$

and so

$$T_c(\mathcal{F}_{\gamma,\beta}(F))(y)=\beta y(T)-i\gamma a(T)$$

Furthermore, using equation (2.1), it follows that for all $y\in C_{a,b}[0,T]$,

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F))(y)=y(T)=F(y).$$

(ii) As proceeding as (i), using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$T_c(F)(y) = y(T) - \frac{\gamma}{\beta}a(T) - \frac{i\gamma}{\beta}a(T)$$

and so

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cF)(y) = \frac{1}{\beta}y(T) - \frac{\gamma}{\beta}a(T).$$

Furthermore, using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cF))(y) = y(T) = F(y).$$

Example 3.6. For $x \in C_{a,b}[0,T]$, let $G(x) = \exp\{x(T)\} = e^{F(x)}$. Then clearly G is an element of $\mathcal{A}$. Now we will calculate all expressions in equation (3.3) and verify equality.

(i) Using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$\mathcal{F}_{\gamma,\beta}(G)(y) = \exp\left\{\beta y(T) + \frac{\gamma^2}{2}b^2(T) + \gamma a(T)\right\}$$

and so

$$T_c(\mathcal{F}_{\gamma,\beta}(G))(y) = \exp\left\{\beta y(T) + \frac{\gamma^2}{2}b^2(T) - i\gamma a(T)\right\}.$$

Furthermore, using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}G))(y) = \exp\{y(T)\} = G(y).$$

(ii) As proceeding as (i), using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$T_c(G)(y) = \exp\left\{y(T) - \frac{\gamma}{\beta}a(T) - \frac{i\gamma}{\beta}a(T)\right\}$$

and so

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cG)(y) = \exp\left\{\frac{1}{\beta}y(T) - \frac{\gamma^2}{2\beta^2}b^2(T) - \frac{\gamma}{\beta}a(T)\right\}.$$

Furthermore, using equation (2.1), it follows that for all $y \in C_{a,b}[0,T]$,

$$\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cG))(y) = \exp\{y(T)\} = G(y).$$

Remark 3.7. In the setting of one-parameter Wiener space $C_0[0,T]$ (i.e., where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ in our research), the operator T_c is the identity operator and so $\mathcal{F}_{\gamma,\beta}^{-1} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}$.

4. Some properties of the inverse integral transform

In Section 3, we established the inverse integral transform. In this section, we use the inverse integral transforms and the operator T_c to investigate some properties with respect to the inverse integral transforms.

The following lemma tells us that the operators $T_c, c \in \mathbb{C}$ and $\mathcal{F}_{\gamma,\beta}$ are commutative with constant weighted.

Lemma 4.1. *Let F be an element of $\mathcal{A}$ and let c be any nonzero complex number. Then for all $(\gamma,\beta) \in \mathcal{G}$,*

$$\mathcal{F}_{\gamma,\beta}(T_{\beta c}(F))(y) = T_c(\mathcal{F}_{\gamma,\beta}(F))(y) \tag{4.1}$$

for $y \in C_{a,b}[0,T]$. That is to say, $\mathcal{F}_{\gamma,\beta} \circ T_{\beta c} = T_c \circ \mathcal{F}_{\gamma,\beta}$.

Proof. Since $F \in \mathcal{A}$, we can easily check that both sides of equation (4.1) exist. Next, using equations (2.1) and (3.1), it follows that for $y \in C_{a,b}[0,T]$

$$\begin{aligned}\mathcal{F}_{\gamma,\beta}(T_c(F))(y) &= \int_{C_{a,b}[0,T]} T_c(F)(\gamma x + \beta y) d\mu(x) \\ &= \int_{C_{a,b}[0,T]} F(\gamma x + \beta y + ca) d\mu(x)\end{aligned}$$

and

$$T_c(\mathcal{F}_{\gamma,\beta}(F))(y) = \mathcal{F}_{\gamma,\beta}(F)(y + ca) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y + \beta ca) d\mu(x)$$

which complete Lemma 4.1 as desired. □

The following theorem is the second main theorem in this paper.

Theorem 4.2. *Let F, γ, β and c be as in Theorem 3.4. Then*

$$\mathcal{F}_{\gamma,\beta}(T_{\beta c}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(F)))(y) = F(y) = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(\mathcal{F}_{\gamma,\beta}(T_{\beta c}(F)))(y) \tag{4.2}$$

for $y \in C_{a,b}[0,T]$. This tells us that the inverse integral transform $\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}$ of integral transform $\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}$ is given by

$$\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} = \mathcal{F}_{\gamma,\beta} \circ T_{\beta c}.$$

Proof. From Theorem 3.4, we established that $\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c$. This implies that

$$\mathcal{F}_{\gamma,\beta} = (\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c)^{-1} = T_c^{-1} \circ \mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}$$

and hence

$$\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} = T_c \circ \mathcal{F}_{\gamma,\beta}.$$

Using equation (4.1) in Lemma 4.1, we establish equation (4.2) as desired. □

From the main theorems outlined in this paper, other versions of the inverse integral transforms can be easily obtained.

Corollary 4.3. *Let F, γ, β and c be as in Theorem 3.4. Then we have the following a table.*

TABLE 1. Some relations

Some relations	Formulas
Inverse transform 1	$\mathcal{F}^{-1}_{\gamma,\beta} = T_{\beta c} \circ \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}$
Inverse transform 2	$\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} = T_c \circ \mathcal{F}_{\gamma,\beta}$
Composition 1	$(\mathcal{F}_{\gamma,\beta} \circ \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}})^{-1}(F)(y) = T_c(F)(y) = F(y + ca)$
Composition 2	$(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ \mathcal{F}_{\gamma,\beta})^{-1}(F)(y) = T_{\beta c}(F)(y) = F(y + \beta ca)$

The following lemma was established in [10, Lemma 3.8].

Lemma 4.4. *Under the hypotheses of Theorem 3.2, we have*

$$\begin{aligned} (4.3) \qquad &\int_{C_{a,b}[0,T]} F(\gamma x + \beta y + ca) d\mu(x) \\ &= \exp\Big\{ -\frac{c^2 + 2c\gamma}{2\gamma^2} (\frac{a'}{b'}, a') \Big\} \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) \exp\Big\{ \frac{c}{\gamma} \langle \frac{a'}{b'}, x \rangle \Big\} d\mu(x). \end{aligned}$$

To obtain expressions just one integral transform with respect to the inverse integral transform, we need the following lemma.

Lemma 4.5. *Let $F \in \mathcal{A}$. Then for all $(\gamma, \beta) \in \mathcal{G}$ and for all complex number c,*

$$(4.4) \quad \mathcal{F}_{\gamma,\beta}(T_c(F))(y) = \exp\Big\{ -\frac{c^2 + 2c\gamma}{2\gamma^2} (\frac{a'}{b'}, a') - \frac{c\beta}{\gamma^2} \langle \frac{a'}{b'}, y \rangle \Big\} \mathcal{F}_{\gamma,\beta}(F^*_{c/\gamma^2})(y)$$

for $y \in C_{a,b}[0,T]$, where $F^*_c(x) = F(x) \exp\{c\langle \frac{a'}{b'}, x \rangle\}$.

Proof. Using equation (4.3), it follows that for $y \in C_{a,b}[0,T]$,

$$\begin{aligned} &\mathcal{F}_{\gamma,\beta}(T_c(F))(y) \\ &= \int_{C_{a,b}[0,T]} F(\gamma x + \beta y + ca) d\mu(x) \\ &= \exp\Big\{ -\frac{c^2 + 2c\gamma}{2\gamma^2} (\frac{a'}{b'}, a') \Big\} \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) \exp\Big\{ \frac{c}{\gamma} \langle \frac{a'}{b'}, x \rangle \Big\} d\mu(x). \end{aligned}$$

On the other hand, using equation (2.1), it follows that for $y \in C_{a,b}[0,T]$,

$$\mathcal{F}_{\gamma,\beta}(F^*_{c/\gamma^2})(y) = \exp\Big\{ \frac{c\beta}{\gamma^2} \langle \frac{a'}{b'}, y \rangle \Big\} \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) \exp\Big\{ \frac{c}{\gamma} \langle \frac{a'}{b'}, x \rangle \Big\} d\mu(x).$$

It implies that

$$\int_{C_{a,b}[0,T]} F(\gamma x + \beta y) \exp\Big\{ \frac{c}{\gamma} \langle \frac{a'}{b'}, x \rangle \Big\} d\mu(x) = \exp\Big\{ -\frac{c\beta}{\gamma^2} \langle \frac{a'}{b'}, y \rangle \Big\} \mathcal{F}_{\gamma,\beta}(F^*_{c/\gamma^2})(y),$$

which completes the proof of Lemma 4.5 as desired. □

We hoped that the inverse integral transform was presented a single integral transform. However it is impossible (as see Section 5 below). But, we can obtain a representation which looks like a single integral transform with some weighted.

Theorem 4.6. *Let F, γ, β and c be as in Theorem 4.2. Then*

$$\mathcal{F}^{-1}_{\gamma,\beta}(F)(y) = \exp\bigg\{ (\frac{a'}{b'}, a') - \frac{1+i}{\gamma}\langle \frac{a'}{b'}, y\rangle \bigg\} \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(F^*_{c_1})(y) \tag{4.5}$$

and

$$\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(F)(y) = \exp\bigg\{ (\frac{a'}{b'}, a') + \frac{\beta(1+i)}{\gamma}\langle \frac{a'}{b'}, y\rangle \bigg\} \mathcal{F}_{\gamma,\beta}(F^*_{c_2})(y) \tag{4.6}$$

for $y \in C_{a,b}[0,T]$, where $c_1 = \frac{\beta(1+i)}{\gamma}$ and $c_2 = -\frac{1+i}{\gamma}$.

Proof. Previous theorems, we established that $\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c$ and $\mathcal{F}^{-1}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} = \mathcal{F}_{\gamma,\beta} \circ T_{c\beta}$. First, by replacing γ and β with $\frac{i\gamma}{\beta}$ and $\frac{1}{\beta}$ in equation (4.4), and then simple calculations with $c = -\frac{\gamma}{\beta}(1+i)$,

$$-\frac{c^2+2c\gamma}{2\gamma^2}(\frac{a'}{b'}, a') - \frac{c\beta}{\gamma^2}\langle \frac{a'}{b'}, y\rangle \text{ and } c/\gamma^2$$

become

$$(\frac{a'}{b'}, a') - \frac{1+i}{\gamma}\langle \frac{a'}{b'}, y\rangle \text{ and } \frac{\beta}{\gamma}(1+i)$$

and hence we obtain equation (4.5) as desired. Similar method in previous proceed, by replacing c with $c\beta$ in equation (4.4), and then simple calculations with $c\beta = -\gamma(1+i)$, we also obtain equation (4.6) as desired. □

Remark 4.7. In the setting of one-parameter Wiener space $C_0[0,T]$ (i.e., where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ in our research), all results and formulas [4, 6, 13, 14] are special cases of our results and formulas in this paper.

5. Further results

We finish this paper by giving several applications as remarks. First, we have three potential formulations for the integral transform. As such, we can ask the following questions;

(1) Is there a pair (γ, β) so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma,\beta}(F)(y) \tag{5.1}$$

for $y \in C_{a,b}[0,T]$?

(2) Is there a pair (γ, β) and complex number c so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma,\beta}(T_c(F))(y) \tag{5.2}$$

for $y \in C_{a,b}[0,T]$?

(3) Are there pairs (γ_1, β_1) and (γ_2, β_2) so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y) \tag{5.3}$$

for $y \in C_{a,b}[0,T]$?

In fact, the answer to questions (1) and (3) is negative. While the answer to question (2) is positive, as we will demonstrate below. For the sake of simplicity, we state formulas without of conditions for the existences.

Remark 5.1. Let $F \in \mathcal{A}$ be a functional defined on $K_{a,b}[0,T]$. Then

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma',\beta'}(T_c(F))(y) \tag{5.4}$$

for $y \in K_{a,b}[0,T]$, with $\gamma' = \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}$, $\beta' = \beta_1\beta_2$ and $c = \gamma_1 + \beta_1\gamma_2 - \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}$. However, γ_1 is not zero and thus c must not be zero. Thus the answer to question (1) is negative and the answer to question (2) is positive.

Remark 5.2. Let $F \in \mathcal{A}$ be a functional defined on $K_{a,b}[0,T]$. Then

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma',\beta'}(T_{c_1}(F))(y) \tag{5.5}$$

and

$$\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y) = \mathcal{F}_{\gamma',\beta'}(T_{c_2}(F))(y) \tag{5.6}$$

for $y \in K_{a,b}[0,T]$, with $\gamma' = \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}$, $\beta' = \beta_1\beta_2$, $\gamma'' = \sqrt{\gamma_2^2 + \beta_2^2\gamma_1^2}$, $\beta'' = \beta_1\beta_2$,$c_1 = \gamma_1 + \beta_1\gamma_2 - \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}$ and $c_2 = \gamma_2 + \beta_2\gamma_1 - \sqrt{\gamma_2^2 + \beta_2^2\gamma_1^2}$. However the system

$$\begin{cases} \gamma_1^2 + \beta_1^2\gamma_2^2 = \gamma_2^2 + \beta_2^2\gamma_1^2 \\ \gamma_1 + \beta_1\gamma_2 - \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2} = \gamma_2 + \beta_2\gamma_1 - \sqrt{\gamma_2^2 + \beta_2^2\gamma_1^2} \end{cases}$$

has no solution except the trivial solution. Thus the answer to question (3) is negative.

The following remark tells us that all answers to questions (1), (2) and (3) are positive in the setting of Wiener space. Theses formulas were appeared in [6, 9].

Remark 5.3. In the setting of Wiener space (i.e., in the case where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ in our research),

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma',\beta'}(F)(y) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y) \tag{5.7}$$

for $y \in C_0[0,T]$, with $\gamma' = \sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}$ and $\beta' = \beta_1\beta_2$ if $\gamma_j^2 + \beta_j^2 = 1$, $j = 1, 2$.

We close this paper by giving some conclusions.

In [14, 16], the authors studied the integral transform of functionals F in $L_2(C_0[0,T])$. They showed that for $F \in L_2(C_0[0,T])$ and nonzero complex numbers α and β with $|\beta| \le 1$, $\beta \ne \pm 1$, $\mathrm{Re}(1-\beta^2) > 0$, $\alpha = \sqrt{1-\beta^2}$ and $-\pi/4 < \arg(\alpha) < \pi/4$,

$$\mathcal{F}_{\alpha',1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y), \quad y \in C_0[0,T] \tag{5.8}$$

where $\alpha' = \sqrt{1 - 1/\beta^2}$. That is to say, "$\mathcal{F}_{\alpha,\beta}^{-1}$" is given by "$\mathcal{F}_{i\alpha/\beta,1/\beta}$". In [7], Chang et al. presented a version of inverse transform of the generalized integral transform $\mathcal{F}_{\alpha,\beta}$ as follows: for appropriate functionals F on $K_{a,b}[0,T]$,

$$\mathcal{F}_{-i\alpha,1}\mathcal{F}_{i\alpha,1}\mathcal{F}_{-\alpha/\beta,1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y) \tag{5.9}$$

for $y \in K_{a,b}[0,T]$, i.e.,

$$\mathcal{F}_{\alpha,\beta}^{-1} = \mathcal{F}_{-i\alpha,1}\mathcal{F}_{i\alpha,1}\mathcal{F}_{-\alpha/\beta,1/\beta}. \tag{5.10}$$

But we pointed out that for any nonzero complex numbers $\alpha_1, \alpha_2, \beta_1$ and β_2, there are no nonzero complex numbers α' and β' such that

$$\mathcal{F}_{\alpha_1,\beta_1}\mathcal{F}_{\alpha_2,\beta_2} = \mathcal{F}_{\alpha',\beta'}. \tag{5.11}$$

Hence the inverse generalized integral transform, $\mathcal{F}_{\alpha,\beta}^{-1}$ can not be expressed as a simple integral transform. It is not easy to verify the existence of the inverse generalized integral transform because the generalized Brownian motion process has a drift term $a(t)$, see Remark 5.1.

Let us return to the inverse integral transforms discussed in this paper. Theorems 3.4 and 4.2 tell us that the inverse integral transform can be expressed as a more simpler integral transform by using the operator T_c defined by equation (3.1) above. In particular, it also can be expressed as a simple integral transform with appropriate weighted functionals see Theorem 4.6. By choosing $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$ and our inverse integral transform and the inverse integral transform introduced in [14, 16] coincide.

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